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Fixed point results on metric and partial metric spaces via simulation functions

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Abstract

We prove existence and uniqueness of fixed point, by using a simulation function and a lower semi-continuous function in the setting of metric space. As consequences of this study, we deduce several related fixed point results, in metric and partial metric spaces. An example is given to support the new theory.

Keywords: Fixed point, metric space, partial metric space, nonlinear contraction, simulation function.
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1. Introduction

Metric fixed point theory is a vivid topic, which furnishes useful methods and notions for dealing with various problems. In particular, we refer to the existence of solutions of mathematical problems reducible to equivalent fixed point problems. Thus we recall that Banach contraction principle [2] is at the foundation of this theory. However, the potentiality of fixed point approaches attracted many scientists and hence there is a wide literature available for interested readers, see for instance [3, 7, 8, 9, 11, 18, 20, 21, 24, 25, 27]. We give some details on the notions and ideas used in this study.

First, the notion of partial metric space was introduced in 1994 by Matthews [12] as a part of the study of denotational semantics of data for networks. Clearly, this setting is a generalization of the classical concept of metric space. Also, some authors discussed the existence of several connections between partial metrics and topological aspects of domain theory, see for instance [5, 6, 13, 17].

Second, the notion of $\mathcal{Z}$-contraction was introduced in 2014 by Khojasteh *et al.* [10]. This concept is a new type of nonlinear contraction defined by using a specific simulation function. Of course, they proved existence and uniqueness of fixed points for $\mathcal{Z}$-contraction mappings. We point out that the advantage of

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this method is in providing a unique point of view for several fixed point problems; see recent results in [1, 23].

Finally, Samet *et al.* [26], and Vetro and Vetro [28] discussed fixed point results by using semi-continuous functions in metric spaces that generalize and improve many existing fixed point theorems in the literature. As an application of presented results, the authors gave some theorems in the setting of partial metric spaces. In this paper, we use the ideas in [26, 28] and the notion of simulation function to establish existence and uniqueness of fixed points. As consequences of this study, we deduce several related fixed point results, in metric and partial metric spaces. Also, an example is given to support the new theory.

2. Preliminaries

In this section we recall some definitions and some properties of partial metric spaces that can be found in [12, 13, 14, 15, 16]. We start with a definition. Precisely, a partial metric on a non-empty set X is a function $p : X \times X \to [0, +\infty[$ such that, for all $x, y, z \in X$, we have

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then, a partial metric space is a pair (X, p), where X is a non-empty set and p is a partial metric on X. Notice that, if $p(x, y) = 0$, then from (p_1) and (p_2) it follows that $x = y$. However, if $x = y$, then $p(x, y)$ may not be 0. A classic example of partial metric space is the pair $([0, +\infty[, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, +\infty[$, see also [12].

Every partial metric $p : X \times X \to [0, +\infty[$ generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Let (X, p) be a partial metric space. A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$. A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

It is easy to see that, every closed subset of a complete partial metric space is complete.

Notice that if p is a partial metric on X, then the function $p^s : X \times X \to [0, +\infty[$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{2.1}$$

is a metric on X. Furthermore, $\lim_{n\to+\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m).$$

Lemma 2.1. *Let (X, p) be a partial metric space and let $\varphi : X \to [0, +\infty[$ be defined by $\varphi(x) = p(x, x)$ for all $x \in X$. Then the function φ is lower semi-continuous in the metric space (X, p^s).*

Proof. Let $\{x_n\} \subset X$ be a sequence which converges to $x \in X$ in the metric space (X, p^s), then

$$p(x,x) = \lim_{n\to+\infty} p(x_n, x_n) = \liminf_{n\to+\infty} p(x_n, x_n).$$

It follows that φ is lower semi-continuous in x and hence in X. $\square$

Lemma 2.2 ([12, 14])**.** *Let (X, p) be a partial metric space. Then*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

Khojasteh *et al.* gave the following definition of simulation function, see [10].

Definition 2.3. A simulation function is a mapping $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ satisfying the following conditions:
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$, for all $t, s > 0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $]0,+\infty[$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = \ell \in]0,+\infty[$, then

$$\limsup_{n\to+\infty} \zeta(t_n, s_n) < 0.$$

Consequently, they proved the following theorem.

Theorem 2.4 ([10])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ, that is,*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \qquad \text{for all } x, y \in X. \tag{2.2}$$

Then T has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

In [1], Argoubi *et al.* note that the condition (ζ_1) was not used for the proof of Theorem 2.4. Also they observed that taking $x = y$ in (2.2), one has $\zeta(0,0) \geq 0$ and hence, if $\zeta(0,0) < 0$, the set of operators $T : X \to X$ satisfying (2.2) is empty.

Therefore, Argoubi *et al.* slightly revised the previous definition, by withdrawing the condition (ζ_1). For the sake of clarity, we consider the following definition.

Definition 2.5. A simulation function is a mapping $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ satisfying the conditions (ζ_2) and (ζ_3).

Note that every simulation function in (original) Khojasteh *et al.* sense (Definition 2.3) is a simulation function in Argoubi *et al.* sense (Definition 2.5), but the converse is not true; see example below.

Example 2.6 ([1], Example 2.4)**.** Let $\zeta_\lambda : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ be the function defined by

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (s,t) = (0,0), \\ \lambda s - t & \text{otherwise,} \end{cases}$$

where $\lambda \in]0,1[$. Then ζ_λ satisfies (ζ_2) and (ζ_3) with $\zeta_\lambda(0,0) > 0$.

For completeness we give some of the most interesting examples of simulation functions.

Example 2.7. Let $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$, be defined by

(i) $\zeta(t,s) = \lambda\, s - t$ for all $t, s \in [0,+\infty[$, where $\lambda \in [0,1[$.

(ii) $\zeta(t,s)=\psi(s)-\varphi(t)$ for all $t,s\in[0,+\infty[$, where $\varphi,\psi:[0,+\infty[\to[0,+\infty[$ are two continuous functions such that $\psi(t)=\varphi(t)=0$ if and only if $t=0$ and $\psi(t)<t\le\varphi(t)$ for all $t>0$.

(iii) $\zeta(t,s)=s-\dfrac{f(t,s)}{g(t,s)}t$ for all $t,s\in[0,+\infty[$, where $f,g:[0,+\infty[\times[0,+\infty[\to]0,+\infty[$ are two continuous functions with respect to each variable such that $f(t,s)>g(t,s)$ for all $t,s>0$.

(iv) $\zeta(t,s)=s-\psi(s)-t$ for all $t,s\in[0,+\infty[$, where $\psi:[0,+\infty[\to[0,+\infty[$ is a lower semi-continuous function such that $\psi(t)=0$ if and only if $t=0$.

(v) $\zeta(t,s)=s\psi(s)-t$ for all $t,s\in[0,+\infty[$, where $\psi:[0,+\infty[\to[0,1[$ is such that $\lim_{t\to r^+}\psi(t)<1$ for all $r>0$.

3. Fixed points via simulation functions

We start this section with the following auxiliary lemma. Also, we use simulation function in the sense of Definition 2.5. Let $X\neq\emptyset$, $T:X\to X$, $x_0\in X$ and $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$. Then $\{x_n\}$ is called sequence of Picard of initial point at x_0.

Lemma 3.1. *Let (X,d) be a metric space and let $T:X\to X$ be a mapping. Suppose that there exist a simulation function ζ and a function $\varphi:X\to[0,+\infty[$ such that*

$$\zeta\left(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty),d(x,y)+\varphi(x)+\varphi(y)\right)\ge 0 \quad \text{for all } x,y\in X. \tag{3.1}$$

Let $\{x_n\}$ be a sequence of Picard of initial point at $x_0\in X$. Suppose that $x_{n-1}\neq x_n$ for all $n\in\mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

Proof. Let x_0 be an arbitrary point in X and let $\{x_n\}$ be a sequence of Picard of initial point at $x_0\in X$. We shall prove the lemma in three steps. The first step is to prove that

$$\lim_{n\to+\infty}d(x_{n-1},x_n)=0 \quad\text{and}\quad \lim_{n\to+\infty}\varphi(x_n)=0. \tag{3.2}$$

From, $x_{n-1}\neq x_n$ for all $n\in\mathbb{N}$, we deduce that

$$d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)>0 \quad \text{for all } n\in\mathbb{N}.$$

Using (3.1) and (ζ_2), with $x=x_{n-1}$ and $y=x_n$, we have

$$\begin{aligned} 0 &\le \zeta(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n))\\ &< d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)-[d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})]\end{aligned}$$

for all $n\in\mathbb{N}$. The above inequality shows that

$$d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})<d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), \quad \text{for all } n\in\mathbb{N},$$

which implies that $\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)\}$ is a decreasing sequence of positive real numbers. Thus, there is some $r\ge 0$ such that

$$\lim_{n\to+\infty}d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)=r. \tag{3.3}$$

Suppose that $r>0$. It follows from the condition (ζ_3), with $t_n=d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})$ and $s_n=d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)$, that

$$0\le\limsup_{n\to+\infty}\zeta\left(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}),d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)\right)<0,$$

which is a contradiction. Then we conclude that $r = 0$ and from (3.3), since $\varphi \geq 0$, we get

$$\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \varphi(x_n) = 0.$$

The second step is to prove that the sequence $\{x_n\}$ is bounded. Let us assume that $\{x_n\}$ is not a bounded sequence. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer greater than n_k such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1$$

and

$$d(x_m, x_{n_k}) \leq 1, \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

By the triangle inequality, we obtain

$$\begin{aligned} 1 < d(x_{n_{k+1}}, x_{n_k}) &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using (3.2), we get

$$\lim_{k\to+\infty} d(x_{n_{k+1}}, x_{n_k}) = 1. \tag{3.4}$$

From

$$1 < d(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})$$

and (3.2), we deduce that we can assume $d(x_{n_{k+1}-1}, x_{n_k-1}) > 0$ for all $k \in \mathbb{N}$.

Again, from (3.1) and (ζ_2), we deduce

$$\begin{aligned} 1 &< d(x_{n_{k+1}}, x_{n_k}) \\ &\leq d(x_{n_{k+1}}, x_{n_k}) + \varphi(x_{n_{k+1}}) + \varphi(x_{n_k}) \\ &< d(x_{n_{k+1}-1}, x_{n_k-1}) + \varphi(x_{n_{k+1}-1}) + \varphi(x_{n_k-1}) \\ &\leq d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1}) + \varphi(x_{n_{k+1}-1}) + \varphi(x_{n_k-1}) \\ &\leq 1 + d(x_{n_k}, x_{n_k-1}) + \varphi(x_{n_{k+1}-1}) + \varphi(x_{n_k-1}). \end{aligned}$$

Letting $k \to +\infty$ in the above inequality, using (3.2), we get

$$\begin{aligned} &\lim_{k\to+\infty} d(x_{n_{k+1}}, x_{n_k}) + \varphi(x_{n_{k+1}}) + \varphi(x_{n_k}) = 1, \\ &\lim_{k\to+\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) + \varphi(x_{n_{k+1}-1}) + \varphi(x_{n_k-1}) = 1. \end{aligned}$$

Then by condition (ζ_3), with $t_k = d(x_{n_{k+1}}, x_{n_k}) + \varphi(x_{n_{k+1}}) + \varphi(x_{n_k})$ and $s_k = d(x_{n_{k+1}-1}, x_{n_k-1}) + \varphi(x_{n_{k+1}-1}) + \varphi(x_{n_k-1})$, we obtain

$$0 \leq \limsup_{k\to+\infty} \zeta\left(d(x_{n_{k+1}}, x_{n_k}) + \varphi(x_{n_{k+1}}) + \varphi(x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1}) + \varphi(x_{n_{k+1}-1}) + \varphi(x_{n_k-1})\right) < 0,$$

which is a contradiction. Thus the sequence $\{x_n\}$ is bounded.

The third step is to prove that the sequence $\{x_n\}$ is Cauchy. To this end, let

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\}, \ n \in \mathbb{N}.$$

Since $\{x_n\}$ is bounded, we know that $C_n < +\infty$ for every $n \in \mathbb{N}$. Note that $\{C_n\}$ is a positive non-increasing sequence and hence there exists some $C \geq 0$ such that

$$\lim_{n\to+\infty} C_n = C. \tag{3.5}$$

Let us suppose that $C > 0$. By the definition of C_n, for every $k \in \mathbb{N}$, there exist $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k.$$

Letting $k \to +\infty$ in the above inequality, we get

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = C. \tag{3.6}$$

By (3.6), we can assume that $d(x_{m_k}, x_{n_k}) > 0$ for all $k \in \mathbb{N}$. From

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}),$$

(3.2) and (3.6), we deduce that we can assume $d(x_{m_k-1}, x_{n_k-1}) > 0$ for all $k \in \mathbb{N}$.

Again, from (3.1), (3.2), (ζ_2) and the definition of C_n, we deduce

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}) \\ &\leq d(x_{m_k-1}, x_{n_k-1}) + \varphi(x_{m_k-1}) + \varphi(x_{n_k-1}) \\ &\leq C_{k-1} + \varphi(x_{m_k-1}) + \varphi(x_{n_k-1}). \end{aligned}$$

Letting $k \to +\infty$ in the above inequality, using (3.2), (3.5) and (3.6), we obtain

$$\begin{aligned} &\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}) = C, \\ &\lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) + \varphi(x_{m_k-1}) + \varphi(x_{n_k-1}) = C. \end{aligned}$$

By condition (ζ_3), with $t_k = d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k})$ and $s_k = d(x_{m_k-1}, x_{n_k-1}) + \varphi(x_{m_k-1}) + \varphi(x_{n_k-1})$, we get

$$0 \leq \limsup_{k\to+\infty} \zeta(d(x_{m_k}, x_{n_k}) + \varphi(x_{m_k}) + \varphi(x_{n_k}), d(x_{m_k-1}, x_{n_k-1}) + \varphi(x_{m_k-1}) + \varphi(x_{n_k-1})) < 0,$$

which is a contradiction. Thus we have $C = 0$ and this ensures that $\{x_n\}$ is a Cauchy sequence. $\square$

Now, we present our first main result inspired to the papers [10, 26, 28].

Theorem 3.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist a simulation function ζ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that (3.1) holds, that is,*

$$\zeta\left(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)\right) \geq 0 \quad \text{for all } x, y \in X.$$

Then T has a unique fixed point z such that $\varphi(z) = 0$.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence of Picard with initial point at x_0. At first, observe that if $x_m = x_{m+1}$ for some $m \in \mathbb{N}$, then $x_m = x_{m+1} = Tx_m$, that is, x_m is a fixed point of T. In this case, the existence of a fixed point is proved. Therefore, we can suppose that $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$.

Now, by Lemma 3.1, the sequence $\{x_n\}$ is Cauchy and since (X, d) is complete, then there exists some $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z. \tag{3.7}$$

Note that first step of Lemma 3.1 and lower semi-continuity of the function φ give

$$0 \leq \varphi(z) \leq \liminf_{n\to+\infty} \varphi(x_n) = 0,$$

that is, $\varphi(z) = 0$.

We claim that z is a fixed point of T. If there exists a subsequence x_{n_k} of x_n such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$ for all $k \in \mathbb{N}$, then z is a fixed point for T. If this does not occur, then we can assume that $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \in \mathbb{N}$. Using (3.1) and (ζ_2) with $x = x_n$ and $y = z$, we deduce that

$$\begin{aligned} 0 &\leq \zeta(d(Tx_n, Tz) + \varphi(Tx_n) + \varphi(Tz), d(x_n, z) + \varphi(x_n) + \varphi(z)) \\ &< d(x_n, z) + \varphi(x_n) + \varphi(z) - [d(Tx_n, Tz) + \varphi(Tx_n) + \varphi(Tz)]. \end{aligned}$$

This implies

$$d(Tx_n, Tz) + \varphi(Tx_n) + \varphi(Tz) < d(x_n, z) + \varphi(x_n) + \varphi(z) \quad \text{for all } n \in \mathbb{N}$$

and consequently

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + d(Tx_n, Tz) + \varphi(Tx_n) + \varphi(Tz) \\ &< d(z, x_{n+1}) + d(x_n, z) + \varphi(x_n) + \varphi(z) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the above inequality, we obtain that $d(z, Tz) = 0$, that is, $z = Tz$.

Now, we establish uniqueness of the fixed point. Suppose that there exists $w \in X$ such that $w = Tw$ and $z \neq w$. Using (3.1) and (ζ_2) with $x = w$ and $y = z$, we get that

$$0 \leq \zeta(d(fw, fz) + \varphi(w) + \varphi(z), d(w, z) + \varphi(w) + \varphi(z)) < d(w, z) - d(w, z) = 0,$$

which ia a contradiction and hence $w = z$. This ends the proof of Theorem 3.2. □

4. Consequences

We show the unifying power of simulation functions by applying Theorem 3.2 to deduce different kinds of contractive conditions in the existing literature.

The following corollary is a result of Banach type [2].

Corollary 4.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist $\lambda \in]0, 1[$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that*

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq \lambda\, d(x, y) + \varphi(x) + \varphi(y) \quad \textit{for all } x, y \in X.$$

Then T has a unique fixed point.

Proof. The result follows from Theorem 3.2, by taking as simulation function

$$\zeta(t, s) = \lambda\, s - t,$$

for all $t, s \geq 0$. □

Note that we obtain Banach contraction principle if we assume $\varphi(x) = 0$ for all $x \in X$.

The following corollary is a result of Rhoades type [22].

Corollary 4.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist two lower semi-continuous functions $\psi : [0, +\infty[\to [0, +\infty[$ with $\psi^{-1}(0) = \{0\}$ and $\varphi : X \to [0, +\infty[$ such that*

$$d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \leq d(x, y) + \varphi(x) + \varphi(y) - \psi(d(x, y) + \varphi(x) + \varphi(y)) \quad \textit{for all } x, y \in X.$$

Then T has a unique fixed point.

Proof. The result follows from Theorem 3.2, by taking as simulation function

$$\zeta(t,s) = s - \psi(s) - t,$$

for all $t, s \geq 0$. □

We have also the following corollary, see [19].

Corollary 4.3. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist a function $\psi : [0,+\infty[\to [0,1[$ with $\limsup_{t\to r^+} \psi(t) < 1$ for all $r > 0$ and a lower semi-continuous function $\varphi : X \to [0,+\infty[$ such that*

$$d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \leq \psi(d(x,y) + \varphi(x) + \varphi(y))\,[d(x,y) + \varphi(x) + \varphi(y)] \quad \text{for all } x,y \in X.$$

Then T has a unique fixed point.

Proof. The result follows from Theorem 3.2, by taking as simulation function

$$\zeta(t,s) = s\psi(s) - t,$$

for all $t, s \geq 0$. □

The following corollary is a result of Boyd-Wong type [4].

Corollary 4.4. *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose that there exist an upper semi-continuous function $\eta : [0,+\infty[\to [0,+\infty[$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ and a lower semi-continuous function $\varphi : X \to [0,+\infty[$ such that*

$$d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq \eta(d(x,y) + \varphi(x) + \varphi(y))) \quad \text{for all } x,y \in X.$$

Then T has a unique fixed point.

Proof. The result follows from Theorem 3.2, by taking as simulation function

$$\zeta(t,s) = \eta(s) - t,$$

for all $t, s \geq 0$. □

Note that we obtain Boyd-Wong result if we assume $\varphi(x) = 0$ for all $x \in X$.

Next example shows that Theorem 3.2 is a proper generalization of both Banach contraction principle and Boyd-Wong result in the setting of metric spaces.

Example 4.5 ([28], Example 4)**.** Let $X = [0,1]$ endowed with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Obviously (X,d) is a complete metric space. Also, fix $r \in [0,1[$, and define $T : X \to X$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \frac{r}{2n} - r\frac{2n-1}{2n}(2nx-1) & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{2n-1}, \\ \frac{r}{2n} + r\frac{2n+1}{2n}(2nx-1) & \text{if } \frac{1}{2n+1} \leq x \leq \frac{1}{2n}. \end{cases}$$

Firstly, we prove that T is not a contraction. In fact, if for odd $n > 1$ we choose $x = \frac{1}{2n-1}$ and $y = \frac{1}{n-1}$, we have

$$d(Tx,Ty) = \frac{r}{n-1} \leq d(x,y) = \frac{n}{(n-1)(2n-1)} \leq \frac{3}{5(n-1)},$$

which is not satisfied for $r > 3/5$. Therefore, both the Banach contraction principle and Boyd-Wong result cannot be applied.

On the other hand, if we consider the function $\phi : X \to [0, +\infty[$ defined by $\phi(x) = x$, then we obtain

$$\begin{aligned} d(Tx, Ty) + \phi(Tx) + \phi(Ty) &= 2\max\{Tx, Ty\} \\ &\le 2\max\{rx, ry\} \\ &= r2\max\{x, y\} \\ &= r[d(x, y) + \phi(x) + \phi(y)], \end{aligned}$$

for all $x, y \in X$. Thus, all the conditions of Corollary 4.1 are satisfied. Therefore, T has a unique fixed point in X.

5. Fixed points in partial metric spaces

In this section, from our Theorem 3.2, we deduce easily various fixed point theorems on partial metric spaces including the Matthews fixed point theorem.

Theorem 5.1. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping. Suppose that there exists a simulation function ζ such that*

$$\zeta\left(p(Tx, Ty), p(x, y)\right) \ge 0 \quad \textit{for all } x, y \in X. \tag{5.1}$$

Then T has a unique fixed point $z \in X$ such that $p(z, z) = 0$.

Proof. From (2.1), we deduce that

$$p(x, y) = \frac{p^s(x, y) + p(x, x) + p(y, y)}{2} \qquad \text{for all } x, y \in X. \tag{5.2}$$

By Lemma 2.2, the metric space $(X, 2^{-1}p^s)$ is complete since (X, p) is complete. Also, by Lemma 2.1, the function $\varphi : X \to [0, +\infty[$ defined by $\varphi(x) = 2^{-1}p(x, x)$ is lower semi-continuous in $(X, 2^{-1}p^s)$. Now, from (5.1) and (5.2), we obtain that the mapping T satisfies the following contractive condition

$$\zeta\left(2^{-1}p^s(Tx, Ty) + \varphi(Tx) + \varphi(Ty), 2^{-1}p^s(x, y) + \varphi(x) + \varphi(y)\right) \ge 0 \quad \text{for all } x, y \in X.$$

Thus the mapping T satisfies all the condition of Theorem 3.2 with respect to the metric space $(X, 2^{-1}p^s)$, and hence has a unique fixed point $z \in X$ such that $p(z, z) = 2\,\varphi(z) = 0$. □

Now, the Matthews fixed point theorem follows immediately from Theorem 5.1 by chosing as simulation function

$$\zeta(t, s) = \lambda s - t \qquad \text{for all } t, s \in [0, +\infty[$$

with $\lambda \in [0, 1[$.

Corollary 5.2. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a mapping. Suppose that there exists a $\lambda \in [0, 1[$ such that*

$$p(Tx, Ty) \le \lambda\, p(x, y) \quad \textit{for all } x, y \in X. \tag{5.3}$$

Then T has a unique fixed point $z \in X$ such that $p(z, z) = 0$.

If we choose as simulation function

$$\zeta(t, s) = \eta(s) - t \qquad \text{for all } t, s \in [0, +\infty[$$

where $\eta : [0, +\infty[\to [0, +\infty[$ is an upper semi-continuous function with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, then we obtain the following result of Boyd-Wong type in the setting of partial metric space.

Corollary 5.3. *Let (X,p) be a complete partial metric space and let $T : X \to X$ be a mapping. Suppose that there exists an upper semi-continuous function $\eta : [0,+\infty[\to [0,+\infty[$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ such that*

$$d(Tx,Ty) \leq \eta(d(x,y) \quad \text{for all } x,y \in X.$$

Then T has a unique fixed point.

Appropriately choosing the simulation function we can obtain other known results of fixed point in partial metric spaces.

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