Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 200–207
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Stability of a nonlinear Volterra integro-differential equation via a fixed point approach

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Communicated by Janusz Brzdek

Abstract

The object of the present paper is to examine the Hyers-Ulam-Rassias stability and the Hyers-Ulam stability of a nonlinear Volterra integro-differential equation by using the fixed point method. ©2016 All rights reserved.

Keywords: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Volterra integro-differential equations, fixed-point method.
2010 MSC: 45J05, 47H10, 45M10.

1. Introduction

This paper is concerned with the following nonlinear Volterra integro-differential equations

$$u'(t) = f(t, u(t)) + \int_0^t k(t, s, u(s))ds, \quad t \in I := [0, T], \tag{1.1}$$

with initial condition $u(0) = \alpha$, where $f(t, u)$ is continuous function with respect to variables t and u on $I \times \mathbb{R}$, $k(t, s, u)$ is continuous with respect to t, s and u on $I \times I \times \mathbb{R}$ and α is a given constant. Volterra integro-differential equations arise widely in the mathematical modeling of physical and biological phenomena.

In 1940, S. M. Ulam posed the following problem: ”Under what conditions does there exists an additive mapping near an approximately additive mapping?” [14]. In the following year, Hyers [8] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. In 1978, Rassias [13] provided a

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Received 2015-04-06

generalization of the result of Hyers by proving the existence of unique linear mappings near approximate additive mappings.

S. M. Jung [9] applied the fixed point method to the investigation of the Volterra integral equation adhering to the notion of Cadariu and Radu [2]. He proved that if a continuous function $u : I \to \mathbb{C}$ satisfies the perturbed Volterra integral equation of second kind

$$\left| u(t) - \int_c^t F(\tau, u(\tau))d\tau \right| \leq \varphi(t)$$

for all $t \in I$, then under some additional conditions, there exist a unique continuous function $u_0 : I \to \mathbb{C}$ and a constant $C > 0$ such that

$$u_0(t) = \int_c^t F(\tau, u_0(\tau))d\tau \quad \text{and} \quad |u(t) - u_0(t)| \leq C\varphi(t)$$

for all $t \in I$. Recently in [11] the authors jointly with S.-M. Jung proved that if $p : I \to \mathbb{R}$, $q : I \to \mathbb{R}$, $K : I \times I \to \mathbb{R}$ and $\varphi : I \to [0, \infty)$ are sufficiently smooth functions and if a continuously differentiable function $u : I \to \mathbb{R}$ satisfies the perturbed Volterra integro-differential equation

$$\left| u'(t) + p(t)u(t) + q(t) + \int_c^t K(t, \tau)u(\tau)d\tau \right| \leq \varphi(t)$$

for all $t \in I$, then there exists a unique solution $u_0 : I \to \mathbb{R}$ of the Volterra integro-differential equation

$$u'(t) + p(t)u(t) + q(t) + \int_c^t K(t, \tau)u(\tau)d\tau = 0,$$

such that

$$|u(t) - u_0(t)| \leq \exp\left\{ -\int_c^t p(\tau)d\tau \right\} \int_t^b \varphi(\xi) \exp\left\{ \int_c^\xi p(\tau)d\tau \right\} d\xi$$

for all $t \in I$. In the past recent years, several authors proved the Hyers-Ulam stability of Volterra equations of other type (we refer to [1, 3, 4, 6, 7, 12]).

Definition 1.1. If for each continuously differentiable function $u(t)$ satisfying

$$\left| u'(t) - f(t, u(t)) - \int_0^t k(t, s, u(s))ds \right| \leq \psi(t),$$

where $\psi(t) \geq 0$ for all t, there exists a solution $u_0(t)$ of the Volterra integro-differential equations (1.1) and a constant $C > 0$ with

$$|u(t) - u_0(t)| \leq C\psi(t)$$

for all t, where C is independent of $u(t)$ and $u_0(t)$, then we say that the equation (1.1) has the Hyers-Ulam-Rassias stability. If $\psi(t)$ is a constant function in the above inequalities, we say that equation (1.1) has the Hyers-Ulam stability.

For a nonempty set X, we introduce the definition of the generalized metric on X.

Definition 1.2. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x, y) = 0$ if and only if $x = y$;

(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We now introduce one of the fundamental results of fixed point theory that will play an important role in proving our main theorems.

Theorem 1.3 ([5])**.** *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the followings are true:*

(a) the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;

(b) x^* is the unique fixed point of Λ in

$$X^* = \left\{ y \in X \;\middle|\; d(\Lambda^k x, y) < \infty \right\}; \tag{1.2}$$

(c)If $y \in X^*$, then

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y). \tag{1.3}$$

The present paper is motivated by the desire to investigate the Hyers-Ulam-Rassias stability and Hyers-Ulam stability for the nonlinear Volterra integro-differential equation (1.1).

2. Hyers-Ulam-Rassias Stability

In this section, we will prove the Hyers-Ulam-Rassias stability of the nonlinear Volterra integro-differential equation (1.1).

Theorem 2.1. *Let $I := [0, T]$ be a given closed and bounded interval, with $T > 0$, and M, L_f and L_k be positive constants with $0 < ML_f + M^2L_k < 1$. Suppose that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition*

$$|f(t, u_1) - f(t, u_2)| \leq L_f |u_1 - u_2|, \quad \forall t \in I, \ \forall u_1, u_2 \in \mathbb{R} \tag{2.1}$$

and $k : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition

$$|k(t, s, u_1) - k(t, s, u_2)| \leq L_k |u_1 - u_2|, \quad \forall t, s \in I, \ \forall u_1, u_2 \in \mathbb{R}. \tag{2.2}$$

If a continuously differentiable function $u : I \to \mathbb{R}$ satisfies

$$\left| u'(t) - f(t, u(t)) - \int_0^t k(t, s, u(s))ds \right| \leq \psi(t), \quad \forall t \in I, \tag{2.3}$$

where $\psi : I \to (0, \infty)$ is a continuous function with

$$\int_0^t \psi(\xi)d\xi \leq M\psi(t) \tag{2.4}$$

for each $t \in I$, then there exists a unique continuous function $u_0 : I \to \mathbb{R}$ such that

$$u_0(t) = \alpha + \int_0^t f(\xi, u_0(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, u_0(\xi))d\xi ds \tag{2.5}$$

and

$$|u(t) - u_0(t)| \leq \frac{M}{1 - (ML_f + M^2L_k)} \psi(t), \quad \forall t \in I. \tag{2.6}$$

Proof. Let X denote the set of all real-valued continuous functions on I. For $v, w \in X$, we set

$$d(v, w) = \inf \{ C \in [0, \infty] \mid \ |v(t) - w(t)| \leq C\psi(t), \ \forall t \in I \}. \tag{2.7}$$

It is easy to see that (X, d) is a complete generalized metric space (see [10]).

Now, consider the operator $\Lambda : X \to X$ defined by

$$(\Lambda v)(t) = \alpha + \int_0^t f(\xi, v(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, v(\xi))d\xi ds,\ \forall t \in I \tag{2.8}$$

for all $v \in X$.

We check that Λ is strictly contractive on X. Let $C_{vw} \in [0, \infty]$ be an discretionary constant with $d(v, w) \leq C_{vw}$ for any $v, w \in X$, that is, by (2.7), we have

$$|v(t) - w(t)| \leq C_{vw}\psi(t),\ \forall t \in I. \tag{2.9}$$

It then follows from (2.1), (2.2), (2.4), (2.8) and (2.9) that

$$\begin{aligned} |(\Lambda v)(t) - (\Lambda w)(t)| &= \left| \int_0^t \{f(\xi, v(\xi)) - f(\xi, w(\xi))\} d\xi \right. \\ &\quad \left. + \int_0^t \int_0^s \{k(t, \xi, v(\xi)) - k(t, \xi, w(\xi))\} d\xi ds \right| \\ &\leq \int_0^t |f(\xi, v(\xi)) - f(\xi, w(\xi))| d\xi \\ &\quad + \int_0^t \int_0^s |k(t, \xi, v(\xi)) - k(t, \xi, w(\xi))| d\xi ds \\ &\leq L_f \int_0^t |v(\xi) - w(\xi)| d\xi + L_k \int_0^t \int_0^s |v(\xi) - w(\xi)| d\xi ds \\ &\leq L_f C_{vw} \int_0^t \psi(\xi)d\xi + L_k C_{vw} \int_0^t \int_0^s \psi(\xi)d\xi ds \\ &\leq C_{vw}\psi(t)(ML_f + M^2 L_k),\ \forall t \in I, \end{aligned}$$

that is, $d(\Lambda v, \Lambda w) \leq C_{vw}\psi(t)(ML_f + M^2 L_k)$. Hence, we can conclude that $d(\Lambda v, \Lambda w) \leq (ML_f + M^2 L_k)d(v, w)$ for any $v, w \in X$, where we note that $0 < ML_f + M^2 L_k < 1$.

It follows from (2.8) that for arbitrary $w_0 \in X$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned} |(\Lambda w_0)(t) - w_0(t)| &= \left| \alpha + \int_0^t f(\xi, w_0(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, u_0(\xi))d\xi ds - w_0(t) \right| \\ &\leq C\psi(t),\ \forall t \in I, \end{aligned}$$

since $f(\xi, w_0(\xi))$, $k(t, \xi, u_0(\xi))$ and $w_0(t)$ are bounded on their domain and $\min_{t\in I}\psi(t) > 0$. Thus, (2.7) implies that

$$d(\Lambda w_0, w_0) < \infty.$$

Therefore, according to Theorem 1.3 (a), there exists a continuous function $u_0 : I \to \mathbb{R}$ such that $\Lambda^n w_0 \to u_0$ in (X, d) and $\Lambda u_0 = u_0$, that is, u_0 satisfies equation (2.5) for every $t \in I$.

Since w and w_0 are bounded on I for any $w \in X$ and $\min_{t\in I}\psi(t) > 0$, there exists a constant $0 < C_w < \infty$ such that

$$|w_0(t) - w(t)| \leq C_w\psi(t)$$

for any $t \in I$. We have $d(w_0, w) < \infty$ for any $w \in X$. Therefore, we obtained that $\{w \in X \mid d(w_0, w) < \infty\}$ is equal to X. From Theorem 1.3 (b), we deduce that u_0, given by (2.5), is the unique continuous function.

From (2.3), we have

$$-\psi(t) \leq u'(t) - f(t, u(t)) - \int_0^t k(t, s, u(s))ds \leq \psi(t), \quad \forall t \in I. \tag{2.10}$$

If each term of the inequality (2.10) is integrated from 0 to t, then

$$\left|u(t) - \alpha - \int_0^t f(\xi, u(\xi))d\xi - \int_0^t \int_0^s k(t, \xi, u(\xi))d\xi ds\right| \leq \int_0^t \psi(\xi)d\xi, \quad \forall t \in I.$$

Thus, by (2.4) and (2.8), we get

$$|u(t) - (\Lambda u)(t)| \leq \int_0^t \psi(\xi)d\xi \leq M\psi(t), \quad \forall t \in I,$$

which implies that

$$d(u, \Lambda u) \leq M. \tag{2.11}$$

By using Theorem 1.3 (c) and (2.11), we conclude that

$$d(u, u_0) \leq \frac{1}{1 - (ML_f + M^2L_k)} d(\Lambda u, u) \leq \frac{M}{1 - (ML_f + M^2L_k)}.$$

Consequently, this yields the inequality (2.6) for all $t \in I$. □

In Theorem 2.1, we have examined the Hyers-Ulam-Rassias stability of the Volterra integro-differential equation (1.1) defined on a bounded and closed interval. We will now show that Theorem 2.1 is also valid for the case unbounded intervals.

Theorem 2.2. *For given nonnegative real numbers T, let I denote either $(-\infty, T]$ or $\mathbb{R}$ or $[0, \infty)$. Let M, L_f and L_k be positive constants with $0 < ML_f + M^2L_k < 1$. Suppose that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition (2.1) for all $t \in I$ and all $u_1, u_2 \in \mathbb{R}$. If a continuously differentiable function $u : I \to \mathbb{R}$ satisfies the differential inequality (2.3) for all $t \in I$, where $\psi : I \to (0, \infty)$ is a continuous function satisfying the condition (2.4) for each $t \in I$, then there exists a unique continuous function $u_0 : I \to \mathbb{R}$ which satisfies (2.5) and (2.6) for all $t \in I$.*

Proof. Let $I = \mathbb{R}$. We first show that u_0 is a continuous function. For any $n \in \mathbb{N}$, we define $I_n = [-n, n]$. In accordance with Theorem 2.1, there exists a unique continuous function $u_n : I_n \to \mathbb{R}$ such that

$$u_n(t) = \alpha + \int_0^t f(\xi, u_n(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, u_n(\xi))d\xi ds \tag{2.12}$$

and

$$|u(t) - u_n(t)| \leq \frac{M}{1 - (ML_f + M^2L_k)} \psi(t) \tag{2.13}$$

for all $t \in I$. The uniqueness of u_n implies that if $t \in I_n$, then

$$u_n(t) = u_{n+1}(t) = u_{n+2}(t) = \cdots . \tag{2.14}$$

For any $t \in \mathbb{R}$, we define $n(t) \in \mathbb{N}$ as

$$n(t) = \min \{n \in \mathbb{N} \mid t \in I_n\}.$$

Moreover, let us define a function $u_0 : \mathbb{R} \to \mathbb{R}$ by

$$u_0(t) = u_{n(t)}(t), \tag{2.15}$$

and we claim that u_0 is continuous. We take the integer $n_1 = n(t_1)$ for an arbitrary $t_1 \in \mathbb{R}$. Then, t_1 belongs to the interior of I_{n_1+1} and there exists an $\varepsilon > 0$ such that $u_0(t) = u_{n_1+1}(t)$ for all t with $t_1 - \varepsilon < t < t_1 + \varepsilon$. Since u_{n_1+1} is continuous at t_1, u_0 is continuous at t_1 for any $t_1 \in \mathbb{R}$.

Now, we will prove that u_0 satisfies (2.5) and (2.7) for all $t \in \mathbb{R}$. Let $n(t)$ be an integer for an arbitrary $t \in \mathbb{R}$. Then, from (2.12) and (2.15), we have $t \in I_{n(t)}$ and

$$\begin{aligned} u_0(t) = u_{n(t)}(t) &= \alpha + \int_0^t f(\xi, u_{n(t)}(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, u_{n(t)}(\xi))d\xi ds \\ &= \alpha + \int_0^t f(\xi, u_0(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, u_0(\xi))d\xi ds. \end{aligned}$$

Since $n(\xi) \leq n(t)$ for any $\xi \in I_{n(t)}$, the last equality be correct and we have

$$u_{n(t)}(\xi) = u_{n(\xi)}(\xi) = u_0(\xi),$$

by virtue (2.14) and (2.15).

Since $t \in I_{n(t)}$ for every $t \in \mathbb{R}$, by (2.13) and (2.15), we have

$$|u(t) - u_0(t)| \leq \left|u(t) - u_{n(t)}(t)\right| \leq \frac{M}{1 - (ML_f + M^2 L_k)}\psi(t)$$

for any $t \in \mathbb{R}$.

Lastly, we prove that u_0 is unique. Assume that $v_0 : \mathbb{R} \to \mathbb{R}$ be another continuous function which satisfies (2.5) and (2.7), with v_0 in place of u_0, for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$ be an discretionary number. Since the restrictions $u_0|_{I_{n(t)}}$ and $v_0|_{I_{n(t)}}$ both satisfy (2.5) and (2.7) for all $t \in I_{n(t)}$, the uniqueness of $u_{n(t)} = u_0|_{I_{n(t)}}$ suggest that

$$u_0(t) = u_0|_{I_{n(t)}}(t) = v_0|_{I_{n(t)}}(t) = v_0(t).$$

The proof can be done similarly for the cases $I = (-\infty, T]$ and $I = [0, \infty)$. □

3. Hyers-Ulam Stability

In this section, we will prove the Hyers-Ulam stability of the nonlinear Volterra integro-differential equation (1.1).

Theorem 3.1. *Let L_f and L_k be positive constants with $0 < TL_f + \frac{T^2}{2}L_k < 1$ and $I := [0, T]$ denote a given closed and bounded interval, with $T > 0$. Suppose that $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition* (2.1) *and $k : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition* (2.2)*. If for $\varepsilon \geq 0$ için a continuously differentiable function $u : I \to \mathbb{R}$ satisfies*

$$\left|u'(t) - f(t, u(t)) - \int_0^t k(t, s, u(s))ds\right| \leq \varepsilon, \quad \forall t \in I, \tag{3.1}$$

then there exists a unique continuous function $u_0 : I \to \mathbb{R}$ satisfying equation (2.5) *and*

$$|u(t) - u_0(t)| \leq \frac{T}{1 - (TL_f + \frac{T^2}{2}L_k)}\varepsilon, \quad \forall t \in I. \tag{3.2}$$

Proof. Initially, let X denote the set of all real-valued continuous functions on I. Furthermore, we define a generalized metric on X by

$$d(v, w) = \inf \{C \in [0, \infty] \mid |v(t) - w(t)| \leq C, \ \forall t \in I\}. \tag{3.3}$$

It is easy to see that (X, d) is a complete generalized metric space (see Jung [10]).

Now, we define the operator $\Lambda : X \to X$ by

$$(\Lambda v)(t) = \alpha + \int_0^t f(\xi, v(\xi))d\xi + \int_0^t \int_0^s k(t, \xi, v(\xi))d\xi ds, \ \forall t \in I \tag{3.4}$$

for all $v \in X$.

We now prove that Λ is strictly contractive on the generalized metric space X. For any $v, w \in X$, let $C_{vw} \in [0, \infty]$ be an arbitrary constant with $d(h, g) \leq C_{gh}$, that is, let us suppose that

$$|v(t) - w(t)| \leq C_{vw}, \ \forall t \in I. \tag{3.5}$$

By using (2.1), (2.2), (3.4) and (3.5), we deduce

$$\begin{aligned}
|(\Lambda v)(t) - (\Lambda w)(t)| &= \left| \int_0^t \{f(\xi, v(\xi)) - f(\xi, w(\xi))\} \, d\xi \right. \\
&\quad \left. + \int_0^t \int_0^s \{k(t, \xi, v(\xi)) - k(t, \xi, w(\xi))\} \, d\xi ds \right| \\
&\leq \int_0^t |f(\xi, v(\xi)) - f(\xi, w(\xi))| \, d\xi \\
&\quad + \int_0^t \int_0^s |k(t, \xi, v(\xi)) - k(t, \xi, w(\xi))| \, d\xi ds \\
&\leq L_f \int_0^t |v(\xi) - w(\xi)| \, d\xi + L_k \int_0^t \int_0^s |v(\xi) - w(\xi)| \, d\xi ds \\
&\leq L_f C_{vw} t + L_k C_{vw} \frac{t^2}{2} \\
&\leq C_{vw}(T L_f + \frac{T^2}{2} L_k), \ \forall t \in I,
\end{aligned}$$

that is, $d(\Lambda v, \Lambda w) \leq C_{vw}(T L_f + \frac{T^2}{2} L_k)$. We conclude that $d(\Lambda v, \Lambda w) \leq (T L_f + \frac{T^2}{2} L_k) d(v, w)$ for any $v, w \in X$.

Let w_0 be any arbitrary element in X. Then there exists a constant $0 < C < \infty$ with

$$\begin{aligned}
|(\Lambda w_0)(t) - w_0(t)| &= \left| \alpha + \int_0^t f(\xi, w_0(\xi)) d\xi + \int_0^t \int_0^s k(t, \xi, w_0(\xi)) d\xi ds - w_0(t) \right| \\
&\leq C, \ \forall t \in I,
\end{aligned}$$

since $f(\xi, w_0(\xi))$, $k(t, \xi, w_0(\xi))$ and $w_0(t)$ are bounded on their domain. Thus, (3.3) implies that

$$d(\Lambda w_0, w_0) < \infty.$$

Therefore, according to Theorem 1.3 (a), there exists a continuous function $u_0 : I \to \mathbb{R}$ such that $\Lambda^n w_0 \to u_0$ in (X, d) as $n \to \infty$, and such that $\Lambda u_0 = u_0$, that is, u_0 satisfies equation (2.5) for every $t \in I$.

As in the proof of Theorem 2.1, it can be verify that $\{w \in X \mid d(w_0, w) < \infty\} = X$. Due to Theorem 1.3 (b), u_0, given by (2.5), is the unique continuous function.

From (2.3), we get

$$-\varepsilon \leq u'(t) - f(t, u(t)) - \int_0^t k(t, s, u(s)) ds \leq \varepsilon, \quad \forall t \in I.$$

If each term of the above inequality is integrated from 0 to t, then

$$\left| u(t) - \alpha - \int_0^t f(\xi, u_0(\xi)) d\xi - \int_0^t \int_0^s k(t, \xi, u_0(\xi)) d\xi ds \right| \leq \varepsilon T, \quad \forall t \in I,$$

that is, it holds that

$$d(u, \Lambda u) \leq \varepsilon T. \tag{3.6}$$

Lastly, Theorem 1.3 (c) together with (2.11) implies that

$$d(u, u_0) \leq \frac{1}{1 - (T L_f + \frac{T^2}{2} L_k)} d(\Lambda u, u) \leq \frac{T}{1 - (T L_f + \frac{T^2}{2} L_k)} \varepsilon,$$

that is the inequality (3.2) be true for all $t \in I$. □

Acknowledgements:

This study was supported by a Grant from Yüzüncü Yıl University, the Directorate of Scientific Research Projects (KONGRE-2014/38).

References

[1] M. Akkouchi, *Hyers-Ulam-Rassias stability of nonlinear Volterra integral equations via a fixed point approach*, Acta Univ. Apulensis Math. Inform., **26** (2011), 257–266. 1

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber., **346** (2004,) 43–52. 1

[3] L. P. Castro, A. Ramos, *Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations*, Banach J. Math. Anal., **3** (2009), 36–43. 1

[4] L. P. Castro, A. Ramos, *Hyers-Ulam and Hyers-Ulam-Rassias stability of Volterra integral equations with delay*, Integral methods in science and engineering, Birkhauser Boston, Inc., Boston, MA, **1** (2010), 85–94. 1

[5] J. B. Diaz, B. Margolis, *A fixed point theorem of the alternative, for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309. 1.3

[6] M. Gachpazan, O. Baghani, *Hyers-Ulam stability of Volterra integral equation*, J. Nonlinear Anal. Appl., **1** (2010), 19–25. 1

[7] M. Gachpazan, O. Baghani, *Hyers-Ulam stability of nonlinear integral equation*, Fixed Point Theory Appl., **2010** (2010), 6 pages. 1

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222–224. 1

[9] S. M. Jung, *A fixed point approach to the stability of a Volterra integral equation*. Fixed Point Theory Appl., **2007** (2007), 9 pages. 1

[10] S. M. Jung, *A fixed point approach to the stability of differential equations* $y' = F(x, y)$, Bull. Malays. Math. Sci. Soc., **33** (2010), 47–56. 2, 3

[11] S.M. Jung, S. Şevgin, H. Şevli. *On the perturbation of Volterra integro-differential equations*, Appl. Math. Lett., **26** (2013), 665–669. 1

[12] J. R. Morales, E. M. Rojas, *Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear integral equations with delay*, Int. J. Nonlinear Anal. Appl., **2** (2011), 1–6. 1

[13] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*. Proc. Amer. Math. Soc., **72** (1978), 297–300. 1

[14] S. M. Ulam, *Problems in Modern Mathematics*,Science Editions John Wiley & Sons, Inc., New York, (1964). 1