Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 705–716
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions

Feixiang Chen[a], Shanhe Wu[b,*]

[a]*Key Laboratory for Nonlinear Science and System Structure, Chongqing Three Georges University, Wanzhou, 404100, P. R. China.*
[b]*Department of Mathematics and Computer Science, Longyan University, Longyan, Fujian, 364012, P. R. China.*

Communicated by Yonghong Yao

Abstract

In this paper, we establish some new Hermite-Hadamard inequalities for s-convex functions via fractional integrals. Some Hermite-Hadamard type inequalities for products of two convex and s-convex functions via Riemann-Liouville integrals are also established.

Keywords: Hermite-Hadamard inequality, s-convex function, Riemann-Liouville fractional integrals.
2010 MSC: 26D15, 26A51.

1. Introduction and preliminaries

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Definition 1.1 ([7])**.** $f : I \subset [0,\infty) \to [0,\infty)$ is said to be s-convex in the second sense, or that f belongs to the class K_s^2, if the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y) \tag{1.2}$$

holds for all $x, y \in I$, $\alpha \in [0,1]$ and for some fixed $s \in (0,1]$.

*Corresponding author
Email addresses: `cfx2002@126.com` (Feixiang Chen), `shanhewu@gmail.com` (Shanhe Wu)

Received 2015-08-10

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

In [6], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.2 ([6]). *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L[a, b]$, then the following inequality holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(t)dt \le \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

In [8], İşcan gave definition of harmonically convexity as follows:

Definition 1.3 ([8]). Let $I \subseteq \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\Big(\frac{xy}{tx+(1-t)y}\Big) \le tf(y) + (1-t)f(x) \tag{1.4}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.4) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

Theorem 1.4 ([8]). *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L(a, b)$ then we have*

$$f\Big(\frac{2ab}{a+b}\Big) \le \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \le \frac{f(a)+f(b)}{2}. \tag{1.5}$$

The definition of harmonically s-convex functions is proposed by İşcan in [9].

Definition 1.5 ([9]). Let $I \subseteq \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be harmonically s-convex, if

$$f\Big(\frac{xy}{tx+(1-t)y}\Big) \le t^s f(y) + (1-t)^s f(x) \tag{1.6}$$

for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. If the inequality in (1.6) is reversed, then f is said to be harmonically s-concave.

The following Hermite-Hadamard inequality for harmonically s-convex functions holds.

Theorem 1.6 ([9]). *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a harmonically s-convex function and $a, b \in I$ with $a < b$. If $f \in L(a, b)$ then we have*

$$2^{s-1} f\Big(\frac{2ab}{a+b}\Big) \le \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \le \frac{f(a)+f(b)}{s+1}. \tag{1.7}$$

In [4], Chen and Wu discussed Fejér and Hermite-Hadamard type inequalities for Harmonically convex functions and presented the following inequality:

Theorem 1.7 ([4]). *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L(a, b)$ then we have*

$$f\Big(\frac{2ab}{a+b}\Big) \int_a^b \frac{p(x)}{x^2} dx \le \int_a^b \frac{f(x)}{x^2} p(x) dx \le \frac{f(a)+f(b)}{2} \int_a^b \frac{p(x)}{x^2} dx, \tag{1.8}$$

where $p : [a, b] \to \mathbb{R}$ is nonnegative, integrable and satisfies

$$p\Big(\frac{ab}{x}\Big) = p\Big(\frac{ab}{a+b-x}\Big). \tag{1.9}$$

Some refinements of the Hermite-Hadamard inequality for convex functions have been extensively investigated by a number of authors (e.g., [1], [2], [3], [4] and [5]).

In [11], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

Theorem 1.8. *Let f and g be real-valued, nonnegative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{1.10}$$

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b), \tag{1.11}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Some Hermite-Hadamard type inequalities for products of two convex and s-convex functions are proposed by Kirmaci *et al.* in [10].

Theorem 1.9 ([10])**.** *Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0,\infty)$, $a < b$, be functions such that g, $fg \in L[a, b]$. If f is convex and nonnegative on $[a, b]$ and g is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{s+2}M(a,b) + \frac{1}{(s+1)(s+2)}N(a,b), \tag{1.12}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.10 ([10])**.** *Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0,\infty)$, $a < b$, be functions such that f, g and $fg \in L[a, b]$. If f is s_1-convex and g is s_2-convex on $[a, b]$ for some fixed $s_1, s_2 \in (0, 1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{s_1+s_2+1}M(a,b) + \beta(s_2+1, s_1+1)N(a,b), \tag{1.13}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.11 ([10])**.** *Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0,\infty)$, $a < b$, be functions such that $fg \in L[a, b]$. If f is convex and nonnegative on $[a, b]$ and g is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then*

$$2^s f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{(s+1)(s+2)}M(a,b) + \frac{1}{s+2}N(a,b), \tag{1.14}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(a)g(b)$.

In [5], Chen and Wu discussed Hermite-Hadamard type inequalities for Harmonically s-convex functions and obtained the following result:

Theorem 1.12 ([5])**.** *Let $f, g : [a, b] \to [0,\infty)$, $a, b \in (0,\infty)$, $a < b$, be functions such that f, g, $fg \in L[a, b]$. If f is harmonically s_1-convex and g is harmonically s_2-convex on $[a, b]$ for some fixed $s_1, s_2 \in (0, 1]$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx \le \frac{1}{1+s_1+s_2}M(a,b) + \frac{\Gamma(1+s_1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}N(a,b),$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.13 ([5])**.** *Let $f, g : [a, b] \to [0,\infty)$, $a, b \in (0,\infty)$, $a < b$, be functions such that f, g, $fg \in L[a, b]$. If f is harmonically s_1-convex and g is harmonically s_2-convex on $[a, b]$ for some fixed $s_1, s_2 \in (0, 1]$, then*

$$2^{s_1+s_2-1}f\Big(\frac{2ab}{a+b}\Big)g\Big(\frac{2ab}{a+b}\Big) \le \frac{ab}{b-a}\int_a^b \frac{f(x)g(x)}{x^2}dx + M(a,b)\frac{\Gamma(1+s_1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} + \frac{1}{s_2+s_1+1}N(a,b),$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(a)g(b)$.

Sarikaya *et al.* [12] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.14 ([12])**.** *Let* $f : [a,b] \to R$ *be a positive function with* $a < b$ *and* $f \in L[a,b]$. *If* f *is a convex function on* $[a,b]$, *then the following inequalities for fractional integrals hold:*

$$f(\frac{a+b}{2}) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)] \le \frac{f(a)+f(b)}{2}, \tag{1.15}$$

with $\alpha > 0$.

We remark that the symbol $J^\alpha_{a^+}$ and $J^\alpha_{b^-}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \ge 0$ with $a \ge 0$ which are defined by

$$J^\alpha_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x > a,$$

and

$$J^\alpha_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

Chen and Wu [3] investigated the Hermite-Hadamard type inequalities for products of two h-convex functions and established the following inequality:

Theorem 1.15. *Let* $f \in SX(h_1, I)$, $g \in SX(h_2, I)$, $a, b \in I$, $a < b$, *be functions such that* $fg \in L[a,b]$, *and* $h_1h_2 \in L_1[0,1]$, *then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b) + J^\alpha_{b^-}f(a)g(a)] \le M(a,b)\int_0^1 t^{\alpha-1}[h_1(t)h_2(t) + h_1(1-t)h_2(1-t)]dt + N(a,b)\int_0^1 t^{\alpha-1}[h_1(t)h_2(1-t) + h_1(1-t)h_2(t)]dt,$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.16. *Let* $f \in SX(h_1, I)$, $g \in SX(h_2, I)$, $a, b \in I$, $a < b$, *be functions such that* $fg \in L_1[a,b]$, *and* $h_1h_2 \in L[0,1]$, *then the following inequality for fractional integrals holds:*

$$\frac{1}{\alpha h_1(\frac{1}{2})h_2(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b) + J^\alpha_{b^-}f(a)g(a)] + M(a,b)\int_0^1 t^{\alpha-1}[h_1(t)h_2(1-t) + h_1(1-t)h_2(t)]dt + N(a,b)\int_0^1 t^{\alpha-1}[h_1(t)h_2(t) + h_1(1-t)h_2(1-t)]dt,$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

In [13], Set *et al.* established the following Hermite-Hadamard inequalities for s-convex functions in the second sense via fractional integrals.

Theorem 1.17 ([13]). *Let $f : [a, b] \to R$ be a positive function with $0 \le a < b$ and $f \in L[a, b]$. If f is s-convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequalities for fractional integrals with $\alpha > 0$ hold:*

$$2^{s-1} f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] \le \Big[\frac{1}{\alpha+s} + \beta(\alpha, s+1)\Big]\frac{f(a)+f(b)}{2}, \tag{1.16}$$

where β is Euler Beta function.

In this paper, we obtain some new Hermite-Hadamard type inequalities for s-convex functions via Riemann-Liouville fractional integrals. Several Hermite-Hadamard type inequalities for products of two convex and s-convex functions are also established.

2. Main results

In order to prove our main theorems, we need the following fundamental integral identity including the second order derivatives of a given function via Riemann-Liouville integrals which was established by Wang *et al.* in [14].

Lemma 2.1 ([14]). *Let $f : [a, b] \to R$ be a twice differentiable mapping on (a, b) with $a < b$. If $f'' \in L[a, b]$, then the following equality for fractional integrals hold:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] - \frac{f(a)+f(b)}{2} = \frac{(b-a)^2}{2}\int_0^1 \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1} f''(ta + (1-t)b)dt. \tag{2.1}$$

Theorem 2.2. *Let $f : [a, b] \to R$ be a twice differentiable mapping on (a, b) with $a < b$ such that $f'' \in L[a, b]$. If $|f''|$ is s-convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] - \frac{f(a)+f(b)}{2}\Big| \\ &\qquad \le \frac{(b-a)^2}{2}\Big(\frac{1}{s+1} - \frac{1}{\alpha+s+2} - \beta(\alpha+2, s+1)\Big)\frac{|f''(a)| + |f''(b)|}{\alpha+1}.\end{aligned}$$

Proof. From lemma 2.1, we get

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] - \frac{f(a)+f(b)}{2}\Big| \\ &\qquad \le \frac{(b-a)^2}{2}\int_0^1 \Big|\frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1}\Big| |f''(ta + (1-t)b)|dt.\end{aligned} \tag{2.2}$$

Because $(1-t)^{\alpha+1} + t^{\alpha+1} \le 1$ for any $t \in [0, 1]$ and $|f''|$ is s-convex on $[a, b]$, we get

$$\begin{aligned}&\int_0^1 \Big|\frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1}\Big| |f''(ta + (1-t)b)|dt \\ &\qquad \le \int_0^1 \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}\Big(t^s|f''(a)| + (1-t)^s|f''(b)|\Big)dt \\ &\qquad = \Big(\frac{1}{s+1} - \frac{1}{\alpha+s+2} - \beta(\alpha+2, s+1)\Big)\frac{|f''(a)| + |f''(b)|}{\alpha+1}.\end{aligned} \tag{2.3}$$

Now by (2.2), (2.3), we can obtain the desired result. □

Theorem 2.3. *Let $f : [a, b] \to R$ be a twice differentiable mapping on (a, b) with $a < b$ such that $f'' \in L[a, b]$. If $|f''|^q$ is s-convex on $[a, b]$ with $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)] - \frac{f(a)+f(b)}{2}\Big| \\ &\qquad \le \frac{(b-a)^2}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}} \times \Big(\frac{1}{s+1} - \beta(\alpha+2, s+1) - \frac{1}{\alpha+s+2}\Big)^{\frac{1}{q}}\Big(|f''(a|^q + |f''(b)|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Proof. From lemma 2.1 and the power mean inequality for q, we have

$$\begin{aligned}
&\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]-\frac{f(a)+f(b)}{2}\Big|\\
&\quad\le\frac{(b-a)^2}{2(\alpha+1)}\Big[\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})dt\Big]^{1-\frac{1}{q}}\\
&\qquad\times\Big(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})|f''(ta+(1-t)b)|^q dt\Big)^{\frac{1}{q}}\\
&\quad\le\frac{(b-a)^2}{2(\alpha+1)}\Big[\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})dt\Big]^{1-\frac{1}{q}}\\
&\qquad\times\Big(\int_0^1(1-(1-t)^{\alpha+1}-t^{\alpha+1})(t^s|f''(a|^q+(1-t)^s|f''(b)|^q)dt\Big)^{\frac{1}{q}}\\
&\quad=\frac{(b-a)^2}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{s+1}-\beta(\alpha+2,s+1)-\frac{1}{\alpha+s+2}\Big)^{\frac{1}{q}}\Big(|f''(a|^q+|f''(b)|^q)\Big)^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 2.4. *Let $f:[a,b]\to R$ be a twice differentiable mapping on (a,b) with $a<b$ such that $f''\in L[a,b]$. If $|f''|^q$ is s-convex on $[a,b]$ with $q>1$, then the following inequality holds:*

$$\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]-\frac{f(a)+f(b)}{2}\Big|\le\frac{(b-a)^2}{2(\alpha+1)}\Big(1-\frac{2}{p(\alpha+1)+1}\Big)^{\frac{1}{p}}\Big(\frac{|f''(a|^q+|f''(b)|^q}{s+1}\Big)^{\frac{1}{q}},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Proof. From lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned}
&\Big|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]-\frac{f(a)+f(b)}{2}\Big|\\
&\quad\le\frac{(b-a)^2}{2}\int_0^1\Big(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\Big)|f''(ta+(1-t)b)|dt\\
&\quad\le\frac{(b-a)^2}{2(\alpha+1)}\Big(\int_0^1\Big(1-(1-t)^{\alpha+1}-t^{\alpha+1}\Big)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1|f''(ta+(1-t)b)|^q dt\Big)^{\frac{1}{q}} \qquad (2.4)\\
&\quad\le\frac{(b-a)^2}{2(\alpha+1)}\Big(\int_0^1\Big(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}\Big)dt\Big)^{\frac{1}{p}}\Big(\int_0^1|f''(ta+(1-t)b)|^q dt\Big)^{\frac{1}{q}}\\
&\quad=\frac{(b-a)^2}{2(\alpha+1)}\Big(1-\frac{2}{p(\alpha+1)+1}\Big)^{\frac{1}{p}}\Big(\int_0^1|f''(ta+(1-t)b)|^q dt\Big)^{\frac{1}{q}}.
\end{aligned}$$

Because $|f''|^q$ is s-convex, we have

$$\int_0^1|f''(ta+(1-t)b)|^q dt\le\frac{|f''(a|^q+|f''(b)|^q}{s+1}. \qquad (2.5)$$

Here we use

$$(1-(1-t)^{\alpha+1}-t^{\alpha+1})^p\le1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}$$

for any $t\in[0,1]$, which follows from

$$(A-B)^p\le A^p-B^p$$

for any $A>B\ge0$ and $p\ge1$. From (2.4) and (2.5), we complete the proof. □

We note that the Beta and Gamma functions are defined, respectively, as follows

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0,$$
$$\beta(x,y) = \int_0^1 (1-t)^{y-1}t^{x-1}dt, \quad x > 0, \quad y > 0,$$

and

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.5. *Let $f, g : [a,b] \to \mathbb{R}$, $a, b \in [0,\infty)$, $a < b$, be functions such that g, $fg \in L[a,b]$. If f is convex and nonnegative and g is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a+}f(b)g(b) + J^\alpha_{b-}f(a)g(a)]\\ &\quad\le \Big(\frac{1}{\alpha+s+1} + \beta(\alpha, s+2)\Big)M(a,b) + \Big(\beta(\alpha+1, s+1) + \frac{1}{(\alpha+s)(\alpha+s+1)}\Big)N(a,b),\end{aligned}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f is convex and g is s-convex on $[a,b]$, then for $t \in [0,1]$ we get

$$f(ta + (1-t)b) \le tf(a) + (1-t)f(b), \tag{2.6}$$

and

$$g(ta + (1-t)b) \le t^s g(a) + (1-t)^s g(b). \tag{2.7}$$

From (2.6) and (2.7), we get

$$f(ta+(1-t)b)g(ta+(1-t)b) \le t^{s+1}f(a)g(a) + (1-t)^{s+1}f(b)g(b) + t(1-t)^s f(a)g(b) + (1-t)t^s f(b)g(a).$$

Similarly, we have

$$f((1-t)a+tb)g((1-t)a+tb) \le (1-t)^{s+1}f(a)g(a) + t^{s+1}f(b)g(b) + (1-t)t^s f(a)g(b) + t(1-t)^s f(b)g(a).$$

So

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb)\\ &\quad\le (t^{s+1} + (1-t)^{s+1})[f(a)g(a) + f(b)g(b)]\\ &\qquad + (t(1-t)^s + (1-t)t^s)[f(a)g(b) + f(b)g(a)].\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)dt\\ &\quad= \int_b^a \Big(\frac{b-u}{b-a}\Big)^{\alpha-1} f(u)g(u)\frac{du}{a-b} + \int_a^b \Big(\frac{v-a}{b-a}\Big)^{\alpha-1} f(v)g(v)\frac{dv}{b-a}\\ &\quad= \frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a+}f(b)g(b) + J^\alpha_{b-}f(a)g(a)]\\ &\quad\le [f(a)g(a) + f(b)g(b)]\int_0^1 t^{\alpha-1}(t^{s+1} + (1-t)^{s+1})dt\end{aligned}$$

$$
\begin{aligned}
&+[f(a)g(b)+f(b)g(a)]\int_0^1 t^{\alpha-1}(t(1-t)^s+(1-t)t^s)dt\\
&=\Big(\frac{1}{\alpha+s+1}+\beta(\alpha,s+2)\Big)[f(a)g(a)+f(b)g(b)]\\
&+\Big(\beta(\alpha+1,s+1)+\frac{1}{(\alpha+s)(\alpha+s+1)}\Big)[f(a)g(b)+f(b)g(a)]\\
&=\Big(\frac{1}{\alpha+s+1}+\beta(\alpha,s+2)\Big)M(a,b)+\Big(\beta(\alpha+1,s+1)+\frac{1}{(\alpha+s)(\alpha+s+1)}\Big)N(a,b).
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)]\\
&\quad\le\Big(\frac{1}{\alpha+s+1}+\beta(\alpha,s+2)\Big)M(a,b)+\Big(\beta(\alpha+1,s+1)+\frac{1}{(\alpha+s)(\alpha+s+1)}\Big)N(a,b),
\end{aligned}
$$

which completes the proof. □

Remark 2.6. Taking $\alpha=1$ in Theorem 2.5, we have

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)dx&\le\Big(\frac{1}{s+2}+\beta(1,s+2)\Big)\frac{M(a,b)}{2}+\Big(\beta(2,s+1)+\frac{1}{(1+s)(s+2)}\Big)\frac{N(a,b)}{2}\\
&=\frac{1}{s+2}M(a,b)+\frac{1}{(1+s)(s+2)}N(a,b)
\end{aligned}
$$

which is the result of (1.12).

Remark 2.7. Choosing $f(x)=1$ for all $x\in[a,b]$ in Theorem 2.5 gives

$$
\begin{aligned}
\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a^+}g(b)+J^\alpha_{b^-}g(a)]&\le\Big(\frac{1}{\alpha+s+1}+\beta(\alpha,s+2)+\beta(\alpha+1,s+1)\\
&\quad+\frac{1}{(\alpha+s)(\alpha+s+1)}\Big)[g(a)+g(b)]\\
&=\Big(\frac{1}{\alpha+s}+\beta(\alpha,s+1)\Big)[g(a)+g(b)],
\end{aligned}
$$

which is the right hand side of (1.16).

Theorem 2.8. *Let $f,g:[a,b]\to\mathbb{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that f, g, $fg\in L[a,b]$. If f is s_1-convex and g is s_2-convex on $[a,b]$ for some fixed $s_1,s_2\in(0,1]$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)]&\le\Big(\frac{1}{\alpha+s_1+s_2}+\beta(\alpha,s_1+s_2+1)\Big)M(a,b)\\
&\quad+\Big(\beta(\alpha+s_1,s_2+1)+\beta(\alpha+s_2,s_1+1)\Big)N(a,b),
\end{aligned}
$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since f is s_1-convex and g is s_2-convex on $[a,b]$, then for $t\in[0,1]$ we get

$$f(ta+(1-t)b)\le t^{s_1}f(a)+(1-t)^{s_1}f(b), \tag{2.8}$$

and

$$g(ta+(1-t)b)\le t^{s_2}g(a)+(1-t)^{s_2}g(b). \tag{2.9}$$

From (2.8) and (2.9), we get

$$f(ta+(1-t)b)g(ta+(1-t)b) \leq t^{s_1+s_2}f(a)g(a)+(1-t)^{s_1+s_2}f(b)g(b)+t^{s_1}(1-t)^{s_2}f(a)g(b)+(1-t)^{s_1}t^{s_2}f(b)g(a).$$

Similarly, we have

$$f((1-t)a+tb)g((1-t)a+tb) \leq (1-t)^{s_1+s_2}f(a)g(a)+t^{s_1+s_2}f(b)g(b)+(1-t)^{s_1}t^{s_2}f(a)g(b)+t^{s_1}(1-t)^{s_2}f(b)g(a).$$

So

$$\begin{aligned}
&f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\quad\leq (t^{s_1+s_2}+(1-t)^{s_1+s_2})[f(a)g(a)+f(b)g(b)]\\
&\qquad+(t^{s_1}(1-t)^{s_2}+(1-t)^{s_1}t^{s_2})[f(a)g(b)+f(b)g(a)].
\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)dt+\int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)dt\\
&=\int_b^a\Big(\frac{b-u}{b-a}\Big)^{\alpha-1}f(u)g(u)\frac{du}{a-b}+\int_a^b\Big(\frac{v-a}{b-a}\Big)^{\alpha-1}f(v)g(v)\frac{dv}{b-a}\\
&=\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J_{a+}^\alpha f(b)g(b)+J_{b-}^\alpha f(a)g(a)]\\
&\leq[f(a)g(a)+f(b)g(b)]\int_0^1 t^{\alpha-1}(t^{s_1+s_2}+(1-t)^{s_1+s_2})dt\\
&\quad+[f(a)g(b)+f(b)g(a)]\int_0^1 t^{\alpha-1}(t^{s_1}(1-t)^{s_2}+(1-t)^{s_1}t^{s_2})dt\\
&=\Big(\frac{1}{\alpha+s_1+s_2}+\beta(\alpha,s_1+s_2+1)\Big)[f(a)g(a)+f(b)g(b)]\\
&\quad+\Big(\beta(\alpha+s_1,s_2+1)+\beta(\alpha+s_2,s_1+1)\Big)[f(a)g(b)+f(b)g(a)]\\
&=\Big(\frac{1}{\alpha+s_1+s_2}+\beta(\alpha,s_1+s_2+1)\Big)M(a,b)+\Big(\beta(\alpha+s_1,s_2+1)+\beta(\alpha+s_2,s_1+1)\Big)N(a,b).
\end{aligned}$$

So

$$\begin{aligned}
\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J_{a+}^\alpha f(b)g(b)+J_{b-}^\alpha f(a)g(a)] &\leq \Big(\frac{1}{\alpha+s_1+s_2}+\beta(\alpha,s_1+s_2+1)\Big)M(a,b)\\
&\quad+\Big(\beta(\alpha+s_1,s_2+1)+\beta(\alpha+s_2,s_1+1)\Big)N(a,b),
\end{aligned}$$

which completes the proof. □

Remark 2.9. Putting $\alpha=1$ in Theorem 2.8 leads to

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \Big(\frac{1}{s_1+s_2+1}+\beta(1,s_1+s_2+1)\Big)\frac{M(a,b)}{2}\\
&\quad+\Big(\beta(1+s_1,s_2+1)+\beta(1+s_2,s_1+1)\Big)\frac{N(a,b)}{2}\\
&=\frac{1}{s_1+s_2+1}M(a,b)+\beta(s_2+1,s_1+1)N(a,b),
\end{aligned}$$

which is the result of (1.13).

Theorem 2.10. *Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0, \infty)$, $a < b$, be functions such that $fg \in L[a, b]$. If f is convex and nonnegative on $[a, b]$ and g is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then*

$$\begin{aligned}2^s f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \le{}& \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+} f(b)g(b) + J^\alpha_{b^-} f(a)g(a)] \\ &+ \frac{1}{2} M(a, b)\Big(\beta(\alpha+1, s+1) + \frac{1}{(\alpha+s)(\alpha+s+1)}\Big) \\ &+ \frac{1}{2} N(a, b)\Big(\beta(\alpha, s+2) + \frac{1}{\alpha+s+1}\Big),\end{aligned}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(a)g(b)$.

Proof. We can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2},$$

so

$$\begin{aligned}&f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) \\ &= f\Big(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\Big) g\Big(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\Big) \\ &\le \frac{1}{2^{s+1}}\Big[f(ta+(1-t)b) + f((1-t)a+tb)\Big]\Big[g(ta+(1-t)b) + g((1-t)a+tb)\Big] \\ &= \frac{1}{2^{s+1}}\Big[f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb) \\ &\quad + f(ta+(1-t)b)g((1-t)a+tb) + f((1-t)a+tb)g(ta+(1-t)b)\Big] \\ &\le \frac{1}{2^{s+1}}\Big[f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb)\Big] \\ &\quad + \frac{1}{2^{s+1}}\Big\{\Big[tf(a) + (1-t)f(b)\Big]\Big[(1-t)^s g(a) + t^s g(b)\Big] \\ &\quad + \Big[(1-t)f(a) + tf(b)\Big]\Big[t^s g(a) + (1-t)^s g(b)\Big]\Big\} \\ &= \frac{1}{2^{s+1}}\Big[f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb)\Big] \\ &\quad + \frac{1}{2^{s+1}}\Big\{\Big[t(1-t)^s + (1-t)t^s\Big]M(a, b) + \Big[(1-t)^{s+1} + t^{s+1}\Big]N(a, b)\Big\}.\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we get

$$\begin{aligned}f\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big)\int_0^1 t^{\alpha-1} dt \le{}& \frac{1}{2^{s+1}}\Big[\int_0^1 t^{\alpha-1} f(ta+(1-t)b)g(ta+(1-t)b)dt \\ &+ \int_0^1 t^{\alpha-1} f((1-t)a+tb)g((1-t)a+tb)dt\Big] \\ &+ \frac{1}{2^{s+1}}\Big\{M(a, b)\int_0^1 t^{\alpha-1}[t(1-t)^s + (1-t)t^s]dt \\ &+ N(a, b)\int_0^1 t^{\alpha-1}[(1-t)^{s+1} + t^{s+1}]dt\Big\}.\end{aligned}$$

That is

$$\begin{aligned}\frac{1}{\alpha}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{1}{2^{s+1}}\Big[\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)]\Big] \\ +\frac{1}{2^{s+1}}\Big\{M(a,b)\int_0^1 t^{\alpha-1}[t(1-t)^s+(1-t)t^s]dt \\ +N(a,b)\int_0^1 t^{\alpha-1}[(1-t)^{s+1}+t^{s+1}]dt\Big\}.\end{aligned}$$

From

$$\int_0^1 t^{\alpha-1}[t(1-t)^s+(1-t)t^s]dt=\beta(\alpha+1,s+1)+\frac{1}{(\alpha+s)(\alpha+s+1)}$$

and

$$\int_0^1 t^{\alpha-1}[(1-t)^{s+1}+t^{s+1}]dt=\beta(\alpha,s+2)+\frac{1}{\alpha+s+1},$$

we get

$$\begin{aligned}2^s f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)] \\ +\frac{1}{2}M(a,b)\Big(\beta(\alpha+1,s+1)+\frac{1}{(\alpha+s)(\alpha+s+1)}\Big) \\ +\frac{1}{2}N(a,b)\Big(\beta(\alpha,s+2)+\frac{1}{\alpha+s+1}\Big),\end{aligned}$$

which completes the proof. □

Remark 2.11. Setting $\alpha=1$ in Theorem 2.10, then

$$\begin{aligned}2^s f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) &\le \frac{1}{b-a}\int_a^b f(x)g(x)dx \\ &+\frac{1}{2}M(a,b)\Big(\beta(2,s+1)+\frac{1}{(s+1)(s+2)}\Big) \\ &+\frac{1}{2}N(a,b)\Big(\beta(1,s+2)+\frac{1}{s+2}\Big) \\ &=\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b),\end{aligned}$$

which is the result of (1.14).

Remark 2.12. Choosing $f(x)=1$ for all $x\in[a,b]$ in Theorem 2.10, we have

$$2^s g\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}g(b)+J^\alpha_{b^-}g(a)]+\Big(\beta(\alpha,s+1)+\frac{1}{\alpha+s}\Big)\frac{g(a)+g(b)}{2}.$$

Acknowledgements:

The authors would like to thank the reviewers and the editors for their valuable suggestions and comments.

The present investigation was supported, in part, by the Natural Science Foundation of Chongqing Municipal Education Commission (No.KJ1501009), and, in part, by the Foundation of Scientific Research Project of Fujian Province Education Department of China (No.JK2013051).

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