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Fixed point results for multivalued mappings on a sequence in a closed ball with applications

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Abstract

In this paper, we establish fixed point results for semi α_*-admissible multivalued mappings satisfying a contractive condition of Reich type only for the elements in a sequence contained in closed ball in a complete dislocated metric space. As an application, we derive some new fixed point theorems for ordered metric space and metric space endowed with a graph. An example has been constructed to demonstrate the novelty of our results. Our results unify, extend, and generalize several comparable results in the existing literature.

Keywords: Fixed point, complete dislocated metric space, closed ball, semi α_*-admissible.
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1. Introduction and preliminaries

Let $S : X \longrightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of S if $x = Sx$. Many results appeared in literature related to the fixed point of mappings which are contractive on the whole domain. It is possible that $S : X \longrightarrow X$ is not a contraction but $S : Y \longrightarrow X$ is a contraction, where Y is a closed ball in X. One can obtain fixed point results for such mapping by using suitable conditions. Recently, Hussain et al. [14] proved a result concerning the existence of fixed points of a mapping satisfying a contractive condition on closed ball (see also [3–6, 28–30]).

The notion of dislocated topologies have useful applications in the context of logic programming semantics (see [12]). Dislocated metric space (metric-like space) (see [2, 17, 25]) is a generalization of partial metric space (see [19, 26]). Karapınar et al. [17] noticed that the notions metric-like space [2] and dislocated metric space [12] are exactly the same. They also discussed the existence and uniqueness of a fixed point of a cyclic mapping in the context of metric-like spaces. Arshad et al. [5, 21, 22] noticed that the closed ball, Cauchy sequence, and completeness defined on these spaces are different from each other.

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They remarked that it is better to find a fixed point in a closed ball in dislocated metric space. They also gave an example of a space which was complete dislocated metric space but was not complete metric-like space.

Nadler [20], introduced a study of fixed point theorems involving multivalued mappings (see also [8, 9]). The existence of fixed points of α-admissible mappings in complete metric spaces has been studied by several researchers (see [18, 23, 27]). Asl et al. [7] generalized these notions by introducing the concepts of α_*-ψ contractive multifunctions, α_*-admissible mapping and obtained some fixed point results for these multifunctions (see also [1, 13, 15]). On the other hand, [24] established some results concerning contraction mappings. In this paper we discuss some new fixed point results for Reich type multivalued mappings in a closed ball in complete dislocated metric space.

The following definitions and results will be needed in the sequel.

Definition 1.1 ([5, 17])**.** Let X be a nonempty set and let $d_l : X \times X \to [0, \infty)$ be a function, called a dislocated metric (or simply d_l-metric), if for any $x, y, z \in X$, the following conditions hold:

(i) if $d_l(x, y) = 0$, then $x = y$;
(ii) $d_l(x, y) = d_l(y, x)$;
(iii) $d_l(x, y) \leqslant d_l(x, z) + d_l(z, y)$.

The pair (X, d_l) is called a dislocated metric space.

It is clear that if $d_l(x, y) = 0$, then from (i), $x = y$. But if $x = y$, $d_l(x, y)$ may not be 0. For $x \in X$ and $\varepsilon > 0$, $\overline{B(x, \varepsilon)} = \{y \in X : d_l(x, y) \leqslant \varepsilon\}$ is a closed ball in (X, d_l).

Example 1.2 ([5])**.** If $X = R^+ \cup \{0\}$, then $d_l(x, y) = x + y$ defines a dislocated metric d_l on X.

Definition 1.3 ([5])**.** Let (X, d_l) be a dislocated metric space.

(i) A sequence $\{x_n\}$ in (X, d_l) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $n, m \geqslant n_0$ we have $d_l(x_m, x_n) < \varepsilon$ or $\lim\limits_{n,m\to\infty} d_l(x_n, x_m) = 0$.
(ii) A sequence $\{x_n\}$ dislocated-converges (for short d_l-converges) to x if $\lim\limits_{n\to\infty} d_l(x_n, x) = 0$. In this case x is called a d_l-limit of $\{x_n\}$.
(iii) (X, d_l) is called complete if every Cauchy sequence in X converges to a point $x \in X$ such that $d_l(x, x) = 0$.

Definition 1.4. Let K be a nonempty subset of dislocated metric space X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d_l(x, K) = d_l(x, y_0), \text{ where } d_l(x, K) = \inf_{y\in K} d_l(x, y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.

We denote $CP(X)$ be the set of all closed proximinal subsets of X. Let Ψ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ. If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

Definition 1.5. Let $S : X \to P(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$, we say that S is semi α_*-admissible on A, whenever $\alpha(x, y) \geqslant 1$ implies that $\alpha_*(Sx, Sy) \geqslant 1$ for all $x, y \in A$, where $\alpha_*(Sx, Sy) = \inf\{\alpha(a, b) : a \in Sx,\ b \in Sy\}$. If $A = X$, then we say that S is α_*-admissible on X.

Definition 1.6. The function $H_{d_l} : P(X) \times P(X) \to X$, defined by

$$H_{d_l}(A, B) = \max\{\sup_{a\in A} d_l(a, B),\ \sup_{b\in B} d_l(A, b)\}$$

is called dislocated Hausdorff-Pompeiu metric on $P(X)$. Also, $(P(X), H_{d_q})$ is known as dislocated quasi Hausdorff-Pompeiu metric space.

Lemma 1.7. *Let (X, d_l) be a dislocated metric space. Let $(P(X), H_{d_l})$ is a dislocated Hausdorff-Pompeiu metric space on $P(X)$. Then for all $A, B \in CP(X)$ and for each $a \in A$ there exists $b_a \in B$ satisfying $d_l(a, B) = d_l(a, b_a)$, then $H_{d_l}(A, B) \geqslant d_l(a, b_a)$.*

Proof. If $H_{d_l}(A, B) = \sup_{a \in A} d_l(a, B)$, then $H_{d_l}(A, B) \geqslant d_l(a, B)$ for each $a \in A$. As B is a proximinal set, so for each $a \in X$, there exists at least one best approximation $b_a \in B$ that satisfies $d_l(a, B) = d_l(a, b_a)$. Now we have, $H_{d_l}(A, B) \geqslant d_l(a, b_a)$. Now $H_{d_l}(A, B) = \sup_{b \in B} d_l(A, b) \geqslant \sup_{a \in A} d_l(a, B)$, hence, the lemma is proved. □

2. Main result

Let (X, d_l) be a dislocated metric space, $x_0 \in X$ and $S : X \to P(X)$ be a multivalued mapping on X. Then there exists $x_1 \in Sx_0$ such that $d_l(x_0, Sx_0) = d_l(x_0, x_1)$. Let $x_2 \in Sx_1$ be such that $d_l(x_1, Sx_1) = d_l(x_1, x_2)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Sx_n$ and $d_l(x_n, Sx_n) = d_l(x_n, x_{n+1})$. We denote this iterative sequence $\{XS(x_n)\}$ and say that $\{XS(x_n)\}$ is a sequence in X generated by x_0.

Theorem 2.1. *Let (X, d_l) be a complete dislocated metric space, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$ $\alpha : X \times X \longrightarrow [0, +\infty)$, $S : X \to P(X)$ be a semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0, r)}$ and $\{XS(x_n)\}$ be a sequence in X generated by x_0, $\alpha(x_0, x_1) \geqslant 1$. Suppose that there exist $a, b \in [0, 1)$ with $a + 2b < 1$ such that*

$$\alpha_*(Sx, Sy) H_{d_l}(Sx, Sy) \leqslant a d_l(x, y) + b\left[d_l(x, Sx) + d_l(y, Sy)\right] \tag{2.1}$$

for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$, and

$$d_l(x_0, Sx_0) \leqslant (1 - \lambda)\, r\,, \text{ where } \lambda = \frac{a + b}{1 - b}. \tag{2.2}$$

Then $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0, r)}$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0, r)}$ and $\alpha(x_n, x_{n+1}) \geqslant 1$ for $x_n, x_{n+1} \in \{XS(x_n)\}$, $n \in \mathbb{N} \cup \{0\}$. Also, if $\alpha(x_n, x^*) \geqslant 1$ or $\alpha(x^*, x_n) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and inequality (2.1) holds for all $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, then S has a fixed point in $\overline{B_{d_l}(x_0, r)}$.

Proof. As $x_0 \in \overline{B_{d_l}(x_0, r)}$, and $S : X \to P(X)$ is a multivalued mapping on X, then there exists $x_1 \in Sx_0$ such that $d_l(x_0, Sx_0) = d_l(x_0, x_1)$. If $x_0 = x_1$, then x_0 is a fixed point in $\overline{B_{d_l}(x_0, r)}$ of S. Let $x_0 \neq x_1$. From (2.2), we get,

$$d_l(x_0, x_1) \leqslant (1 - \lambda)\, r < r.$$

It follows that $x_1 \in \overline{B_{d_l}(x_0, r)}$. As $\alpha(x_0, x_1) \geqslant 1$ and S is semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0, r)}$, so $\alpha_*(Sx_0, Sx_1) \geqslant 1$. As $\alpha_*(Sx_0, Sx_1) \geqslant 1$, $x_1 \in Sx_0$ and $x_2 \in Sx_1$, so $\alpha(x_1, x_2) \geqslant 1$. As S is semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0, r)}$, thus, we have $\alpha_*(Sx_1, Sx_2) \geqslant 1$. As $\alpha_*(Sx_1, Sx_2) \geqslant 1$, we have $\alpha(x_2, x_3) \geqslant 1$, which further implies $\alpha_*(Sx_2, Sx_3) \geqslant 1$. Continuing this process, we have $\alpha_*(Sx_{j-1}, Sx_j) \geqslant 1$. Now,

$$\begin{aligned}
d_l(x_j, x_{j+1}) \leqslant H_{d_l}(Sx_{j-1}, Sx_j) &\leqslant \alpha_*(Sx_{j-1}, Sx_j) H_{d_l}(Sx_{j-1}, Sx_j) \\
&\leqslant a d_l(x_{j-1}, x_j) + b\left[d_l(x_{j-1}, Sx_{j-1}) + d_l(x_j, Sx_j)\right] \\
&= a d_l(x_{j-1}, x_j) + b d_l(x_{j-1}, x_j) + b d_l(x_j, x_{j+1}) \\
&\leqslant (a + b)\, d_l(x_{j-1}, x_j) + b d_l(x_j, x_{j+1}) \\
&\leqslant \frac{a + b}{1 - b} d_l(x_{j-1}, x_j) = \lambda d_l(x_{j-1}, x_j) \leqslant \cdots \leqslant \lambda^j d_l(x_0, x_1),
\end{aligned}$$

which implies,

$$d_l(x_j, x_{j+1}) \leqslant \lambda^j d_l(x_0, x_1). \tag{2.3}$$

Now,

$$\begin{aligned} d_l(x_0,x_{j+1}) &\leqslant d_l(x_0,x_1)+\cdots+d_l(x_j,x_{j+1}) \\ &\leqslant d_l(x_0,x_1)+\cdots+\lambda^j d_l(x_0,x_1) \\ &= \left(1+\lambda+\cdots+\lambda^j\right) d_l(x_0,x_1) \\ &\leqslant (1-\lambda)\left(1+\lambda+\cdots+\lambda^j\right) r < r. \end{aligned}$$

Thus $x_{j+1} \in \overline{B_{d_l}(x_0,r)}$. Hence by induction, $x_n \in \overline{B_{d_l}(x_0,r)}$ and $\alpha(x_n,x_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$. As S is semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0,r)}$, therefore $\alpha_*(Sx_n,Sx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$. Now inequality (2.3) can be written as

$$d_l(x_n,x_{n+1}) \leqslant \lambda^n d_l(x_0,x_1) \text{ for all } n \in N. \tag{2.4}$$

Now,

$$d_l(x_n,x_{n+i}) \leqslant d_l(x_n,x_{n+1})+\ldots+d_l(x_{n+i-1},x_{n+i}) \leqslant \frac{\lambda^n(1-\lambda^i)}{1-\lambda} d_l(x_0,x_1) \longrightarrow 0 \text{ as } n\to\infty.$$

Thus, we proved that $\{x_n\}$ is a Cauchy sequence in $(\overline{B_{d_l}(x_0,r)},d_l)$. As every closed ball in a complete dislocated metric space is complete, so there exists $x^* \in \overline{B_{d_l}(x_0,r)}$ such that $x_n \to x^*$, and

$$\lim_{n\to\infty} d_l(x_n,x^*) = 0. \tag{2.5}$$

Hence $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$ generated by x_0 and $\{XS(x_n)\}\to x^* \in \overline{B_{d_l}(x_0,r)}$ and $\alpha(x_n,x_{n+1}) \geqslant 1$ for $x_n,x_{n+1} \in \{XS(x_n)\}$, $n \in \mathbb{N}\cup\{0\}$. As $\alpha_*(Sx_n,Sx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$, we have $\alpha(x_{n+1},x_{n+2}) \geqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$. By assumption, we have $\alpha(x_n,x^*) \geqslant 1$ for all $n \in \mathbb{N}\cup\{0\}$. Thus $\alpha_*(Sx_n,Sx^*) \geqslant 1$. Now,

$$\begin{aligned} d_l(x^*,Sx^*) &\leqslant d_l(x^*,x_{n+1})+d_l(x_{n+1},Sx^*) \\ &\leqslant d_l(x^*,x_{n+1})+H_{d_l}(Sx_n,Sx^*) \\ &\leqslant d_l(x^*,x_{n+1})+\alpha_*(Sx_n,Sx^*)H_{d_l}(Sx_n,Sx^*) \\ &\leqslant d_l(x^*,x_{n+1})+a d_l(x_n,x^*)+b\left[d_l(x_n,Sx_n)+d_l(x^*,Sx^*)\right]. \end{aligned}$$

Letting $n\to\infty$ in the previous inequality, by using inequality (2.4) and (2.5), we get

$$(1-b)\, d_l(x^*,Sx^*) \leqslant 0.$$

Similarly, if $\alpha(x^*,x_n) \geqslant 1$ for all $n \in N\cup\{0\}$, thus $\alpha_*(Sx^*,Sx_n) \geqslant 1$. Now,

$$(1-b)\, d_l(Sx^*,x^*) \leqslant 0.$$

We obtain, $d_l(Sx^*,x^*) = 0$. Hence $x^* \in Sx^*$. So S has a fixed point in $\overline{B_{d_l}(x_0,r)}$.

Let X be a nonempty set. Then $(X,\preceq,d_l)$ is called a preordered dislocated metric space if d_l is a dislocated metric on X and is a preorder on X. Let $(X,\preceq,d_l)$ be a preordered metric space and $A,B \subseteq X$. We say that $A \preceq B$ whenever for each $a \in A$ there exists such that $a \preceq b$. Also, we say that $A \preceq_r B$ whenever for each $a \in A$ and $b \in B$, we have $a \preceq b$. □

Corollary 2.2. *Let $(X,\preceq,d_l)$ be a preordered complete dislocatedmetric space, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0,r)}$, $S : X \to P(X)$ and $\{XS(x_n)\}$ be a sequence in X generated by x_0 with $x_0 \preceq x_1$. Suppose there exist $a,b \in [0,1)$ with $a+2b<1$ such that*

$$H_{d_l}(Sx,Sy) \leqslant a d_l(x,y)+b\left[d_l(x,Sx)+d_l(y,Sy)\right] \tag{2.6}$$

for all x, y *in* $\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$ *with* $x \preceq y$, *and*

$$d_l(x_0, Sx_0) \leqslant (1-\lambda)\, r, \text{ where } \lambda = \frac{a+b}{1-b}.$$

If $x \preceq y$ *implies* $Sx \preceq_r Sy$ *for all* $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$, *then* $\{XS(x_n)\}$ *is a sequence in* $\overline{B_{d_l}(x_0, r)}$, $x_n \preceq x_{n+1}$ *and* $\{XS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0, r)}$. *Also if* $x^* \preceq x_n$ *or* $x_n \preceq x^*$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and inequality* (2.6) *holds for all* $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, *then* x^* *is a fixed point of* S *in* $\overline{B_{d_l}(x_0, r)}$.

Let $f : X \longrightarrow X$ be a self-mapping of a set X and $\alpha : X \times X \to [0, +\infty)$ be a mapping, then the mapping f is called semi α-admissible if, $A \subseteq X$, $x, y \in A$, $\alpha(x, y) \geqslant 1$ implies $\alpha(fx, fy) \geqslant 1$. If $A = X$, then the mapping f is called α-admissible.

Corollary 2.3. *Let* (X, d_l) *be a complete dislocated metric space and* $S : X \to X$, $r > 0$ *and* x_0 *be an arbitrary point in* $\overline{B_{d_l}(x_0, r)}$ *and* $\{x_n\}$ *be a Picard sequence in* X *with initial guess* x_0. *Let* $\alpha : X \times X \to [0, +\infty)$ *be a semi* α*-admissible mapping on* $\overline{B_{d_l}(x_0, r)}$ *with* $\alpha(x_0, x_1) \geqslant 1$. *For* $a, b \in [0, 1)$ *with* $a + 2b < 1$, *assume that,*

$$x, y \in \overline{B_{d_l}(x_0, r)},\ \alpha(x, y) \geqslant 1, \text{ implies } d_l(Sx, Sy) \leqslant a d_l(x, y) + b\,[d_l(x, Sx) + d_l(y, Sy)]\,, \tag{2.7}$$

and

$$d_l(x_0, Sx_0) \leqslant (1-\lambda)\, r\ , \text{ where } \lambda = \frac{a+b}{1-b}.$$

Then $\{x_n\}$ *is a sequence in* $\overline{B_{d_l}(x_0, r)}$ *and* $x_n \to x^* \in \overline{B_{d_l}(x_0, r)}$ *and* $\alpha(x_n, x_{n+1}) \geqslant 1$ *for all* $n \in \mathbb{N} \cup \{0\}$. *Also, if* $\alpha(x_n, x^*) \geqslant 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *and inequality* (2.7) *holds for all* $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, *then* x^* *is a fixed point of* S *in* $\overline{B_{d_l}(x_0, r)}$.

Recall that if $(X, \preceq)$ is a preordered set and $T : X \to X$ is such that for $x, y \in X$, with $x \preceq y$ implies $Tx \preceq Ty$, then the mapping T is said to be non-decreasing.

Corollary 2.4. *Let* (X, d_l) *be a complete dislocatedmetric space,* $S : X \to X$ *be nondecreasing maping,* $r > 0$ *and* x_0 *be an arbitrary point in* $\overline{B_{d_l}(x_0, r)}$, $\{x_n\}$ *be a Picard sequence in* X *with initial guess* x_0 *and* $x_0 \preceq x_1$. *For* $a, b \in [0, 1)$ *with* $a + 2b < 1$ *such that*

$$d_{d_l}(Sx, Sy) \leqslant a d_l(x, y) + b\,[d_l(x, Sx) + d_l(y, Sy)] \tag{2.8}$$

for all x, y *in* $\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$ *with* $x \preceq y$, *and*

$$d_l(x_0, Sx_0) \leqslant (1-\lambda)\, r, \text{ where } \lambda = \frac{a+b}{1-b}.$$

Then $\{x_n\}$ *is a sequence in* $\overline{B_{d_l}(x_0, r)}$, $x_n \preceq x_{n+1}$ *and* $\{x_n\} \to x^* \in \overline{B_{d_l}(x_0, r)}$. *Also if* $x^* \preceq x_n$ *or* $x_n \preceq x^*$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and inequality* (2.8) *holds for all* $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, *then* x^* *is a fixed point of* S *in* $\overline{B_{d_l}(x_0, r)}$.

Example 2.5. Let $X = R^+ \cup \{0\}$ and let $d_l : X \times X \to X$ be the complete dislocated metric on X defined by,

$$d_l(x, y) = x + y \text{ for all } x, y \in X.$$

Define the multivalued mapping $S : X \to P(X)$ by

$$Sx = \begin{cases} [\frac{2}{3}x, \frac{1}{2}], & \text{if } x \in [0, 1), \\ [x, x+2], & \text{if } x \in (1, \infty). \end{cases}$$

Consider $x_0 = 1$, $r = 21$, $a = \frac{1}{2}$, $b = \frac{1}{5}$, then $\lambda = \frac{7}{8}$, $\overline{B_{d_l}(x_0, r)} = [0, 20]$ and $(1-\lambda) r = \frac{5}{2} > d_l(x_0, Sx_0) = \frac{5}{3}$. So we obtain a sequence $\{XS(x_n)\} = \left\{1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots\right\}$ in X generated by x_0. Define the mapping,

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ \frac{3}{2}, & \text{otherwise.} \end{cases}$$

Now,

$$\alpha_*(S4, S6) H_{d_l}(S4, S6) = (\frac{3}{2})12 > \frac{1}{2} d_l(4,6) + \frac{1}{5}\left[d_l(4, S4) + d_l(6, S6)\right] = 9.$$

So the contractive condition does not hold on X. Clearly, the contractive condition does not hold for all $x, y \in X$ and for all $x, y \in \overline{B_{d_l}(x_0, r)}$. Now for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$, we have

$$\begin{aligned}
\alpha_*(Sx, Sy) H_{d_l}(Sx, Sy) &= 1 \left[\max\left\{\sup_{a \in Sx} d_l(a, Sy), \sup_{b \in Sy} d_l(Sx, b)\right\}\right] \\
&= \max\left\{\sup_{a \in Sx} d_l(a, \left[\frac{2y}{3}, \frac{3y}{4}\right]), \sup_{b \in Sy} d_l(\left[\frac{2x}{3}, \frac{3x}{4}\right], b)\right\} \\
&= \max\left\{d_l(\frac{3x}{4}, \left[\frac{2y}{3}, \frac{3y}{4}\right]), d_l(\left[\frac{2x}{3}, \frac{3x}{4}\right], \frac{3y}{4})\right\} \\
&= \max\left\{d_l(\frac{3x}{4}, \frac{2y}{3}), d_l(\frac{2x}{3}, \frac{3y}{4})\right\} \\
&= \max\left\{\frac{3x}{4} + \frac{2y}{3}, \frac{2x}{3} + \frac{3y}{4})\right\} \\
&\leqslant \frac{5}{6}x + \frac{5}{6}y = \frac{1}{2}(x+y) + \frac{1}{5}\left[x + \frac{2x}{3} + y + \frac{2y}{3}\right] \\
&= \frac{1}{2}(x+y) + \frac{1}{5}\left[d_l\left(x, \left[\frac{2x}{3}, \frac{3x}{4}\right]\right) + d_l\left(y, \left[\frac{2y}{3}, \frac{3y}{4}\right]\right)\right] \\
&= a d_l(x,y) + b\left[d_l(x, Sx) + d_l(y, Sy)\right].
\end{aligned}$$

So the contractive condition holds on $\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$. Hence all the conditions of Theorem 2.1 are satisfied. Now, we have $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0, r)}$, $\alpha(x_n, x_{n+1}) \geqslant 1$ and $\{XS(x_n)\} \longrightarrow 0 \in \overline{B_{d_l}(x_0, r)}$. Also, $\alpha(x_n, 0) \geqslant 1$ or $\alpha(0, x_n) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, S has a fixed point 0.

3. Fixed point results for graphic contractions

Consistent with Jachymski [16], let (X, d) be a metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [16]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected (see for details [10, 11, 16]).

Definition 3.1 ([31]). Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$, and let $T : X \to CB(X)$. T is said to be graph preserving if it satisfies the following:

- if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$.

In this section, we give fixed point results on a dislocated metric space endowed with a graph.

Theorem 3.2. *Let* (X, d_l) *be a complete dislocated metric space endowed with a graph* G, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$, $S : X \to P(X)$ *and* $\{XS(x_n)\}$ *be a sequence in* X *generated by* x_0 *with* $(x_0, x_1) \in E(G)$. *Assume the following conditions hold:*

(i) S *is graph preserving for all* $x, y \in \overline{B_{d_l}(x_0, r)}$,

(ii) *there exist* $a, b \in [0, 1)$ *with* $a + 2b < 1$, *such that*

$$H_{d_l}(Sx, Sy) \leqslant a d_l(x, y) + b\,[d_l(x, Sx) + d_l(y, Sy)] \tag{3.1}$$

for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$ *and* $(x, y) \in E(G)$,

(iii) *there exists* $x_0 \in \overline{B_{d_l}(x_0, r)}$, *such that* $d_l(x_0, Sx_0) \leqslant (1 - \lambda)\, r$, *where* $\lambda = \dfrac{a+b}{1-b}$.

Then $\{XS(x_n)\}$ *is a sequence in* $\overline{B_{d_l}(x_0, r)}$, $(x_n, x_{n+1}) \in E(G)$ *and* $\{XS(x_n)\} \to x^*$. *Also if* $(x_n, x^*) \in E(G)$ *or* $(x^*, x_n) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and inequality* (3.1) *holds for all* $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, *then* x^* *is a fixed point of* S *in* $\overline{B_{d_l}(x_0, r)}$.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

As $\{XS(x_n)\}$ is a sequence in X generated by x_0 with $(x_0, x_1) \in E(G)$, we have $\alpha(x_0, x_1) \geqslant 1$. Let, $\alpha(x, y) \geqslant 1$, then $(x, y) \in E(G)$. From (i), we have $(u, v) \in E(G)$ for all $u \in Sx$ and $v \in Sy$. This implies that $\alpha(u, v) = 1$ for all $u \in Sx$ and $v \in Sy$. This further implies that $\inf\{\alpha(u, v) : u \in Sx,\ v \in Sy\} = 1$. Thus S is a semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0, r)}$. Also, if $(x, y) \in E(G)$, we have $\alpha(x, y) = 1$ and hence, $\alpha_*(Sx, Sy) = 1$. Now, condition (ii) can be written as

$$\alpha_*(Sx, Sy) H_{d_l}(Sx, Sy) = H_{d_l}(Sx, Sy) \leqslant a d_l(x, y) + b\,[d_l(x, Sx) + d_l(y, Sy)]$$

for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$. By including condition (iii) we obtain all the conditions of Theorem 2.1 are satisfied. Now, by Theorem 2.1, we have $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0, r)}$, $\alpha(x_n, x_{n+1}) \geqslant 1$, that is, $(x_n, x_{n+1}) \in E(G)$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0, r)}$. Also if $(x_n, x^*) \in E(G)$ or $(x^*, x_n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and inequality (3.1) holds for all $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, then, we have $\alpha(x_n, x^*) \geqslant 1$ or $\alpha(x^*, x_n) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$ and inequality (2.1) holds for all $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$. Again, by Theorem 2.1, S has a fixed point x^* in $\overline{B_{d_l}(x_0, r)}$. □

Corollary 3.3. *Let* (X, d_l) *be a complete dislocated metric space endowed with a graph* G, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$, $S : X \to P(X)$ *and* $\{XS(x_n)\}$ *be a sequence in* X *generated by* x_0 *with* $(x_0, x_1) \in E(G)$. *Assume the following conditions hold:*

(i) S *is graph preserving for all* $x, y \in \overline{B_{d_l}(x_0, r)}$;

(ii) *there exists* $b \in \left[0, \frac{1}{2}\right)$, *such that*

$$H_{d_l}(Sx, Sy) \leqslant b\,[d_l(x, Sx) + d_l(y, Sy)] \tag{3.2}$$

for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$ *and* $(x, y) \in E(G)$;

(iii) *there exists* $x_0 \in \overline{B_{d_l}(x_0, r)}$, *such that* $d_l(x_0, Sx_0) \leqslant (1 - \lambda)\, r$, *where* $\lambda = \dfrac{b}{1-b}$.

Then $\{XS(x_n)\}$ *is a sequence in* $\overline{B_{d_l}(x_0, r)}$, $(x_n, x_{n+1}) \in E(G)$ *and* $\{XS(x_n)\} \to x^*$. *Also if* $(x_n, x^*) \in E(G)$ *or* $(x^*, x_n) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and inequality* (3.2) *holds for all* $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, *then* x^* *is a fixed point of* S *in* $\overline{B_{d_l}(x_0, r)}$.

Proof. In Theorem 3.2, take $a = 0$ to get fixed point $x^* \in \overline{B(x_0, r)}$ such that $x^* \in Sx^*$. □

Corollary 3.4. *Let (X, d_l) be a complete dislocated metric space endowed with a graph G, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$, $S : X \to P(X)$ and $\{XS(x_n)\}$ be a sequence in X generated by x_0 with $(x_0, x_1) \in E(G)$. Assume the following conditions hold:*

(i) *S is graph preserving for all $x, y \in \overline{B_{d_l}(x_0, r)}$,*

(ii) *there exists $a \in [0, 1)$, such that*

$$H_{d_l}(Sx, Sy) \leqslant a d_l(x, y) \tag{3.3}$$

for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$ and $(x, y) \in E(G)$,

(iii) *there exists $x_0 \in \overline{B_{d_l}(x_0, r)}$, such that $d_l(x_0, Sx_0) \leqslant (1 - a) r$.*

Then $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0, r)}$, $(x_n, x_{n+1}) \in E(G)$ and $\{XS(x_n)\} \to x^*$. Also if $(x_n, x^*) \in E(G)$ or $(x^*, x_n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and inequality (3.3) holds for all $x, y \in \left(\overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}\right) \cup \{x^*\}$, then x^* is a fixed point of S in $\overline{B_{d_l}(x_0, r)}$.

Proof. In Theorem 3.2, take $b = 0$ to get fixed point $x^* \in \overline{B(x_0, r)}$ such that $x^* \in Sx^*$. □

Remark 3.5. We can obtain the metric version of all the theorems which are still not presented in the literature.

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