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Multi-soliton solutions of the BBM equation arisen in shallow water

O. Alsayyed[a], H. M. Jaradat[b,c,*], M. M. M. Jaradat[d], Zead Mustafa[d], Feras Shatat[e]

[a]*Department of Mathematics, Hashemite University, Jordan.*
[b]*Department of Mathematics, Al al-Bayt University, Jordan.*
[c]*Department of Mathematics and Applied Sciences, Dhofar University, Salalah, Oman.*
[d]*Department of Mathematics, Statistics and Physics, Qatar University, Doha, Qatar.*
[e]*General Courses Department, Emirates College of Technology, Abu-Dhabi, Emirates.*

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Abstract

In this work, multiple soliton solutions and multiple singular soliton solutions are formally derived for the BBM equation. A novel transformation method combined with the Hirota's bilinear method are used to determine the two sets of solutions, where each set has a distinct structure. The resonance phenomenon does not exist for the model under the study.

Keywords: N-soliton solutions, BBM equation, shallow water waves.
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1. Introduction and Preliminaries

Long waves in shallow water is a subject of broad interest and has a long colorful history. Physically, it has a rich variety of phenomenological manifestation, especially the existence of waves permanent in form and robust in maintaining their entities through mutual interaction and collision as well as the remarkable property of exhibiting recurrences of initial data when circumstances should prevail. These characteristics are due to the intimate interplay between the roles of nonlinearity and dispersion.

The Benjamin–Bona–Mahony (BBM) equation [2]

$$u_t + au_x - bu_{xxt} + c\left(u^2\right)_x = 0, \tag{1.1}$$

where a, b, c are arbitrary constants, was derived as a model for the unidirectional propagation of long-

*Corresponding author

Email addresses: o.alsayyed@yahoo.com (O. Alsayyed), husseinjaradat@yahoo.com (H. M. Jaradat), mmjst4@qu.edu.qa (M. M. M. Jaradat), zead@qu.edu.qa (Zead Mustafa), feras.shatat@ect.ac.ae (Feras Shatat)

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crested, surface water waves. In addition to shallow water waves, the equation is applicable to the study of drift waves in plasma or the Rossby waves in rotating fluids. The equation can also characterize the hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals and acoustic-gravity waves in compressible fluids [2]. Apart from these applications, solutions to the BBM equation are interesting in themselves. As a result, many methods were used for the solutions of the BBM equation [1, 3, 4, 12, 13, 14].

The BBM equation has been investigated as a regularized version of the Korteweg–de Vries equation for shallow water waves. It incorporates nonlinear dispersive and dissipative effects. In certain theoretical investigations the equation is superior as a model for long waves, and the word "regularized" refers to the fact that, from the standpoint of existence and stability, the equation offers considerable technical advantages over the Korteweg–de Vries equation.

The investigation of exact travelling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. Solitons are the most important solutions among travelling wave solutions. The existence of multi-soliton, especially two-soliton solutions, is crucial for information technology: it makes it possible undisturbed simultaneous propagation of many pulses in both directions. There are many powerful methods such as, the inverse scattering method, the Bäcklund transformation method [15], and the Hirota bilinear method [7, 8, 9, 10, 11] were thoroughly used to derive the multiple soliton solutions of these equations.

The Hirota bilinear method possesses significant features that make it practical for the determination of multiple soliton solutions. The Hirota bilinear method relies on a transformation for considered equation to a bilinear form. The bilinear forms are usually used to enable us deriving the auxiliary function. It is remarkable to mention that it is not easy for us to find such a transformation for some equations and sometimes it requires the introduction of new dependent and sometimes even independent variables. However, Hereman et al. [5, 6], formally introduced the simplified algorithm to derive the auxiliary functions without using the bilinear forms.

It is the aim of this work to derive multiple soliton solutions and multiple singular soliton solutions for the BBM equation (1.1). An efficient transformation combined with the Hereman's method will be used to achieve the goals set for this work.

2. Method description

To derive N-soliton solutions, we will mainly use the Hirota's direct method combined with the simplified version of Hereman and Zhuang [5, 6] where it was shown that soliton solutions are just polynomials of exponentials. We only summarize the necessary steps, the details of which can be found in [16, 17, 18, 19, 20, 21].

We first substitute

$$u(x,t) = e^{k_i x - \omega_i t}$$

into the linear terms of the equation under discussion to determine the dispersion relation between k_i and ω_i. To obtain the single soliton solution, we use a suitable transformation formula, such as

$$u(x,t) = R\,(\ln f)_{xx}\,,$$

into the equation under discussion, where the auxiliary function $f(x,t)$ is given by

$$f(x,t) = 1 + e^{\theta_1},$$

where

$$\theta_i = k_i x - \omega_i t, \qquad i = 1, 2, \cdots, N.$$

Then solving the resulting equation to determine the numerical value for R. The N-soliton solutions can be obtained by using the following forms for $f(x,t)$:

- For dispersion relation, we use

$$u(x,t) = e^{k_i x - \omega_i t}, \qquad \theta_i = k_i x - \omega_i t.$$

- For single soliton solution, we use

$$f(x,t) = 1 + e^{\theta_1}.$$

- For two-soliton solutions, we use

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}.$$

- For three-soliton solutions, we use

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1+\theta_2} + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3} + a_{123} e^{\theta_1+\theta_2+\theta_3}.$$

Notice that if we find that $b_{123} = a_{12}a_{13}a_{23}$, then three-soliton solutions are obtained.

The determination of three-soliton solutions confirms the fact that N-soliton solutions exist for any order.

However, for the multiple singular soliton solutions, we follow the following steps:

- For dispersion relation, we use

$$u(x,t) = e^{k_i x - \omega_i t}, \qquad \theta_i = k_i x - \omega_i t.$$

- For single soliton, we use

$$f(x,t) = 1 - e^{\theta_1}.$$

- For two-soliton solutions, we use

$$f(x,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12} e^{\theta_1+\theta_2}.$$

- For three-soliton solutions, we use

$$f(x,t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12} e^{\theta_1+\theta_2} + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3} - a_{123} e^{\theta_1+\theta_2+\theta_3}.$$

3. Soliton solutions for the BBM equation

In this section, we derive the solution for the BBM equation. In the first subsection we deal with the single soliton solution and in the second one we treat the single singular soliton solution.

3.1. Single soliton solution for the BBM equation

We first substitute

$$u(x,t) = e^{k_i x - \omega_i t} \tag{3.1}$$

into the linear terms of the BBM equation (1.1) to determine the dispersion relation between k_i and ω_i as follows

$$\omega_i = \frac{a k_i}{1 - b k_i^2}. \tag{3.2}$$

It is obvious that the dispersion relation (3.2) depends on the coefficient k_i of x. As a result we obtain

$$\theta_i = k_i x - \frac{a k_i}{1 - b k_i^2} t. \tag{3.3}$$

The single soliton solution of (1.1) is assumed to be

$$u(x,t) = R\left(\ln f\right)_{xx}, \tag{3.4}$$

where the auxiliary function $f(x,t)$, for the single soliton solution, is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - \frac{ak_1}{1-bk_1^2}t}. \tag{3.5}$$

Substituting (3.4) into (1.1) and solving for R we find

$$R = \frac{6ab}{c\left(1-bk_1^2\right)}. \tag{3.6}$$

Substituting (3.5) into (3.4) and using (3.6) gives the single soliton solution

$$u(x,t) = \frac{6abk_1^2}{c\left(1-bk_1^2\right)}\frac{e^{\eta}}{\left(1+e^{\eta}\right)^2} = \frac{3abk_1^2}{2c\left(1-bk_1^2\right)}\sec h^2(\frac{\eta}{2}), \tag{3.7}$$

where

$$\eta = \left(\frac{k_1 - bk_1^3}{1-bk_1^2}\right)x - \left(\frac{a}{1-bk_1^2}\right)t. \tag{3.8}$$

Solution (3.8) is a bell-shaped solitary wave solution of the BBM Eq. (1.1), which has the maximum amplitude $\frac{3abk_1^2}{2c\left(1-bk_1^2\right)}$. The profile of this solution is depicted in Figure 1.

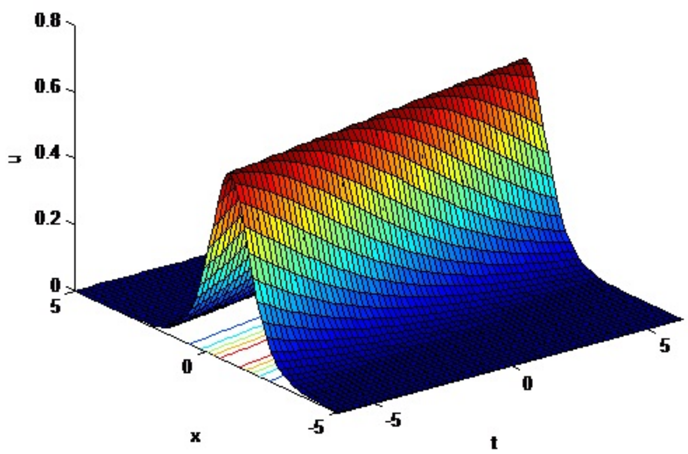

Figure 1: The profile of the bell-shaped solitary wave solution (3.11) with $a=-1, b=2, c=3, k_1=2$.

3.2. Single singular soliton solution for the BBM equation

To obtain a single singular soliton solution, we substitute

$$u(x,t) = e^{k_i x - \omega_i t} \tag{3.9}$$

into the linear terms of the BBM Eq. (1.1). This gives the dispersion relation by

$$\omega_i = \frac{ak_i}{1-bk_i^2} \tag{3.10}$$

and as a result we obtain

$$\theta_i = k_i x - \frac{ak_i}{1-bk_i^2}t. \tag{3.11}$$

The singular single soliton solution of (1.1) is assumed to be

$$u(x,t) = R\left(\ln f\right)_{xx}, \tag{3.12}$$

where the auxiliary function $f(x,t)$ is given by

$$f(x,t) = 1 - e^{\theta_1} = 1 - e^{k_1 x - \frac{ak_1}{1-bk_1^2}t}. \tag{3.13}$$

Substituting (3.12) into (1.1) and solving for R we find

$$R = \frac{6ab}{c\left(1-bk_1^2\right)}. \tag{3.14}$$

Substituting (3.13) into (3.12) gives the single soliton solution

$$u(x,t) = \frac{6abk_1^2}{c\left(bk_1^2-1\right)}\frac{e^{\phi}}{\left(e^{\phi}-1\right)^2} = \frac{3abk_1^2}{2c\left(1-bk_1^2\right)}\operatorname{csc}h^2(\frac{\phi}{2}), \tag{3.15}$$

where

$$\phi = \left(\frac{bk_1^3-k_1}{bk_1^2-1}\right)x + \left(\frac{a}{bk_1^2-1}\right)t. \tag{3.16}$$

The profile of the solution (3.15) is depicted in Figure 2.

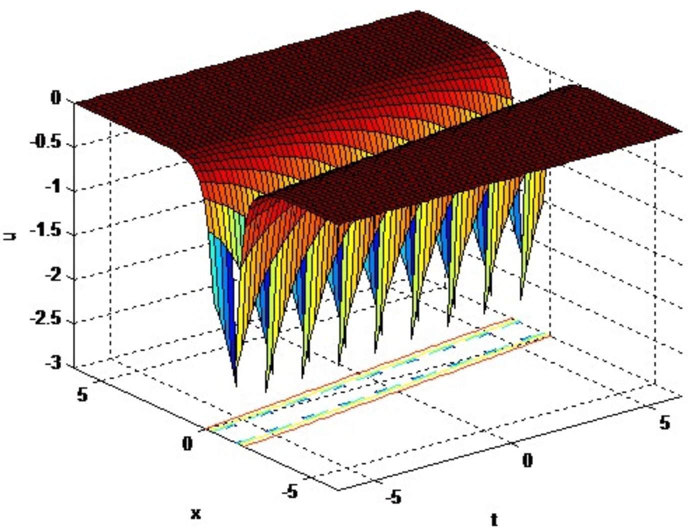

Figure 2: The profile of the singular solitary wave solution (3.11) with $a=-1, b=2, c=3, k_1=2$.

3.3. Multiple soliton solutions for the BBM equation

More generally, the N-soliton solutions for Eq. (1.1) can be expressed in the following form:

$$u(x,t) = \frac{6ab}{c\left(1-bk_1^2\right)}\left(\ln f\right)_{xx}, \tag{3.17}$$

$$f(x,t) = \sum_{\mu=0,1} \exp\left(\sum_{i>j}^{N} M_{ij}\mu_i\mu_j + \sum_{j=1}^{N}\mu_j\theta_j\right),$$

where $\theta j = k_j x - \frac{ak_j}{1-bk_j^2}t$ and $\exp\left(M_{ij}\right) = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}$, while k_j is the parameter characterizing the jth soliton, $\sum\limits_{i>j}^{N}$ denotes the summation over all possible combinations of $\mu_j = 0, 1$ and $\sum\limits_{i>j}^{N}$ is the summation over all possible pairs chosen from N elements under the condition $i > j$.

For two-soliton solutions, we substitute

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + \frac{\left(k_1-k_2\right)^2}{\left(k_1+k_2\right)^2}e^{\theta_1+\theta_2}, \tag{3.18}$$

where θ_1 and θ_2 are defined in (3.3), into

$$u(x,t) = \frac{6ab}{c\left(1-bk_1^2\right)}\left(\ln f\right)_{xx}. \tag{3.19}$$

It worth to point out that the BBM Eq. (1.1) does not show any resonant phenomenon because the phase shift term $a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$ in (3.18) cannot be 0 or ∞ for $|k_1| \neq |k_2|$. It is well known that a two-soliton solution can degenerate into a resonant triad under the conditions

$$a_{12} = 0, \qquad \text{or} \qquad \left(a_{12}\right)^{-1} = 0 \qquad \text{for } \left|k_1\right| \neq \left|k_2\right|.$$

The profile of the two-soliton solution (3.17) is depicted in Figure 3.

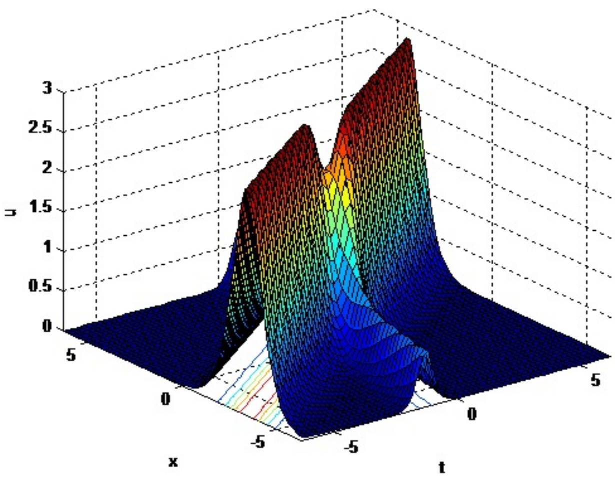

Figure 3: The profile of the bell-shaped solitary wave solution (3.11) with $a = -5, b = 2, c = 3, k_1 = 2, k_2 = 1$

To determine the three-soliton solutions we set

$$\begin{aligned} f(x,t) =& 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\theta_1+\theta_2} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{\theta_1+\theta_3} \\ &+ \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{\theta_2+\theta_3} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \frac{(k_1-k_3)^2}{(k_1+k_3)^2} \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{3.20}$$

The three-soliton solutions are obtained by substituting (3.20) into (3.17). This shows that the N-soliton solutions can be obtained for finite N, where $N \geq 1$.

3.4. *Multiple singular soliton solutions for the BBM equation*

Multiple singular soliton solutions for Eq. (1.1) can be expressed in the following form:

$$u(x,t) = \frac{6ab}{c\left(1-bk_1^2\right)} \left(\ln f\right)_{xx}, \tag{3.21}$$

where $\theta j = k_j x - \frac{ak_j}{1-bk_j^2} t$.

To determine the two singular soliton solutions explicitly, we substitute

$$f(x,t) = 1 - e^{\theta_1} - e^{\theta_2} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\theta_1+\theta_2}, \tag{3.22}$$

where θ_1 and θ_2 are defined in (3.3), into (3.21).

For the three-soliton solutions we substitute

$$\begin{aligned} f(x,t) =& 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{\theta_1+\theta_2} + \frac{(k_1-k_3)^2}{(k_1+k_3)^2} e^{\theta_1+\theta_3} \\ &+ \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{\theta_2+\theta_3} - \frac{(k_1-k_2)^2}{(k_1+k_2)^2} \frac{(k_1-k_3)^2}{(k_1+k_3)^2} \frac{(k_2-k_3)^2}{(k_2+k_3)^2} e^{\theta_1+\theta_2+\theta_3}, \end{aligned}$$

where θ_i $(i = 1, 2, 3)$ are defined in (3.3), into (3.21).

4. Conclusion

Under consideration in this paper is the BBM equation for the propagation of long waves in shallow water. In this paper, the simplified form of the Hirota bilinear method has been employed to derive the multi-soliton solutions and multiple singular soliton solutions. The solutions are formally obtained without any need to derive the bilinear forms. The derived solutions show the interaction of standard solitons and the singular behavior of the singular solitons. The analysis confirms the fact that certain equations which have N-soliton solutions, have simultaneously, N-singular soliton solutions. To our knowledge, the solutions that we have constructed are new and different from those in the existing papers.

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