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Inverse eigenvalue problems for centrosymmetric matrices under a central principal submatrix constraint

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Abstract

This article considers an inverse eigenvalue problem for centrosymmetric matrices under a central principal submatrix constraint and the corresponding optimal approximation problem. We first discuss the specified structure of centrosymmetric matrices and their central principal submatrices. Then we give some necessary and sufficient conditions for the solvability of the inverse eigenvalue problem, and we derive an expression for its general solution. Finally, we obtain an expression for the solution to the corresponding optimal approximation problem.

Keywords: Centrosymmetric matrix, central principal submatrix, inverse eigenvalue problem, optimal approximation problem.
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1. Introduction

We first introduce some notations. Let $R^{n\times m}$ denote the set of all $n \times m$ real matrices. Let $OR^{n\times n}$ be the set of all $n \times n$ orthogonal matrices. Let A^T, rank(A) and A^+ represent the transpose, rank and Moore-Penrose generalized inverse of matrix A, respectively. Let I_n denote the identity matrix of order n, and 0 be a zero matrix or vector of size implied by context. We use $< A, B >= \text{trace}(B^T A)$ to define the inner product of matrices A and B in $R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix generated by the inner product is the Frobenius norm $\|\cdot\|$, that is, $\|A\| = \sqrt{< A, A >} = (\text{trace}(A^T A))^{\frac{1}{2}}$.

Definition 1.1. A real $n \times n$ matrix $A = (a_{i,j})$ is called a centrosymmetric matrix if its elements satisfy the properties

$$a_{i,j} = a_{n-i+1,n-j+1}, \quad \text{for } 1 \leqslant i, j \leqslant n.$$

The set of all $n \times n$ centrosymmetric matrices is denoted by $CSR^{n\times n}$.

Centrosymmetric matrices play an important role in areas such as the numerical solution to certain differential equations [1], various engineering problems [5], and the study of some Markov processes [16].

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In particular, Kimura [12] studied a Markov process whose transition matrix R is an $(n+1)\times(n+1)$ centrosymmetric matrix. The matrix R is given when $\alpha = 2$ by

$$R_{i+1,j+1} = \binom{(n-i)\alpha}{n-j}\binom{i\alpha}{j}\Big/\binom{n\alpha}{n}, \quad 0 \leqslant i,j \leqslant n.$$

In fact, symmetric Toeplitz matrices and persymmetric Hankel matrices are two useful examples of centrosymmetric matrices. Good [8] pointed out that "Toeplitz matrices arise as discrete approximations to kernels $k(x,t)$ of integral equations when these kernels are functions of $|x-t|$. Similarly if a kernel is an even function of its vector argument (x,t), that is $k(x,t) = k(-x,-t)$, then it can be discretely approximated by a centrosymmetric matrix."

Several interesting results are available in the literature on centrosymmetric matrices [14, 20]. In particular, Peng et al. discussed the linear constrained problem of centrosymmetric matrices with a leading principal submatrix constraint [14]. The problem, finding solutions of a matrix equation under a principal submatrix constraint, comes from a practical subsystem expansion problem. Therefore, researchers have great interest in studying a variety of problems under submatrices constraint of late years [6, 14, 17]. Because of the specified structure of centrosymmetric matrices, it is unfit for discussing centrosymmetric matrices under their leading principal submatrices constraint, for it destroys the special symmetric of centrosymmetric matrices. Therefore, we present a different concept, a central principal submatrix, which was first defined in [18]. The definition is as follows.

Definition 1.2. If $n-k$ is even, then the k-square central principal submatrix $A_c(k)$ of a given matrix $A \in R^{n\times n}$ is a k-square submatrix obtained by deleting the first and last $\frac{n-k}{2}$ rows and columns of A, that is

$$A_c(k) = (0,\ I_k,\ 0)A(0,\ I_k,\ 0)^T, \quad 0 \in R^{k\times\frac{n-k}{2}}.$$

It is intuitive and obvious that a matrix of odd (even) order only has central principal submatrices of odd (even) order.

Throughout this paper, we denote by $A[k]$ the right-bottom $k\times k$ principal submatrix of $A \in R^{n\times n}$, that is

$$A[k] = (a_{i,j})_{i,j=n-k+1}^{n} = (0,\ I_k)A(0,\ I_k)^T.$$

In particular, an inverse eigenvalue problem is as follows. Given vectors $x_1, \cdots, x_l$ and scalars $\lambda_1, \cdots, \lambda_l$ where $1 \leqslant l < n$, construct an $n\times n$ matrix $A \in S$ such that

$$Ax_j = \lambda_j x_j, \quad j = 1, \cdots, l, \tag{1.1}$$

where S is a subset of $R^{n\times n}$ with specified structures.

It may arise from the design of Hopfield networks and the mass-spring system [13], in a remarkable variety of applications such as applied mechanics and structure design [11], in the discrete analogue of the inverse Sturm-Liouville problem [19] and in vibration design [15]. Studies of inverse eigenvalue problems have ranged from engineering application to algebraic theorization [2–4, 10, 20]. The inverse eigenvalue problem for centrosymmetric matrices under a central principal submatrix constraint is presented in light of the extended matrix preserves the centrosymmetric structure, thus this problem has practical applications. Furthermore, it allows us to extend the existing solutions. However, it has not been considered yet. In this paper, we will discuss this problem and its optimal approximation. We first discuss the special structure of eigenvalues and eigenvectors for a real matrix in real number field.

If a real matrix A has a complex eigenvalue $\lambda_j = \alpha_j + i*\beta_j$ (α_j, β_j are real numbers, $1 \leqslant j \leqslant l$, $i = \sqrt{-1}$), and the associated complex eigenvector is $x_j = \xi_j + i*\eta_j$ (ξ_j, η_j are real vectors), then $\overline{\lambda}_j = \alpha_j - i*\beta_j$ is also an eigenvalue of A with the associated eigenvector $\overline{x}_j = \xi_j - i*\eta_j$, and

$$A(\xi_j,\ \eta_j) = (\xi_j,\ \eta_j)\begin{pmatrix}\alpha_j & \beta_j\\ -\beta_j & \alpha_j\end{pmatrix}. \tag{1.2}$$

Thus, if x_j is a real eigenvector associated with a real eigenvalue λ_j of A, then let

$$\widetilde{\Lambda}_j = \lambda_j, \quad \widetilde{X}_j = x_j.$$

If $x_j = \xi_j + i * \eta_j$ is a complex eigenvector associated with a complex eigenvalue $\lambda_j = \alpha_j + i * \beta_j$ of A, then set

$$\widetilde{\Lambda}_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}, \quad \widetilde{X}_j = (\xi_j, \eta_j). \tag{1.3}$$

Hence, for given eigenvalues $\lambda_1, \cdots, \lambda_l$ and the associated eigenvectors $x_1, \cdots, x_l$, write

$$X = (\widetilde{X}_1, \cdots, \widetilde{X}_l) \in R^{n\times m}, \quad \Lambda = \mathrm{diag}(\widetilde{\Lambda}_1, \cdots, \widetilde{\Lambda}_l) \in R^{m\times m}, \tag{1.4}$$

where $m \geqslant l$, then (1.1) is equal to $AX = X\Lambda$. Therefore, the problem discussed in this paper can be expressed as follows.

Problem 1.3. Given $X \in R^{n\times m}$ and $\Lambda \in R^{m\times m}$ with the form as in (1.4), $A_0 \in CSR^{k\times k}$, for $k < n$, find an extended matrix $A \in CSR^{n\times n}$ such that

$$AX = X\Lambda, \text{ and } A_c(k) = A_0.$$

Problem 1.4. Given an estimate $\widetilde{A} \in R^{n\times n}$, find a matrix $A^* \in S_A$ such that

$$\|A^* - \widetilde{A}\| = \min_{\forall A \in S_A} \|A - \widetilde{A}\|,$$

where S_A is the solution set of Problem 1.3.

This paper is organized as follows. First, we discuss the specified properties and structure of centrosymmetric matrices and their central principal submatrices. Next, we study eigenvalues and eigenvectors of centrosymmetric matrices, and obtain that the eigenvectors of a centrosymmetric matrix can be expressed in a special form. After that, we convert Problem 1.3 into two inverse eigenvalue problems of half-sized independent real matrices under themselves right-bottom principal submatrices constraint trickily, which is a special feature of this paper. Meanwhile, this simplifies and is crucial to solve Problem 1.3. Later on, we derive the solvability conditions of Problem 1.3 and an expression for its general solution. Finally, we prove that Problem 1.4 has a unique solution and give an expression for it.

2. The properties of centrosymmetric matrices and their central principal submatrices

Denote by e_i the i-th $(i = 1, 2, \cdots, n)$ column of I_n, and let $S_n = (e_n, e_{n-1}, \cdots, e_1)$, then

$$S_n = S_n^T, \quad S_n S_n^T = I_n.$$

Lemma 2.1 ([7]). *A matrix $A \in CSR^{n\times n}$ if and only if $S_n A S_n = A$.*

Let $r = [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the maximum integer which is not greater than $\frac{n}{2}$. And let

$$D_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & I_r \\ S_r & -S_r \end{pmatrix} (n = 2r), \quad D_n = \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & 0 & I_r \\ 0 & \sqrt{2} & 0 \\ S_r & 0 & -S_r \end{pmatrix} (n = 2r+1).$$

It is easy verified that the above matrices D_n are orthogonal matrices.

Lemma 2.2 ([20]). *$A \in CSR^{2r\times 2r}$ if and only if there exist $M, N \in R^{r\times r}$ such that*

$$A = \begin{pmatrix} M & NS_r \\ S_r N & S_r M S_r \end{pmatrix} = D_{2r} \begin{pmatrix} M+N & 0 \\ 0 & M-N \end{pmatrix} D_{2r}^T. \tag{2.1}$$

$A \in CSR^{(2r+1)\times(2r+1)}$ if and only if there exist $M, N \in R^{r\times r}$, $u, v \in R^r$ and $\alpha \in R$ such that

$$A = \begin{pmatrix} M & u & NS_r \\ v^T & \alpha & v^T S_r \\ S_r N & S_r u & S_r M S_r \end{pmatrix} = D_{2r+1} \begin{pmatrix} M+N & \sqrt{2}u & 0 \\ \sqrt{2}v^T & \alpha & 0 \\ 0 & 0 & M-N \end{pmatrix} D_{2r+1}^T. \tag{2.2}$$

Furthermore, when $n = 2r$, *let*

$$A_{11} = M + N, \quad A_{22} = M - N, \tag{2.3}$$

and when $n = 2r + 1$, *set*

$$A_{11} = \begin{pmatrix} M+N & \sqrt{2}u \\ \sqrt{2}v^T & \alpha \end{pmatrix}, \quad A_{22} = M - N, \tag{2.4}$$

then $A \in CSR^{n\times n}$ *if and only if there exist* $A_{11} \in R^{(n-r)\times(n-r)}$ *and* $A_{22} \in R^{r\times r}$, *whether* n *is odd or even, such that*

$$A = D_n \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D_n^T. \tag{2.5}$$

Now we give the special properties of the k-central principal submatrix of a centrosymmetric matrix, that is the submatrix having the same symmetric properties and structure as the given centrosymmetric matrix. Hence they have similar expressions, which is crucial to solve Problem 1.3, for it provides a reasoned way to convert Problem 1.3 to two inverse eigenvalue problems of half-sized independent real matrices under themselves right-bottom principal submatrices constraint. Here, we always assume that $t = [\frac{k}{2}]$.

Lemma 2.3. *Let* $A \in CSR^{n\times n}$ *have the form as in* (2.5). *Then the* k*-square central principal submatrix of* A *can be expressed as*

$$A_c(k) = D_k \begin{pmatrix} A_{11}[k-t] & 0 \\ 0 & A_{22}[t] \end{pmatrix} D_k^T. \tag{2.6}$$

Proof. When $n = 2r$, from (2.1) and the properties of central principal submatrices, that is, a matrix of even order only having central principal submatrices of even order, we have $k = 2t$, and

$$A_c(k) = \begin{pmatrix} M[t] & N[t]S_t \\ S_t N[t] & S_t M[t] S_t \end{pmatrix}.$$

Thus,

$$\begin{aligned} D_k^T A_c(k) D_k &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & S_t \\ I_t & -S_t \end{pmatrix} \cdot \begin{pmatrix} M[t] & N[t]S_t \\ S_t N[t] & S_t M[t] S_t \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & I_t \\ S_t & -S_t \end{pmatrix} \\ &= \begin{pmatrix} M[t]+N[t] & 0 \\ 0 & M[t]-N[t] \end{pmatrix}, \end{aligned}$$

and (2.3) implies $M[t] + N[t] = A_{11}[t]$ and $M[t] - N[t] = A_{22}[t]$. It says that the k-square central principal submatrix of A may be expressed as

$$A_c(k) = D_k \begin{pmatrix} A_{11}[t] & 0 \\ 0 & A_{22}[t] \end{pmatrix} D_k^T. \tag{2.7}$$

When $n = 2r + 1$, from (2.2) and the properties of central principal submatrices, that is, a matrix of odd order only having central principal submatrices of odd order, we have $k = 2t + 1$, and

$$A_c(k) = \begin{pmatrix} M[t] & u_t & N[t]S_t \\ v_t^T & \alpha & v_t^T S_t \\ S_t N[t] & S_t u_t & S_t M[t] S_t \end{pmatrix}, \quad u_t = (0, I_t)u, \quad v_t = (0, I_t)v.$$

Hence,

$$D_k^T A_c(k) D_k = \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & 0 & S_t \\ 0 & \sqrt{2} & 0 \\ I_t & 0 & -S_t \end{pmatrix} \cdot \begin{pmatrix} M[t] & u_t & N[t]S_t \\ v_t^T & \alpha & v_t^T S_t \\ S_t N[t] & S_t u_t & S_t M[t] S_t \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} I_t & 0 & I_t \\ 0 & \sqrt{2} & 0 \\ S_t & 0 & -S_t \end{pmatrix}$$

$$= \begin{pmatrix} M[t]+N[t] & \sqrt{2}u_t & 0 \\ \sqrt{2}v_t^T & \alpha & 0 \\ 0 & 0 & M[t]-N[t] \end{pmatrix},$$

and (2.4) implies $\begin{pmatrix} M[t]+N[t] & \sqrt{2}u_t \\ \sqrt{2}v_t^T & \alpha \end{pmatrix} = A_{11}[t+1]$ and $M[t]-N[t] = A_{22}[t]$. It means that the k-square central principal submatrix of A may be written as

$$A_c(k) = D_k \begin{pmatrix} A_{11}[t+1] & 0 \\ 0 & A_{22}[t] \end{pmatrix} D_k^T. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain that the k-square central principal submatrix of A has the form as in (2.6). □

It is easy to verify the following lemma from Lemma 2.3.

Lemma 2.4. *Suppose that $A \in CSR^{n\times n}$ has the form as in (2.5). Partition $A_0 \in CSR^{k\times k}$ as*

$$A_0 = D_k \begin{pmatrix} A_{10} & 0 \\ 0 & A_{20} \end{pmatrix} D_k^T, \quad A_{10} \in R^{(k-t)\times(k-t)}, \quad A_{20} \in R^{t\times t}, \tag{2.9}$$

then A_0 is a central principal submatrix of A if and only if $A_{10} = A_{11}[k-t]$ and $A_{20} = A_{22}[t]$.

3. Expression for the general solution to Problem 1.3

In this section, we first discuss the properties of eigenvalues and eigenvectors of a centrosymmetric matrix, and obtain that the eigenvectors of a centrosymmetric matrix can be expressed in a special form. Next, utilizing the special expression forms of a centrosymmetric matrix and its central principal submatrices, as discussed in Section 2, we convert Problem 1.3 to two inverse eigenvalue problems of half-sized independent real matrices under themselves right-bottom principal submatrices constraint trickily, which is a special feature of this paper. Finally, we solve Problem 1.3 completely, that is, we provide necessary and sufficient conditions for the existence of a solution to Problem 1.3 and give an expression for the general solution. Now, we investigate the expressions of eigenvectors of a centrosymmetric matrix in the real number field.

Definition 3.1. Let $x \in R^n$. x is called a symmetric vector if $S_n x = x$. x is called an anti-symmetric vector if $S_n x = -x$.

Given $A \in CSR^{n\times n}$, if $\lambda_j (1 \leqslant j \leqslant l)$ is a real eigenvalue of A, and x_j is a real eigenvector associated with λ_j, that is $Ax_j = \lambda_j x_j$. Then we have, from Lemma 2.1,

$$AS_n x_j = S_n A x_j = \lambda_j S_n x_j,$$

thus $S_n x_j$ is also an eigenvector of A associated with λ_j. Therefore, $x_j \pm S_n x_j$ are eigenvectors associated with λ_j, where $x_j + S_n x_j$ is a symmetric vector, and $x_j - S_n x_j$ is an anti-symmetric vector.

If $\lambda_j = \alpha_j + i * \beta_j$ is a complex eigenvalue of A, $x_j = \xi_j + i * \eta_j$ is an associated eigenvector, then (1.2) and (1.3) imply $A\widetilde{X}_j = \widetilde{X}_j \widetilde{\Lambda}_j$. From Lemma 2.1, we obtain

$$AS_n \widetilde{X}_j = S_n A \widetilde{X}_j = S_n \widetilde{X}_j \widetilde{\Lambda}_j,$$

thus

$$A(\widetilde{X}_j \pm S_n \widetilde{X}_j) = (\widetilde{X}_j \pm S_n \widetilde{X}_j)\widetilde{\Lambda}_j,$$

where the columns of $\widetilde{X}_j + S_n \widetilde{X}_j = (\xi_j + S_n \xi_j, \ \eta_j + S_n \eta_j)$ are symmetric vectors, and the columns of $\widetilde{X}_j - S_n \widetilde{X}_j = (\xi_j - S_n \xi_j, \ \eta_j - S_n \eta_j)$ are anti-symmetric vectors.

According to the previous analysis, without loss of generality, we may suppose that X and Λ have the following forms in Problem 1.3,

$$X = \begin{pmatrix} Z_1 & Y_1 \\ S_r Z_1 & -S_r Y_1 \end{pmatrix} (n = 2r), \quad X = \begin{pmatrix} Z_1 & Y_1 \\ \sqrt{2}c^T & 0 \\ S_r Z_1 & -S_r Y_1 \end{pmatrix} (n = 2r+1), \tag{3.1}$$

$$\Lambda = \mathrm{diag}(\Lambda_1,\ \Lambda_2), \quad \Lambda_1 \in R^{s\times s}, \quad \Lambda_2 \in R^{(m-s)\times(m-s)}, \tag{3.2}$$

where $Z_1 \in R^{r\times s}$, $Y_1 \in R^{r\times(m-s)}$, $c \in R^s$, and Λ_1, Λ_2 are block diagonal matrices, for the block is a square matrix of order 1 or 2. Then $D_n^T X$ has the following form:

$$D_n^T X = \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & S_r \\ I_r & -S_r \end{pmatrix} \cdot \begin{pmatrix} Z_1 & Y_1 \\ S_r Z_1 & -S_r Y_1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}Z_1 & 0 \\ 0 & \sqrt{2}Y_1 \end{pmatrix} (n = 2r),$$

$$D_n^T X = \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & 0 & S_r \\ 0 & \sqrt{2} & 0 \\ I_r & 0 & -S_r \end{pmatrix} \cdot \begin{pmatrix} Z_1 & Y_1 \\ \sqrt{2}c^T & 0 \\ S_r Z_1 & -S_r Y_1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}Z_1 & 0 \\ \sqrt{2}c^T & 0 \\ 0 & \sqrt{2}Y_1 \end{pmatrix} (n = 2r+1).$$

If $n = 2r$, let $X_1 = Z_1$, and if $n = 2r+1$, let $X_1 = \begin{pmatrix} Z_1 \\ c^T \end{pmatrix}$, then for all arbitrary n, $D_n^T X$ may be written as

$$D_n^T X = \begin{pmatrix} \sqrt{2}X_1 & 0 \\ 0 & \sqrt{2}Y_1 \end{pmatrix}, \quad X_1 \in R^{(n-r)\times s}, \quad Y_1 \in R^{r\times(m-s)}. \tag{3.3}$$

Lemma 3.2 ([10]). *Let $X \in R^{n\times m}$ and $\Lambda \in R^{m\times m}$ as in (1.4), then there exists a matrix $A \in R^{n\times n}$ such that $AX = X\Lambda$ if and only if $X\Lambda X^+ X = X\Lambda$. Moreover, its general solution can be expressed as $A = X\Lambda X^+ + G(I_n - XX^+)$, where $G \in R^{n\times n}$ is arbitrary.*

Lemma 3.3 ([9]). *Given $Y \in R^{k\times n}$, $X \in R^{m\times l}$ and $B \in R^{k\times l}$, denote*

$$S_1 \equiv \{A \in R^{n\times m} \mid f_1(A) = \|YAX - B\| = \min\},$$

then every element in S_1 has the form as

$$A = Y^+ B X^+ + G - Y^+ Y G X X^+, \quad \forall\, G \in R^{n\times m}. \tag{3.4}$$

In particular, $f_1(A) = 0$ has solutions in $R^{n\times m}$ if and only if $YY^+BX^+X = B$, and the general solution has the same form as in (3.4).

We can obtain the following lemma easily from Lemma 3.3.

Lemma 3.4. *Given $X,\ B \in R^{m\times l}$, denote $S_2 \equiv \left\{A \in R^{n\times m} | f_2(A) = \|AX - B\| = \min\right\}$, then every element in S_2 has the form as*

$$A = BX^+ + G(I_m - XX^+), \quad \forall\, G \in R^{n\times m}. \tag{3.5}$$

In particular, $f_2(A) = 0$ has solutions in $R^{n\times m}$ if and only if $BX^+X = B$, and the general solution has the same form as in (3.5).

Theorem 3.5. *Given $X \in R^{n\times m}$ as in (3.1), $\Lambda \in R^{m\times m}$ as in (3.2), partition $D_n^T X$ as in (3.3). Given $A_0 \in CSR^{k\times k}$, partition A_0 as in (2.9). Set*

$$\begin{aligned} Z_1 &= (I_{n-r} - X_1 X_1^+)(0,\ I_{k-t})^T, \quad K_1 = A_{10} - (0,\ I_{k-t}) X_1 \Lambda_1 X_1^+ (0,\ I_{k-t})^T, \\ Z_2 &= (I_r - Y_1 Y_1^+)(0,\ I_t)^T, \quad K_2 = A_{20} - (0,\ I_t) Y_1 \Lambda_2 Y_1^+ (0,\ I_t)^T. \end{aligned} \tag{3.6}$$

Problem 1.3 is solvable if and only if

$$X_1\Lambda_1X_1^+X_1 = X_1\Lambda_1, \quad Y_1\Lambda_2Y_1^+Y_1 = Y_1\Lambda_2, \tag{3.7}$$

$$K_1Z_1^+Z_1 = K_1, \quad K_2Z_2^+Z_2 = K_2. \tag{3.8}$$

Furthermore, every matrix A in the solution set S_A may be written as

$$A = D_n \begin{pmatrix} X_1\Lambda_1X_1^+ + G_1(I_{n-r} - X_1X_1^+) & 0 \\ 0 & Y_1\Lambda_2Y_1^+ + G_2(I_r - Y_1Y_1^+) \end{pmatrix} D_n^T, \tag{3.9}$$

where

$$G_1 = \begin{pmatrix} E_1 \\ K_1Z_1^+ + E_2(I_{n-r} - Z_1Z_1^+) \end{pmatrix}, \quad G_2 = \begin{pmatrix} F_1 \\ K_2Z_2^+ + F_2(I_r - Z_2Z_2^+) \end{pmatrix}, \tag{3.10}$$

where $E_1 \in R^{(n-r-k+t)\times(n-r)}$, $E_2 \in R^{(k-t)\times(n-r)}$, $F_1 \in R^{(r-t)\times r}$ and $F_2 \in R^{t\times r}$ are arbitrary.

Proof. By Lemmas 2.2 and 2.4, Problem 1.3 is equivalent to find $A_{11} \in R^{(n-r)\times(n-r)}$ and $A_{22} \in R^{r\times r}$ such that

$$A = D_n \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D_n^T,$$

where A_{11} and A_{22} must satisfy

$$A_{11}X_1 = X_1\Lambda_1, \quad A_{22}Y_1 = Y_1\Lambda_2, \tag{3.11}$$

$$\begin{aligned} A_{10} &= A_{11}[k-t] = (0,\ I_{k-t})A_{11}(0,\ I_{k-t})^T, \\ A_{20} &= A_{22}[t] = (0,\ I_t)A_{22}(0,\ I_t)^T. \end{aligned} \tag{3.12}$$

From Lemma 3.2, (3.11) holds if and only if

$$X_1\Lambda_1X_1^+X_1 = X_1\Lambda_1, \quad Y_1\Lambda_2Y_1^+Y_1 = Y_1\Lambda_2,$$

which means that (3.7) holds. Moreover A_{11} and A_{22} can be written as

$$A_{11} = X_1\Lambda_1X_1^+ + G_1(I_{n-r} - X_1X_1^+), \quad A_{22} = Y_1\Lambda_2Y_1^+ + G_2(I_r - Y_1Y_1^+), \tag{3.13}$$

where $G_1 \in R^{(n-r)\times(n-r)}$ and $G_2 \in R^{r\times r}$ are arbitrary matrices. Substituting (3.13) into (3.12), and noticing (3.6), the definitions of Z_1, Z_2, K_1 and K_2, then G_1 and G_2 satisfy

$$(0,\ I_{k-t})G_1Z_1 = K_1, \quad (0,\ I_t)G_2Z_2 = K_2. \tag{3.14}$$

Lemma 3.3 implies that (3.14) holds if and only if

$$(0,\ I_{k-t})(0,\ I_{k-t})^+K_1Z_1^+Z_1 = K_1, \quad (0,\ I_t)(0,\ I_t)^+K_2Z_2^+Z_2 = K_2. \tag{3.15}$$

We know from $(0,\ I_{k-t})^+ = (0,\ I_{k-t})^T$ and $(0,\ I_t)^+ = (0,\ I_t)^T$ that $(0,\ I_{k-t})(0,\ I_{k-t})^+ = I_{k-t}$ and $(0,\ I_t)(0,\ I_t)^+ = I_t$. Hence, (3.15) is equivalent to (3.8), and G_1, G_2 can be expressed as

$$\begin{aligned} G_1 &= (0,\ I_{k-t})^+K_1Z_1^+ + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} - (0,\ I_{k-t})^+(0,\ I_{k-t}) \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} Z_1Z_1^+ \\ &= \begin{pmatrix} E_1 \\ K_1Z_1^+ + E_2(I_{n-r} - Z_1Z_1^+) \end{pmatrix}, \quad \forall E_1 \in R^{(n-r-k+t)\times(n-r)}, \quad \forall E_2 \in R^{(k-t)\times(n-r)}, \\ G_2 &= (0,\ I_t)^+K_2Z_2^+ + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} - (0,\ I_t)^+(0,\ I_t) \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} Z_2Z_2^+ \\ &= \begin{pmatrix} F_1 \\ K_2Z_2^+ + F_2(I_r - Z_2Z_2^+) \end{pmatrix}, \quad \forall F_1 \in R^{(r-t)\times r}, \quad \forall F_2 \in R^{t\times r}. \end{aligned}$$

Thus, the general solution to Problem 1.3 may be written as in (3.9). □

4. The solution to Problem 1.4

When the solution set of Problem 1.3 is nonempty, it is easy to verify that S_A is a closed convex set, therefore there exists a unique solution A^* to Problem 1.4. Now we give the expression for A^*.

Lemma 4.1. *Given $X \in R^{n\times m}$, $I_n - XX^+$ and XX^+ are orthogonal projection matrices, that is*

$$(I_n - XX^+)^2 = I_n - XX^+ = (I_n - XX^+)^T, \quad (XX^+)^2 = XX^+ = (XX^+)^T.$$

Moreover, we have $(I_n - XX^+)XX^+ = 0$.

Lemma 4.1 can be verified easily upon computation.

Theorem 4.2. *Let $X \in R^{n\times m}$ form as in (3.1). Let $\Lambda \in R^{m\times m}$ form as in (3.2). Given $\widetilde{A} \in R^{n\times n}$ and $A_0 \in CSR^{k\times k}$, denote*

$$D_n^T \widetilde{A} D_n = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}, \quad \widetilde{A}_{11} \in R^{(n-r)\times(n-r)}, \quad \widetilde{A}_{22} \in R^{r\times r},$$

and partition $\widetilde{A}_{11}$ and $\widetilde{A}_{22}$ as

$$\widetilde{A}_{11} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad \widetilde{A}_{22} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{4.1}$$

where $P_1 \in R^{(n-r-k+t)\times(n-r)}$, $P_2 \in R^{(k-t)\times(n-r)}$, $Q_1 \in R^{(r-t)\times r}$ and $Q_2 \in R^{t\times r}$. Set

$$\begin{aligned} \widetilde{P}_2 &= (P_2 - K_1 Z_1^+)(I_{n-r} - X_1 X_1^+), \quad \widetilde{Z}_1 = (I_{n-r} - Z_1 Z_1^+)(I_{n-r} - X_1 X_1^+), \\ \widetilde{Q}_2 &= (Q_2 - K_2 Z_2^+)(I_r - Y_1 Y_1^+), \quad \widetilde{Z}_2 = (I_r - Z_2 Z_2^+)(I_r - Y_1 Y_1^+). \end{aligned} \tag{4.2}$$

If Problem 1.3 is solvable, then Problem 1.4 has a unique solution A^*, which can be written as

$$A^* = D_n \begin{pmatrix} X_1 \Lambda_1 X_1^+ + \widetilde{G}_1 & 0 \\ 0 & Y_1 \Lambda_2 Y_1^+ + \widetilde{G}_2 \end{pmatrix} D_n^T, \tag{4.3}$$

where $\widetilde{G}_1 = \begin{pmatrix} P_1(I_{n-r} - X_1X_1^+) \\ K_1 Z_1^+(I_{n-r} - X_1 X_1^+) + \widetilde{P}_2 \widetilde{Z}_1^+ \widetilde{Z}_1 \end{pmatrix}$, *and* $\widetilde{G}_2 = \begin{pmatrix} Q_1(I_r - Y_1Y_1^+) \\ K_2 Z_2^+(I_r - Y_1 Y_1^+) + \widetilde{Q}_2 \widetilde{Z}_2^+ \widetilde{Z}_2 \end{pmatrix}$.

Proof. Suppose that A is an arbitrary solution to Problem 1.3, then by (3.9), we have

$$\begin{aligned} \left\| A - \widetilde{A} \right\|^2 &= \left\| D_n \begin{pmatrix} X_1\Lambda_1 X_1^+ + G_1(I_{n-r} - X_1X_1^+) & 0 \\ 0 & Y_1 \Lambda_2 Y_1^+ + G_2(I_r - Y_1Y_1^+) \end{pmatrix} D_n^T - \widetilde{A} \right\|^2 \\ &= \left\| X_1\Lambda_1X_1^+ + G_1(I_{n-r} - X_1X_1^+) - \widetilde{A}_{11} \right\|^2 + \left\| \widetilde{A}_{12} \right\|^2 \\ &\quad + \left\| \widetilde{A}_{21} \right\|^2 + \left\| Y_1\Lambda_2Y_1^+ + G_2(I_r - Y_1Y_1^+) - \widetilde{A}_{22} \right\|^2. \end{aligned}$$

Hence $\left\| A - \widetilde{A} \right\| = \min\limits_{A\in S_A}$ is equivalent to

$$\left\| G_1(I_{n-r} - X_1X_1^+) - (\widetilde{A}_{11} - X_1\Lambda_1X_1^+) \right\| = \min_{G_1 \in R^{(n-r)\times(n-r)}}, \tag{4.4}$$

$$\left\| G_2(I_r - Y_1Y_1^+) - (\widetilde{A}_{22} - Y_1\Lambda_2Y_1^+) \right\| = \min_{G_2 \in R^{r\times r}}. \tag{4.5}$$

Utilizing (3.10), (4.1), (4.2), and noticing Lemma 4.1, we get

$$\begin{aligned}
&\left\|G_1(I_{n-r}-X_1X_1^+)-(\widetilde{A}_{11}-X_1\Lambda_1X_1^+)\right\|^2\\
&\quad=\left\|\left[G_1(I_{n-r}-X_1X_1^+)-(\widetilde{A}_{11}-X_1\Lambda_1X_1^+)\right](I_{n-r}-X_1X_1^+)\right\|^2\\
&\qquad+\left\|\left[G_1(I_{n-r}-X_1X_1^+)-(\widetilde{A}_{11}-X_1\Lambda_1X_1^+)\right]X_1X_1^+\right\|^2\\
&\quad=\left\|G_1(I_{n-r}-X_1X_1^+)-\widetilde{A}_{11}(I_{n-r}-X_1X_1^+)\right\|^2+\left\|\widetilde{A}_{11}X_1X_1^+-X_1\Lambda_1X_1^+\right\|^2\\
&\quad=\left\|\begin{pmatrix} E_1\\ K_1Z_1^++E_2(I_{n-r}-Z_1Z_1^+)\end{pmatrix}(I_{n-r}-X_1X_1^+)-\begin{pmatrix}P_1\\P_2\end{pmatrix}(I_{n-r}-X_1X_1^+)\right\|^2\\
&\qquad+\left\|\widetilde{A}_{11}X_1X_1^+-X_1\Lambda_1X_1^+\right\|^2\\
&\quad=\|E_1(I_{n-r}-X_1X_1^+)-P_1(I_{n-r}-X_1X_1^+)\|^2+\left\|E_2\widetilde{Z}_1-\widetilde{P}_2\right\|^2+\left\|\widetilde{A}_{11}X_1X_1^+-X_1\Lambda_1X_1^+\right\|^2.
\end{aligned}$$

Hence (4.4) is equivalent to

$$\|E_1(I_{n-r}-X_1X_1^+)-P_1(I_{n-r}-X_1X_1^+)\|=\min,\qquad \left\|E_2\widetilde{Z}_1-\widetilde{P}_2\right\|=\min. \tag{4.6}$$

We know from Lemma 3.4 that (4.6) holds, which implies that (4.4) holds, if and only if

$$E_1=P_1(I_{n-r}-X_1X_1^+)+\widetilde{E}_1X_1X_1^+,\quad E_2=\widetilde{P}_2\widetilde{Z}_1^++\widetilde{E}_2(I_{n-r}-\widetilde{Z}_1\widetilde{Z}_1^+),$$

where $\widetilde{E}_1\in R^{(n-r-k+t)\times(n-r)}$ and $\widetilde{E}_2\in R^{(k-t)\times(n-r)}$ are arbitrary matrices.

We can prove in a similar way that (4.5) holds if and only if

$$F_1=Q_1(I_r-Y_1Y_1^+)+\widetilde{F}_1Y_1Y_1^+,$$
$$F_2=\widetilde{Q}_2\widetilde{Z}_2^++\widetilde{F}_2(I_r-\widetilde{Z}_2\widetilde{Z}_2^+),$$

where $\widetilde{F}_1\in R^{(r-t)\times r}$ and $\widetilde{F}_2\in R^{t\times r}$ are arbitrary matrices.

Substituting E_1, E_2 and F_1, F_2 into (3.10), we get that the unique solution to Problem 1.4 can be expressed as in (4.3) as desired.

□

Algorithm 4.3.

(1) Input $X\in R^{n\times m}$ as in (3.1), $\Lambda\in R^{m\times m}$ as in (3.2), $\widetilde{A}\in R^{n\times n}$ and $A_0\in CSR^{k\times k}$.

(2) Partition A_0 as in (2.9) to get A_{10} and A_{20}.

(3) Obtain X_1 and Y_1 according to (3.3).

(4) Follow (3.6) to calculate Z_1, Z_2, K_1 and K_2.

(5) If (3.7) and (3.8) hold, then continue; otherwise stop.

(6) According to Theorem 4.2 calculate $\widetilde{A}_{11}$, $\widetilde{A}_{22}$, P_1, Q_1, $\widetilde{P}_2$, $\widetilde{Z}_1$, $\widetilde{Q}_2$, $\widetilde{Z}_2$ and A^*.

Exemple 4.4. Assume $n=10$, $k=4$, $m=4$. Given

$$\Lambda=\begin{pmatrix}-2.7645&0&0&0\\0&0.8744&0&0\\0&0&-2.8382&0\\0&0&0&-1.3716\end{pmatrix},\quad A_0=\begin{pmatrix}-1.65&0.55&0.55&0.25\\0.35&-0.6&-0.2&0.45\\0.45&-0.2&-0.6&0.35\\0.25&0.55&0.55&-1.65\end{pmatrix},$$

$$X = \begin{pmatrix} -0.1371 & 0.2076 & -0.2167 & 0.1560 \\ -0.1338 & 0.4773 & -0.0028 & 0.4014 \\ 0.0780 & 0.1427 & 0.4728 & 0.0788 \\ -0.5429 & 0.0952 & -0.4434 & 0.3097 \\ 0.4030 & 0.4469 & -0.1812 & -0.4609 \\ 0.4030 & 0.4469 & 0.1812 & 0.4609 \\ -0.5429 & 0.0952 & 0.4434 & -0.3097 \\ 0.0780 & 0.1427 & -0.4728 & -0.0788 \\ -0.1338 & 0.4773 & 0.0028 & -0.4014 \\ -0.1371 & 0.2076 & 0.2167 & -0.1560 \end{pmatrix},$$

$$\widetilde{A} = \begin{pmatrix} 1.7643 & -0.6475 & 0.7996 & -1.4946 & -1.0485 & 0.2050 & -0.2997 & 0.3591 & 1.1636 & 0.1017 \\ 0.9457 & 0.8022 & -0.4446 & -0.7732 & 1.3888 & -0.5330 & 0.2510 & -0.9312 & -0.0488 & -0.5711 \\ -0.2314 & -0.1201 & 0.1656 & 1.1546 & 0.4174 & -0.3023 & -0.3856 & 1.5166 & -0.1959 & 0.4041 \\ -0.0590 & -0.2892 & 1.2945 & -1.5725 & 0.6320 & 0.5720 & 0.3005 & 0.0652 & -0.1688 & -0.8455 \\ -0.5931 & 1.8189 & -0.5775 & 0.4104 & -0.5819 & -0.1916 & 0.5068 & -0.8231 & 0.1822 & 0.6014 \\ 0.5694 & 0.2275 & -0.8018 & 0.4816 & -0.1903 & -0.5641 & 0.4142 & -0.5604 & 1.8448 & -0.6054 \\ -0.7796 & -0.1209 & 0.0171 & 0.2933 & 0.5531 & 0.6233 & -1.6293 & 1.2396 & -0.2793 & -0.1009 \\ 0.4152 & -0.1291 & 1.5941 & -0.3157 & -0.3990 & 0.4602 & 1.1340 & 0.1692 & -0.1511 & -0.1840 \\ -0.6168 & -0.0053 & -0.9043 & 0.2626 & -0.5033 & 1.3451 & -0.7673 & -0.4762 & 0.8445 & 0.9005 \\ 0.1522 & 1.1798 & 0.3175 & -0.2727 & 0.2888 & -1.0087 & -1.4385 & 0.7949 & -0.5679 & 1.7742 \end{pmatrix},$$

then we can obtain the best approximate solution A^* to Problem 1.4 by Algorithm 4.3, where

$$A^* = \begin{pmatrix} 1.7418 & -0.6746 & 0.7747 & -1.4808 & -1.0768 & 0.1823 & -0.2974 & 0.3248 & 1.1168 & 0.1012 \\ 0.8863 & 0.7497 & -0.4751 & -0.7970 & 1.2891 & -0.5704 & 0.2258 & -0.9470 & -0.1194 & -0.6331 \\ -0.2421 & -0.2142 & 0.1429 & 1.1080 & 0.4041 & -0.4036 & -0.3916 & 1.5472 & -0.2300 & 0.3731 \\ -0.0915 & -0.3337 & 1.2344 & -1.6500 & 0.5500 & 0.5500 & 0.2500 & 0.0143 & -0.2126 & -0.8356 \\ -0.6369 & 1.7404 & -0.5895 & 0.3500 & -0.6000 & -0.2000 & 0.4500 & -0.8821 & 0.1326 & 0.5793 \\ 0.5793 & 0.1326 & -0.8821 & 0.4500 & -0.2000 & -0.6000 & 0.3500 & -0.5895 & 1.7404 & -0.6369 \\ -0.8356 & -0.2126 & 0.0143 & 0.2500 & 0.5500 & 0.5500 & -1.6500 & 1.2344 & -0.3337 & -0.0915 \\ 0.3731 & -0.2300 & 1.5472 & -0.3916 & -0.4036 & 0.4041 & 1.1080 & 0.1429 & -0.2142 & -0.2421 \\ -0.6331 & -0.1194 & -0.9470 & 0.2258 & -0.5704 & 1.2891 & -0.7970 & -0.4751 & 0.7497 & 0.8863 \\ 0.1012 & 1.1168 & 0.3248 & -0.2974 & 0.1823 & -1.0768 & -1.4808 & 0.7747 & -0.6746 & 1.7418 \end{pmatrix}.$$

Exemple 4.5. Assume $n = 10$, $k = 4$, $m = 5$. Let A_0 be the same matrix as in Example 4.4, and

$$\Lambda = \begin{pmatrix} -2.7645 & 0 & 0 & 0 & 0 \\ 0 & 0.8744 & 0 & 0 & 0 \\ 0 & 0 & 1.6888 & 0.6148 & 0 \\ 0 & 0 & -0.6148 & -1.6888 & 0 \\ 0 & 0 & 0 & 0 & -1.3716 \end{pmatrix}, \quad X = \begin{pmatrix} -0.1371 & 0.2076 & -0.5872 & 0.0000 & 0.1560 \\ -0.1338 & 0.4773 & -0.1075 & 0.1681 & 0.4014 \\ 0.0780 & 0.1427 & 0.1565 & -0.0217 & 0.0788 \\ -0.5429 & 0.0952 & -0.0595 & -0.0017 & 0.3097 \\ 0.4030 & 0.4469 & 0.2920 & 0.0398 & -0.4609 \\ 0.4030 & 0.4469 & -0.2920 & -0.0398 & 0.4609 \\ -0.5429 & 0.0952 & 0.0595 & 0.0017 & -0.3097 \\ 0.0780 & 0.1427 & -0.1565 & 0.0217 & -0.0788 \\ -0.1338 & 0.4773 & 0.1075 & -0.1681 & -0.4014 \\ -0.1371 & 0.2076 & 0.5872 & 0 & -0.1560 \end{pmatrix},$$

$$\widetilde{A} = \begin{pmatrix} 1.7572 & -0.6165 & 0.7680 & -1.4552 & -1.0340 & 0.2842 & -0.2699 & 0.3469 & 1.1695 & 0.1680 \\ 0.9292 & 0.8229 & -0.4083 & -0.7349 & 1.3295 & -0.5869 & 0.2496 & -0.9913 & -0.0295 & -0.5985 \\ -0.2080 & -0.1178 & 0.1257 & 1.1170 & 0.4950 & -0.3811 & -0.3742 & 1.5829 & -0.1819 & 0.4022 \\ -0.0967 & -0.2652 & 1.2886 & -1.5969 & 0.6194 & 0.5654 & 0.3233 & 0.0686 & -0.1478 & -0.8397 \\ -0.5574 & 1.7665 & -0.5080 & 0.4134 & -0.5793 & -0.1971 & 0.4617 & -0.8733 & 0.1796 & 0.6497 \\ 0.6399 & 0.1528 & -0.8700 & 0.4514 & -0.1445 & -0.5991 & 0.4246 & -0.5031 & 1.7963 & -0.6141 \\ -0.7955 & -0.1045 & 0.0073 & 0.2970 & 0.6379 & 0.6096 & -1.5690 & 1.2184 & -0.2075 & -0.0875 \\ 0.4401 & -0.1320 & 1.5767 & -0.3114 & -0.3442 & 0.4609 & 1.1745 & 0.1300 & -0.1784 & -0.2107 \\ -0.6448 & -0.0139 & -0.9915 & 0.2114 & -0.5248 & 1.3919 & -0.7663 & -0.4589 & 0.8001 & 0.8508 \\ 0.1809 & 1.2439 & 0.3729 & -0.2557 & 0.2895 & -1.0266 & -1.4416 & 0.7236 & -0.5593 & 1.7545 \end{pmatrix}.$$

In this case, $\widetilde{Z}_2 = 0$, thus we can simplify $\widetilde{G}_2 = \begin{pmatrix} Q_1(I_r - Y_1Y_1^+) \\ K_2Z_2^+(I_r - Y_1Y_1^+) + \widetilde{Q}_2\widetilde{Z}_2^+\widetilde{Z}_2 \end{pmatrix}$ in Theorem 4.2 to $\widetilde{G}_2 = \begin{pmatrix} Q_1 \\ K_2Z_2^+ \end{pmatrix}(I_r - Y_1Y_1^+)$, and the unique solution A^* to Problem 1.4 is

$$A^* = \begin{pmatrix} 1.7162 & -0.6778 & 0.7252 & -1.4836 & -1.1027 & 0.2208 & -0.2919 & 0.3335 & 1.1176 & 0.1318 \\ 0.8660 & 0.7810 & -0.4566 & -0.7759 & 1.3222 & -0.6060 & 0.2012 & -0.9996 & -0.1181 & -0.6573 \\ -0.2321 & -0.2322 & 0.1201 & 1.0968 & 0.4102 & -0.3935 & -0.3691 & 1.5516 & -0.2269 & 0.3700 \\ -0.1327 & -0.3110 & 1.2116 & -1.6500 & 0.5500 & 0.5500 & 0.2500 & 0.0116 & -0.2110 & -0.8327 \\ -0.5930 & 1.7138 & -0.5619 & 0.3500 & -0.6000 & -0.2000 & 0.4500 & -0.8619 & 0.1138 & 0.6070 \\ 0.6070 & 0.1138 & -0.8619 & 0.4500 & -0.2000 & -0.6000 & 0.3500 & -0.5619 & 1.7138 & -0.5930 \\ -0.8327 & -0.2110 & 0.0116 & 0.2500 & 0.5500 & 0.5500 & -1.6500 & 1.2116 & -0.3110 & -0.1327 \\ 0.3700 & -0.2269 & 1.5516 & -0.3691 & -0.3935 & 0.4102 & 1.0968 & 0.1201 & -0.2322 & -0.2321 \\ -0.6573 & -0.1181 & -0.9996 & 0.2012 & -0.6060 & 1.3222 & -0.7759 & -0.4566 & 0.7810 & 0.8660 \\ 0.1318 & 1.1176 & 0.3335 & -0.2919 & 0.2208 & -1.1027 & -1.4836 & 0.7252 & -0.6778 & 1.7162 \end{pmatrix}.$$

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