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# Common fixed point theorems for four mappings on cone $b$-metric spaces over Banach algebras

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**Abstract**

The purpose of this paper is to obtain several common fixed point theorems for four mappings in the setting of cone $b$-metric spaces over Banach algebras. The obtained results generalize, complement, and improve some results in the literature. Moreover, we give some supportive examples for our conclusions. In addition, an application in the solution of a class of equations is given to illustrate the superiority of the main results. 

*Keywords:* Cone $b$-metric space over Banach algebra, $c$-sequence, weakly compatible, common fixed point.
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## 1. Introduction and Preliminaries

In 2007, Huang and Zhang [15] introduced the concept of cone metric space, as a generalization of usual metric space, in which the distance $d(x, y)$ of $x$ and $y$ is defined by a vector in an ordered Banach space, replacing the usual real line. They proved that the well-known Banach contraction principle is also true in such spaces. Since then, a large number of fixed point results have appeared in cone metric spaces. The reader refers to [1–3, 17, 18, 27, 28] and the references therein. Wherein, some authors extend cone metric spaces into several more general cases. The most famous ones of them are three cases as follows: *tvs*-cone

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metric spaces or vector spaces valued cone metric spaces (see [3, 4, 24]), cone $b$-metric spaces or cone metric type spaces (see [6, 16, 20]), and *tvs*-cone $b$-metric spaces (see [23]). We have the following diagram for these four classes of abstract metric spaces including cone metric spaces:

$$\begin{tikzcd}
\boxed{\text{cone metric space}} \arrow[r] \arrow[d] & \boxed{\textit{tvs}\text{-cone metric space}} \arrow[d] \\
\boxed{\text{cone } b\text{-metric space}} \arrow[r] & \boxed{\textit{tvs}\text{-cone } b\text{-metric space}}
\end{tikzcd}$$

Here arrows stand for inclusions. The inverse inclusions do not hold. It is well-known that there exists a *tvs*-cone metric space (resp. *tvs*-cone $b$-metric space) which is not a cone metric space (resp. cone $b$-metric space). There also exists a *tvs*-cone $b$-metric space (resp. cone $b$-metric space) which is not *tvs*-cone metric space (resp. cone metric space).

However, in recent years it is not popular since some authors give an answer to the natural problem that whether cone metric spaces or cone $b$-metric spaces are equivalent to metric spaces or $b$-metric spaces (see [29]), respectively, in terms of the existence of the fixed points of the involved mappings. Concretely, they appeal to the fact that any cone metric space or cone $b$-metric space is just equivalent to a metric space or $b$-metric space, respectively, if the metric or $b$-metric function is defined by a nonlinear scalarization function $\xi_e$ or by a Minkowski functional $q_e$ (see [7–10, 13, 19, 21]). Based on this finding, people start to lose interest in studying fixed point theorems in cone metric spaces or cone $b$-metric spaces. Fortunately, very recently, Liu and Xu [22] introduced the concept of cone metric space over Banach algebra by replacing Banach space with Banach algebra and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constant $k$ by means of spectral radius and pointed out that it is significant to introduce this concept because it can be proved that cone metric spaces over Banach algebras are not equivalent to metric spaces in terms of the existence of the fixed points of the generalized Lipschitz mappings. By utilizing the similar ideas, Huang and Radenović [11, 12] introduced the concept of cone $b$-metric space over Banach algebra and coped with the non-equivalence between cone $b$-metric spaces over Banach algebras and $b$-metric spaces regarding the existence of the fixed points of the corresponding mappings. According to these evidences, we make a conclusion that the fixed point results of vectorial versions are never equivalent to the ones of scalar versions under some hypotheses. Similar to the work of [11, 12, 22], lots of fixed point theorems of vectorial versions in different spaces have been presented (see [5, 14, 25, 30]). Throughout this paper, we present several common fixed theorems in the framework of cone $b$-metric spaces over Banach algebras. Our results simplify, improve and complement some recent results from several papers. Further, by using our results, we obtain the existence and uniqueness of solution for a class of nonlinear integral equations.

For the sake of the reader, we recall some notions and lemmas as follows.

**Definition 1.1** ([24])**.** Let $E$ be a topological vector space (for example, locally convex Hausdorff space) with its zero vector $\theta$. A nonempty subset $P$ of $E$ is called a proper, closed and convex pointed cone (for short, a cone) if:

(i) $P$ is closed and $P \neq \{\theta\}$;

(ii) $\lambda, \mu \in \mathbb{R}$, $\lambda,\ \mu \geq 0$ and $x, y \in P$ imply $\lambda x + \mu y \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P$ we define a partial ordering "$\preceq$" with respect to $P$ by $x \preceq y$ if $y - x \in P$. We also define a partial ordering "$\ll$" with respect to $P$ by $x \ll y$ if $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ stands for the set of all interiors of $P$. If $\mathrm{int}P \neq \emptyset$, then $P$ is called a solid cone. The cone $P$ is called normal if there is a real number $M > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. The least positive number satisfying above is called the normal constant of $P$.

In the following, unless otherwise specified, we always assume that $P$ is a solid cone, and "$\preceq$" and "$\ll$" are partial orderings with respect to $P$.

**Definition 1.2** ([7])**.** Let $X$ be a nonempty set and $E$ a real locally convex Hausdorff space. A vector-valued function $d : X \times X \to E$ is said to be a *tvs*-cone $b$-metric function on $X$ with the constant $K \geq 1$ if the following conditions are satisfied:

(b1) $\theta \preceq d(x,y)$, for all $x,y \in X$, and $d(x,y) = \theta$ if and only if $x = y$;

(b2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(b3) $d(x,z) \preceq K[d(x,y) + d(y,z)]$ for all $x,y,z \in X$.

The pair $(X,d)$ is called a *tvs*-cone $b$-metric space or *tvs*-cone metric type space. If $K = 1$, then $(X,d)$ is called a *tvs*-cone metric space. In the case when $E$ is an ordered real Banach space, then $(X,d)$ is called a cone metric space (see [11]).

In [22], the authors modified Definition 1.1 and gave the following notion:

**Definition 1.3** ([22])**.** Let $\mathcal{A}$ be a Banach algebra with a unit $e$, and $\theta$ the zero element of $\mathcal{A}$. A nonempty closed convex subset $P$ of $\mathcal{A}$ is called a cone if $\{\theta, e\} \subset P$, $P^2 = PP \subset P$, $P \cap (-P) = \{\theta\}$ and $\lambda P + \mu P \subset P$ for all $\lambda, \mu \geq 0$.

**Definition 1.4** ([11])**.** Let $X$ be a nonempty set, $K \geq 1$ be a constant, and $\mathcal{A}$ be a Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies for all $x,y,z \in X$,

(d1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,z) \preceq K[d(x,y) + d(y,z)]$.

Then $d$ is called a cone $b$-metric on $X$, and $(X,d)$ is called a cone $b$-metric space over Banach algebra.

*Remark* 1.5. In Definition 1.4, if $E$ is a real locally convex Hausdorff space and $\mathcal{A}$ is the corresponding locally convex Hausdorff algebra, then $(X,d)$ is called a *tvs*-cone $b$-metric space over locally convex Hausdorff algebra, which generalizes the notion of cone $b$-metric space over Banach algebra.

**Definition 1.6** ([11])**.** Let $(X,d)$ be a cone $b$-metric space over Banach algebra $\mathcal{A}$, $x \in X$, $\{x_n\}$ a sequence in $X$ and $\{u_n\}$ a sequence in $\mathcal{A}$. Then

(i) $\{x_n\}$ converges to $x$ whenever for every $c \gg \theta$ there is a natural number $N$ such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$;

(ii) $\{x_n\}$ is a Cauchy sequence whenever for each $c \gg \theta$ there is a natural number $N$ such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$;

(iii) $(X,d)$ is complete if every Cauchy sequence is convergent;

(iv) $\{u_n\}$ is a $c$-sequence if for each $c \gg \theta$, there is a natural number $N$ such that $u_n \ll c$ for all $n \geq N$.

**Example 1.7.** Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$. Take multiplication in $\mathcal{A}$ as just pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with a unit $e = 1$ ($e(t) = 1$ for all $t \in [0,1]$). The set $P = \{x \in \mathcal{A} : x(t) \geq 0 \text{ for all } t \in [0,1]\}$ is a cone in $\mathcal{A}$. Moreover, $P$ is a non-normal solid cone (see [17]). Let $X = \{1,2,3\}$. Define $d : X \times X$ by $d(1,2)(t) = d(2,1)(t) = e^t$, $d(2,3)(t) = d(3,2)(t) = 2e^t$, $d(1,3)(t) = d(3,1)(t) = 4e^t$ and $d(x,x)(t) = \theta$ for all $t \in [0,1]$ and each $x \in X$. We have that $(X,d)$ is a complete cone $b$-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K = \frac{4}{3}$.

**Definition 1.8** ([1])**.** Let $S, F : X \to X$ be mappings on a set $X$.

(1) If $y = Sx = Fx$ for some $x \in X$, then $x$ is called a coincidence point of $S$ and $F$, and $y$ is called a point of coincidence of $S$ and $F$.

(2) The pair $\{S, F\}$ is called weakly compatible if $S$ and $F$ commute at all of their coincidence points, that is, $SFx = FSx$ for all $x \in \{x \in X : Sx = Fx\}$.

**Lemma 1.9** ([26])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit $e$, then the spectral radius $\rho(u)$ of $u \in \mathcal{A}$ holds*

$$\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \inf \|u^n\|^{\frac{1}{n}}.$$

*If $\rho(u) < |C|$ and $C$ is a complex constant, then $Ce - u$ is invertible in $\mathcal{A}$, moreover,*

$$(Ce-u)^{-1} = \sum_{i=0}^{\infty} \frac{u^i}{C^{i+1}}.$$

**Lemma 1.10** ([26])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit $e$, $u, v \in \mathcal{A}$. If $u$ commutes with $v$, then*

$$\rho(u+v) \leq \rho(u) + \rho(v), \qquad \rho(uv) \leq \rho(u)\rho(v).$$

**Lemma 1.11** ([11])**.** *Let $\{u_n\}$ be a sequence in $\mathcal{A}$ with $u_n \to \theta$ $(n \to \infty)$. Then $\{u_n\}$ is a c-sequence.*

**Lemma 1.12** ([17])**.** *Let $E$ be a Banach space.*

*(i) If $a, b, c \in E$ and $a \preceq b \ll c$, then $a \ll c$.*

*(ii) If $\theta \preceq a \ll c$ for each $c \gg \theta$, then $a = \theta$.*

**Lemma 1.13** ([30])**.** *Let $P$ be a solid cone in a Banach algebra $\mathcal{A}$ and $\{u_n\}$ be a c-sequence in $P$. If $\beta \in P$ is an arbitrarily given vector, then $\{\beta u_n\}$ is a c-sequence.*

**Lemma 1.14** ([1])**.** *Let $S$ and $F$ be weakly compatible self maps of a set $X$. If $S$ and $F$ have a unique point of coincidence $w$, then $w$ is the unique common fixed point of $S$ and $F$.*

**Lemma 1.15** ([11])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit $e$. Let $\alpha \in \mathcal{A}$ and $\rho(\alpha) < 1$. Then $\{\alpha^n\}$ is a c-sequence.*

**Lemma 1.16** ([11])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit $e$ and $u \in \mathcal{A}$. If $\rho(u) < |C|$ and $C$ is a complex constant, then*

$$\rho\big((Ce-u)^{-1}\big) \leq \frac{1}{|C| - \rho(u)}.$$

## 2. Main results

In this section, we offer two lemmas, which will be used constantly in the sequel. Then we acquire some common fixed theorems and their corollaries for four mappings in cone $b$-metric spaces over Banach algebras. We also present two examples to support our conclusions. In addition, we give some remarks to account for the usability of our results.

**Lemma 2.1.** *Let $\mathcal{A}$ be a Banach algebra with a unit $e$ and $P$ be a solid cone in $\mathcal{A}$. Let $u, \alpha, \beta \in P$ hold $\alpha \preceq \beta$ and $u \preceq \alpha u$. If $\rho(\beta) < 1$, then $u = \theta$.*

*Proof.* By Lemma 1.15, it is valid that $\{\beta^n\}$ is a $c$-sequence, and then by Lemma 1.13, $\{\beta^n u\}$ is also a $c$-sequence. As $\alpha \preceq \beta$ leads to $u \preceq \alpha u \preceq \alpha^2 u \preceq \cdots \preceq \alpha^n u \preceq \beta^n u$, thus by Lemma 1.12 it follows that $u = \theta$. □

**Lemma 2.2.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that $\alpha \in \mathcal{A}$, $\rho(\alpha) < \frac{1}{K}$, and $\{z_n\}$ is a sequence in $X$ satisfying the following inequality:*

$$d(z_n, z_{n+1}) \preceq \alpha d(z_{n-1}, z_n). \tag{2.1}$$

*Then $\{z_n\}$ is a Cauchy sequence in $X$.*

*Proof.* Making full use of (2.1), we have that

$$d(z_n, z_{n+1}) \preceq \alpha d(z_{n-1}, z_n) \preceq \alpha^2 d(z_{n-2}, z_{n-1}) \preceq \cdots \preceq \alpha^n d(z_0, z_1).$$

Since $\rho(\alpha) < \frac{1}{K}$ leads to $\rho(K\alpha) = K\rho(\alpha) < 1$, then by Lemma 1.9, we get that $e - K\alpha$ is invertible and $(e - K\alpha)^{-1} = \sum_{i=0}^{\infty}(K\alpha)^i$. Thus for any $n > m$, it follows that

$$\begin{aligned}
d(z_m, z_n) &\preceq K[d(z_m, z_{m+1}) + d(z_{m+1}, z_n)] \\
&\preceq Kd(z_m, z_{m+1}) + K^2[d(z_{m+1}, z_{m+2}) + d(z_{m+2}, z_n)] \\
&\preceq Kd(z_m, z_{m+1}) + K^2 d(z_{m+1}, z_{m+2}) + K^3[d(z_{m+2}, z_{m+3}) + d(z_{m+3}, z_n)] \\
&\preceq Kd(z_m, z_{m+1}) + K^2 d(z_{m+1}, z_{m+2}) + K^3 d(z_{m+2}, z_{m+3}) \\
&\quad + \cdots + K^{n-m-1} d(z_{n-2}, z_{n-1}) + K^{n-m-1} d(z_{n-1}, z_n) \\
&\preceq K\alpha^m d(z_0, z_1) + K^2\alpha^{m+1} d(z_0, z_1) + K^3\alpha^{m+2} d(z_0, z_1) \\
&\quad + \cdots + K^{n-m-1}\alpha^{n-2} d(z_0, z_1) + K^{n-m-1}\alpha^{n-1} d(z_0, z_1) \\
&\preceq K\alpha^m (e + K\alpha + K^2\alpha^2 + \cdots + K^{n-m-2}\alpha^{n-m-2} + K^{n-m-1}\alpha^{n-m-1}) d(z_0, z_1) \\
&\preceq K\alpha^m \left(\sum_{i=0}^{\infty}(K\alpha)^i\right) d(z_0, z_1) \\
&= K\alpha^m (e - K\alpha)^{-1} d(z_0, z_1).
\end{aligned}$$

Note that $\rho(\alpha) < \frac{1}{K} \leq 1$ and Lemma 1.15, it is easy to see that $\{\alpha^m\}$ is a $c$-sequence. Therefore, using Lemma 1.13 and Lemma 1.12 (i), we claim that $\{z_n\}$ is a Cauchy sequence. □

**Theorem 2.3.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $F, G, S, T : X \to X$ satisfy $SX \subseteq GX$, $TX \subseteq FX$, and that for some vector $\lambda \in P$ with $\rho(\lambda) \in \left(0, \frac{2}{K^2+K}\right)$, for all $x, y \in X$ there exists*

$$u(x, y) \in \left\{ d(Fx, Gy), d(Fx, Sx), d(Gy, Ty), \frac{d(Fx, Ty) + d(Gy, Sx)}{2} \right\}, \tag{2.2}$$

*such that the following inequality*

$$d(Sx, Ty) \preceq \lambda u(x, y) \tag{2.3}$$

*holds. If one of $SX, TX, FX$ or $GX$ is a complete subspace of $X$, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then $F, G, S,$ and $T$ have a unique common fixed point.*

*Proof.* For any arbitrary point $x_0 \in X$, construct sequences $\{x_n\}$ and $\{z_n\}$ as follows:

$$z_{2n} = Sx_{2n} = Gx_{2n+1}, \qquad z_{2n+1} = Tx_{2n+1} = Fx_{2n+2}. \tag{2.4}$$

First we prove that

$$d(z_n, z_{n+1}) \preceq \alpha d(z_{n-1}, z_n), \tag{2.5}$$

where $\alpha \in \{\lambda, (2e - K\lambda)^{-1}K\lambda\}$.

To show the inequality (2.5), we need to consider the following cases.

For $n = 2l+1,\ l \in \mathbb{N}_0$, we have $d(z_{2l+1}, z_{2l+2}) = d(Sx_{2l+2}, Tx_{2l+1})$, and from (2.2), there exists

$$\begin{aligned} u(x_{2l+2}, x_{2l+1}) \in \Big\{ & d(Fx_{2l+2}, Gx_{2l+1}), d(Fx_{2l+2}, Sx_{2l+2}), d(Gx_{2l+1}, Tx_{2l+1}), \\ & \frac{d(Fx_{2l+2}, Tx_{2l+1}) + d(Gx_{2l+1}, Sx_{2l+2})}{2} \Big\} \\ = & \Big\{ d(z_{2l+1}, z_{2l}), d(z_{2l+1}, z_{2l+2}), \frac{d(z_{2l}, z_{2l+2})}{2} \Big\}, \end{aligned}$$

such that $d(z_{2l+1}, z_{2l+2}) \preceq \lambda u(x_{2l+2}, x_{2l+1})$. Thus we have the following three cases:

(i) $d(z_{2l+1}, z_{2l+2}) \preceq \lambda d(z_{2l+1}, z_{2l}) = \alpha d(z_{2l}, z_{2l+1})$ (Here $\alpha = \lambda$);

(ii) $d(z_{2l+1}, z_{2l+2}) \preceq \lambda d(z_{2l+1}, z_{2l+2})$. By Lemma 2.1, so $d(z_{2l+1}, z_{2l+2}) = \theta$;

(iii) $d(z_{2l+1}, z_{2l+2}) \preceq (\lambda/2) d(z_{2l}, z_{2l+2})$, then

$$d(z_{2l+1}, z_{2l+2}) \preceq \frac{K\lambda}{2}[d(z_{2l}, z_{2l+1}) + d(z_{2l+1}, z_{2l+2})]. \tag{2.6}$$

Note that $\rho(K\lambda) = K\rho(\lambda) < \frac{2K}{K^2+K} = \frac{2}{K+1} \leq 1 < 2$, then by Lemma 1.9 it establishes that $2e - K\lambda$ is invertible. So by (2.6), we arrive at

$$d(z_{2l+1}, z_{2l+2}) \preceq (2e - K\lambda)^{-1} K\lambda d(z_{2l}, z_{2l+1}).$$

Thus, (2.5) holds in this case. Here $\alpha = (2e - K\lambda)^{-1} K\lambda$.

For $n = 2l,\ l \in \mathbb{N}$, we have $d(z_{2l}, z_{2l+1}) = d(Sx_{2l}, Tx_{2l+1})$, and from (2.2), there exists

$$\begin{aligned} u(x_{2l}, x_{2l+1}) \in \Big\{ & d(Fx_{2l}, Gx_{2l+1}), d(Fx_{2l}, Sx_{2l}), d(Gx_{2l+1}, Tx_{2l+1}), \\ & \frac{d(Fx_{2l}, Tx_{2l+1}) + d(Gx_{2l+1}, Sx_{2l})}{2} \Big\} \\ = & \Big\{ d(z_{2l-1}, z_{2l}), d(z_{2l}, z_{2l+1}), \frac{d(z_{2l-1}, z_{2l+1})}{2} \Big\}, \end{aligned}$$

such that $d(z_{2l}, z_{2l+1}) \preceq \lambda u(x_{2l}, x_{2l+1})$. Hence we have the following three cases:

(i) $d(z_{2l}, z_{2l+1}) \preceq \lambda d(z_{2l-1}, z_{2l}) = \alpha d(z_{2l-1}, z_{2l})$ (Here $\alpha = \lambda$);

(ii) $d(z_{2l}, z_{2l+1}) \preceq \lambda d(z_{2l}, z_{2l+1})$. By Lemma 2.1, then $d(z_{2l}, z_{2l+1}) = \theta$;

(iii) $d(z_{2l}, z_{2l+1}) \preceq (\lambda/2) d(z_{2l-1}, z_{2l+1})$, then

$$d(z_{2l}, z_{2l+1}) \preceq \frac{K\lambda}{2}[d(z_{2l-1}, z_{2l}) + d(z_{2l}, z_{2l+1})],$$

which implies that

$$d(z_{2l}, z_{2l+1}) \preceq (2e - K\lambda)^{-1} K\lambda d(z_{2l-1}, z_{2l}).$$

Accordingly, (2.5) is satisfied in this case, too. Here $\alpha = (2e - K\lambda)^{-1} K\lambda$.

Next we shall prove $\rho(\alpha) < \frac{1}{K}$. Indeed, if $\alpha = \lambda$, then $\rho(\alpha) = \rho(\lambda) < \frac{2}{K^2+K} \leq \frac{1}{K}$. If $\alpha = (2e-K\lambda)^{-1}K\lambda$, then by Lemma 1.10 and Lemma 1.16, we speculate that

$$\begin{aligned}\rho(\alpha) &= \rho((2e-K\lambda)^{-1}K\lambda) \leq \rho((2e-K\lambda)^{-1})\rho(K\lambda)\\ &\leq \frac{K\rho(\lambda)}{2-K\rho(\lambda)} < \frac{K\frac{2}{K^2+K}}{2-K\frac{2}{K^2+K}} = \frac{1}{K}.\end{aligned}$$

So from (2.5), by using Lemma 2.2, we claim that $\{z_n\}$ is a Cauchy sequence.

Without loss of generality, let us suppose that $SX$ is a complete subspace of $X$. Then there exists some point $z \in SX \subseteq GX$ such that $z_n \to z = Gu$ for some $u \in X$. Of course, the subsequences $\{z_{2n}\}$ and $\{z_{2n-1}\}$ also converge to $z$. Let us prove $z = Tu$. From (2.3) we obtain that

$$d(Tu,z) \preceq Kd(Tu,Sx_{2n}) + Kd(Sx_{2n},z) \preceq K\lambda u(x_{2n},u) + Kd(z_{2n},z),$$

where

$$\begin{aligned}u(x_{2n},u) \in \Big\{&d(Fx_{2n},Gu), d(Fx_{2n},Sx_{2n}), d(Gu,Tu),\\ &\frac{d(Fx_{2n},Tu)+d(Gu,Sx_{2n})}{2}\Big\}\\ = \Big\{&d(z_{2n-1},z), d(z_{2n-1},z_{2n}), d(z,Tu), \frac{d(z_{2n-1},Tu)+d(z,z_{2n})}{2}\Big\}.\end{aligned}$$

Thus for each $c \gg \theta$, making full use of Lemma 1.13, we have the following four cases:

(i) $d(Tu,z) \preceq K\lambda d(z_{2n-1},z) + Kd(z_{2n},z) \ll c$;

(ii) $d(Tu,z) \preceq K\lambda d(z_{2n-1},z_{2n}) + Kd(z_{2n},z) \ll c$;

(iii) $d(Tu,z) \preceq K\lambda d(z,Tu) + Kd(z_{2n},z)$, that is, $d(Tu,z) \preceq (e-K\lambda)^{-1}Kd(z_{2n},z) \ll c$;

(iv) $d(Tu,z) \preceq K\lambda\frac{d(z_{2n-1},Tu)+d(z,z_{2n})}{2} + Kd(z_{2n},z)$, hence,

$$d(Tu,z) \preceq K\lambda\frac{Kd(z_{2n-1},z)+Kd(Tu,z)+d(z,z_{2n})}{2} + Kd(z_{2n},z),$$

which yields that

$$(2e-K^2\lambda)d(Tu,z) \preceq K^2\lambda d(z_{2n-1},z) + K(\lambda+2e)d(z_{2n},z). \tag{2.7}$$

On account of $\rho(K^2\lambda) = K^2\rho(\lambda) < \frac{2K^2}{K^2+K} < 2$, so by Lemma 1.9, $2e - K^2\lambda$ is invertible. Then by (2.7), it is valid that

$$d(Tu,z) \preceq (2e-K^2\lambda)^{-1}[K^2\lambda d(z_{2n-1},z) + K(\lambda+2e)d(z_{2n},z)] \ll c.$$

Consider the above cases, it follows from Lemma 1.12 (ii) that $z = Tu$. As a result, $Tu = Gu = z$. That is to say, $u$ is a coincidence point and $z$ is a point of coincidence of $T$ and $G$.

Since $TX \subseteq FX$, there exists $v \in X$ such that $z = Fv$. Let us prove that $z = Sv$. From (2.3), we get that

$$d(Sv,z) \preceq Kd(Sv,Tx_{2n+1}) + Kd(Tx_{2n+1},z) \preceq K\lambda u(v,x_{2n+1}) + Kd(z_{2n+1},z),$$

where

$$u(v,x_{2n+1}) \in \Big\{d(Fv,Gx_{2n+1}), d(Fv,Sv), d(Gx_{2n+1},Tx_{2n+1}),$$

$$\left.\frac{d(Fv,Tx_{2n+1})+d(Gx_{2n+1},Sv)}{2}\right\}$$
$$=\left\{d(z,z_{2n}),d(z,Sv),d(z_{2n},z_{2n+1}),\frac{d(z,z_{2n+1})+d(z_{2n},Sv)}{2}\right\}.$$

Thus for each $c \gg \theta$, taking advantage of Lemma 1.13, we have the following four cases:

(i) $d(Sv,z) \preceq K\lambda d(z,z_{2n}) + Kd(z_{2n+1},z) \ll c$;

(ii) $d(Sv,z) \preceq K\lambda d(z,Sv) + Kd(z_{2n+1},z)$, i.e., $d(Sv,z) \preceq (e-K\lambda)^{-1}Kd(z_{2n+1},z) \ll c$;

(iii) $d(Sv,z) \preceq K\lambda d(z_{2n},z_{2n+1}) + Kd(z_{2n+1},z) \ll c$;

(iv) $d(Sv,z) \preceq K\lambda\frac{d(z,z_{2n+1})+d(z_{2n},Sv)}{2} + Kd(z_{2n+1},z)$, then

$$d(Sv,z) \preceq K\lambda\frac{d(z,z_{2n+1})+Kd(z_{2n},z)+Kd(z,Sv)}{2} + Kd(z_{2n+1},z),$$

which establishes that

$$d(Sv,z) \preceq (2e-K^2\lambda)^{-1}[K(\lambda+2e)d(z_{2n+1},z)+K^2\lambda d(z_{2n},z)] \ll c.$$

Uniting the above cases together with Lemma 1.12 (ii), we get $Sv = z$. As a consequence, $Sv = Fv = z$. In other words, $v$ is a coincidence point and $z$ is a point of coincidence of $S$ and $F$.

In the following, we prove that $z$ is the unique point of coincidence of pairs $\{S,F\}$ and $\{T,G\}$. We suppose for absurd that there exists another point of coincidence $z^*$ of these four mappings. That is, $Sv^* = Fv^* = Tu^* = Gu^* = z^*$ (say). From (2.3), we acquire that

$$d(z,z^*) = d(Sv,Tu^*) \preceq \lambda u(v,u^*),$$

where

$$u(v,u^*) \in \left\{d(Fv,Gu^*),d(Fv,Sv),d(Gu^*,Tu^*),\right.$$
$$\left.\frac{d(Fv,Tu^*)+d(Gu^*,Sv)}{2}\right\}$$
$$=\{d(z,z^*),\theta\}.$$

Again by Lemma 2.1, we deduce that $z = z^*$.

Finally, if $\{S,F\}$ and $\{T,G\}$ are weakly compatible pairs, then by Lemma 1.14, we claim that $F,G,S$, and $T$ have a unique common fixed point.

Similarly, we can prove the statement in the case when $FX$, $GX$ or $TX$ is complete. □

**Corollary 2.4.** *Let $(X,d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $F,S,T : X \to X$ satisfy $SX \cup TX \subseteq FX$, and that for some vector $\lambda \in P$ with $\rho(\lambda) \in \left(0,\frac{2}{K^2+K}\right)$, for all $x,y \in X$ there exists*

$$u(x,y) \in \left\{d(Fx,Fy),d(Fx,Sx),d(Fy,Ty),\frac{d(Fx,Ty)+d(Fy,Sx)}{2}\right\},$$

*such that the following inequality*

$$d(Sx,Ty) \preceq \lambda u(x,y)$$

*holds. If one of $SX,TX$ or $FX$ is a complete subspace of $X$, then $\{S,F\}$ and $\{T,F\}$ have a unique point of coincidence in $X$. Moreover, if $\{S,F\}$ and $\{T,F\}$ are weakly compatible pairs, then $F,S$, and $T$ have a unique common fixed point.*

**Corollary 2.5.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $S, T : X \to X$ satisfy that for some vector $\lambda \in P$ with $\rho(\lambda) \in \left(0, \frac{2}{K^2+K}\right)$, for all $x, y \in X$ there exists*

$$u(x,y) \in \left\{ d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2} \right\},$$

*such that the following inequality*

$$d(Sx,Ty) \preceq \lambda u(x,y)$$

*holds. If one of $SX$ or $TX$ is a complete subspace of $X$, then $S$ and $T$ have a unique common fixed point.*

**Corollary 2.6.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $F, S : X \to X$ satisfy $SX \subseteq FX$, and that for some vector $\lambda \in P$ with $\rho(\lambda) \in \left(0, \frac{2}{K^2+K}\right)$, for all $x, y \in X$ there exists*

$$u(x,y) \in \left\{ d(Fx,Fy), d(Fx,Sx), d(Fy,Sy), \frac{d(Fx,Sy)+d(Fy,Sx)}{2} \right\},$$

*such that the following inequality*

$$d(Sx,Sy) \preceq \lambda u(x,y)$$

*holds. If one of $SX$ or $FX$ is a complete subspace of $X$, then $S$ and $F$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ is a weakly compatible pair, then $S$ and $F$ have a unique common fixed point.*

**Corollary 2.7.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mapping $S : X \to X$ satisfies that for some vector $\lambda \in P$ with $\rho(\lambda) \in \left(0, \frac{2}{K^2+K}\right)$, for all $x, y \in X$ there exists*

$$u(x,y) \in \left\{ d(x,y), d(x,Sx), d(y,Sy), \frac{d(x,Sy)+d(y,Sx)}{2} \right\},$$

*such that the following inequality*

$$d(Sx,Sy) \preceq \lambda u(x,y)$$

*holds. If $SX$ is a complete subspace of $X$, then $S$ has a unique fixed point.*

**Corollary 2.8.** *Let $(X, d)$ be a cone b-metric space with the coefficient $K \geq 1$. Suppose that self-mappings $F, G, S, T : X \to X$ satisfy $SX \subseteq GX$, $TX \subseteq FX$, and that for some real constant $\lambda$ with $\lambda \in \left(0, \frac{2}{K^2+K}\right)$, for all $x, y \in X$ there exists*

$$u(x,y) \in \left\{ d(Fx,Gy), d(Fx,Sx), d(Gy,Ty), \frac{d(Fx,Ty)+d(Gy,Sx)}{2} \right\}, \tag{2.8}$$

*such that the following inequality*

$$d(Sx,Ty) \preceq \lambda u(x,y) \tag{2.9}$$

*holds. If one of $SX, TX, FX$ or $GX$ is a complete subspace of $X$, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then $F, G, S$, and $T$ have a unique common fixed point.*

*Remark* 2.9. Our conclusions never consider the normality of cones, which may bring us more convenience in applications. Moreover, they greatly generalize the previous results from several references. For instance, Corollary 2.6 generalizes Theorem 2.1 and Theorem 2.4 from [1]. Corollary 2.7 generalizes Theorem 1 and Theorem 4 from [15], Theorem 2.1 and Theorem 2.2 from [22], and Theorem 3.1 and Theorem 3.2 from [30]. Corollary 2.8 generalizes Theorem 2.2 from [2].

*Remark* 2.10. Corollary 2.8 extend, unite and improve Theorem 3.1 and Theorem 3.3 from [6] in several sides. Firstly, Corollary 2.8 contains these two theorems. This is because our condition $K \geq 1$ includes $1 \leq K \leq 2$ of Theorem 3.1 and $K \geq 2$ of Theorem 3.3. Secondly, our conditions (2.8) and (2.9) are much simpler than (3.1) and (3.2), respectively, from Theorem 3.1. Thirdly, we correct some mistakes in the proof of Theorem 3.1. Indeed, (3.3) of Theorem 3.1 should satisfy $0 < \alpha < \frac{1}{K}$, otherwise, (3.9) of Theorem 3.1 is incorrect since $1 - K\alpha$ is not necessarily greater than 0.

**Example 2.11.** Under the hypotheses of Example 1.7, define two mappings $S, F : X \to X$ as follows:

$$S1 = S2 = 2,\ S3 = 1; \qquad F1 = 1,\ F2 = 2,\ F3 = 3.$$

Put $\lambda = \frac{1}{8}t + \frac{1}{2} \in \mathcal{A}$. Simple calculations show that

$$d(Sx, Sy) \preceq \lambda d(Fx, Fy)$$

for all $x, y \in X$. As a result, the conditions of Corollary 2.6 are satisfied. Therefore, $S$ and $F$ have a unique common fixed point $x = 2$.

**Theorem 2.12.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $F, G, S, T : X \to X$ satisfy $SX \subseteq GX$, $TX \subseteq FX$, and that one of $SX, TX, FX$ or $GX$ is a complete subspace of $X$. Suppose that*

$$\begin{aligned} d(Sx, Ty) \preceq{}& \lambda_1 d(Fx, Gy) + \lambda_2 d(Fx, Sx) + \lambda_3 d(Gy, Ty) \\ &+ \lambda_4[d(Fx, Ty) + d(Gy, Sx)] \end{aligned} \tag{2.10}$$

*for all $x, y \in X$, where $\lambda_i \in P$ are some vectors with $\lambda_i\lambda_j = \lambda_j\lambda_i$ $(i = 1, 2, 3, 4)$. If $\rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4) < 1$ and $\rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4) < 1$, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then $F, G, S,$ and $T$ have a unique common fixed point.*

*Proof.* For arbitrary point $x_0 \in X$, construct the same sequences $\{x_n\}$ and $\{z_n\}$ in $X$ as in the proof of Theorem 2.3. By utilizing (2.10), then on the one hand, we have that

$$\begin{aligned} d(z_{2n}, z_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\preceq \lambda_1 d(Fx_{2n}, Gx_{2n+1}) + \lambda_2 d(Fx_{2n}, Sx_{2n}) + \lambda_3 d(Gx_{2n+1}, Tx_{2n+1}) \\ &\quad + \lambda_4[d(Fx_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, Sx_{2n})] \\ &= \lambda_1 d(z_{2n-1}, z_{2n}) + \lambda_2 d(z_{2n-1}, z_{2n}) + \lambda_3 d(z_{2n}, z_{2n+1}) \\ &\quad + \lambda_4 d(z_{2n-1}, z_{2n+1}) \\ &\preceq (\lambda_1 + \lambda_2 + K\lambda_4) d(z_{2n-1}, z_{2n}) + (\lambda_3 + K\lambda_4) d(z_{2n}, z_{2n+1}), \end{aligned}$$

which means that

$$d(z_{2n}, z_{2n+1}) \preceq (e - \lambda_3 - K\lambda_4)^{-1}(\lambda_1 + \lambda_2 + K\lambda_4) d(z_{2n-1}, z_{2n}). \tag{2.11}$$

On the other hand, we obtain that

$$\begin{aligned} d(z_{2n+1}, z_{2n+2}) &= d(Sx_{2n+2}, Tx_{2n+1}) \\ &\preceq \lambda_1 d(Fx_{2n+2}, Gx_{2n+1}) + \lambda_2 d(Fx_{2n+2}, Sx_{2n+2}) + \lambda_3 d(Gx_{2n+1}, Tx_{2n+1}) \\ &\quad + \lambda_4[d(Fx_{2n+2}, Tx_{2n+1}) + d(Gx_{2n+1}, Sx_{2n+2})] \\ &= \lambda_1 d(z_{2n+1}, z_{2n}) + \lambda_2 d(z_{2n+1}, z_{2n+2}) + \lambda_3 d(z_{2n}, z_{2n+1}) \\ &\quad + \lambda_4 d(z_{2n}, z_{2n+2}) \\ &\preceq (\lambda_1 + \lambda_3 + K\lambda_4) d(z_{2n}, z_{2n+1}) + (\lambda_2 + K\lambda_4) d(z_{2n+1}, z_{2n+2}), \end{aligned}$$

which implies that

$$d(z_{2n+1}, z_{2n+2}) \preceq (e - \lambda_2 - K\lambda_4)^{-1}(\lambda_1 + \lambda_3 + K\lambda_4)d(z_{2n}, z_{2n+1}). \tag{2.12}$$

Using Lemma 1.10 and Lemma 1.16, we arrive at

$$\begin{aligned}\rho((e - \lambda_3 - K\lambda_4)^{-1}(\lambda_1 + \lambda_2 + K\lambda_4)) &\leq \rho((e - \lambda_3 - K\lambda_4)^{-1})\rho(\lambda_1 + \lambda_2 + K\lambda_4)\\ &\leq \frac{\rho(\lambda_1 + \lambda_2 + K\lambda_4)}{1 - \rho(\lambda_3 + K\lambda_4)} < \frac{1}{K},\end{aligned}$$

and

$$\begin{aligned}\rho((e - \lambda_2 - K\lambda_4)^{-1}(\lambda_1 + \lambda_3 + K\lambda_4)) &\leq \rho((e - \lambda_2 - K\lambda_4)^{-1})\rho(\lambda_1 + \lambda_3 + K\lambda_4)\\ &\leq \frac{\rho(\lambda_1 + \lambda_3 + K\lambda_4)}{1 - \rho(\lambda_2 + K\lambda_4)} < \frac{1}{K}.\end{aligned}$$

Hence, from (2.11) and (2.12), by Lemma 2.2, we demonstrate that $\{z_n\}$ is a Cauchy sequence.

Assume that $SX$ is a complete subspace of $X$. Then there exists some point $z \in SX \subseteq GX$ such that $z_n \to z = Gu$ for some $u \in X$. Of course, the subsequences $\{z_{2n}\}$ and $\{z_{2n-1}\}$ also converge to $z$. Let us prove $z = Tu$. From (2.10) we get that

$$\begin{aligned}d(Tu, z) &\preceq K[d(Sx_{2n}, Tu) + d(Sx_{2n}, z)]\\ &\preceq K\{\lambda_1 d(Fx_{2n}, Gu) + \lambda_2 d(Fx_{2n}, Sx_{2n}) + \lambda_3 d(Gu, Tu)\\ &\quad + \lambda_4[d(Fx_{2n}, Tu) + d(Gu, Sx_{2n})] + d(Sx_{2n}, z)\}\\ &\preceq K\{\lambda_1 d(z_{2n-1}, z) + \lambda_2 d(z_{2n-1}, z_{2n}) + \lambda_3 d(z, Tu) + d(z_{2n}, z)\\ &\qquad + \lambda_4[Kd(z_{2n-1}, z) + Kd(z, Tu) + d(z, z_{2n})]\} + K\lambda_1 d(z, Tu),\end{aligned}$$

which yields that

$$\begin{aligned}(e - K\lambda_1 - K\lambda_3 - K^2\lambda_4)d(Tu, z) &\preceq K\{\lambda_1 d(z_{2n-1}, z) + \lambda_2 d(z_{2n-1}, z_{2n}) + d(z_{2n}, z)\\ &\qquad + \lambda_4[Kd(z_{2n-1}, z) + d(z, z_{2n})]\}.\end{aligned} \tag{2.13}$$

Since

$$\rho(K\lambda_1 + K\lambda_3 + K^2\lambda_4) = K\rho(\lambda_1 + \lambda_3 + K\lambda_4) \leq \rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4) < 1,$$

implies that $e - K\lambda_1 - K\lambda_3 - K^2\lambda_4$ is invertible, then from (2.13) we obtain that

$$\begin{aligned}d(Tu, z) &\preceq (e - K\lambda_1 - K\lambda_3 - K^2\lambda_4)^{-1}K\{\lambda_1 d(z_{2n-1}, z) + \lambda_2 d(z_{2n-1}, z_{2n})\\ &\qquad + d(z_{2n}, z) + \lambda_4[Kd(z_{2n-1}, z) + d(z, z_{2n})]\}.\end{aligned} \tag{2.14}$$

By Lemma 1.13, it is clear that the right side of the inequality (2.14) is a $c$-sequence, this means $z = Tu$. As a result, $Tu = Gu = z$. That is to say, $u$ is a coincidence point and $z$ is a point of coincidence of $T$ and $G$.

Since $TX \subseteq FX$, there exists $v \in X$ such that $z = Fv$. Let us prove $z = Sv$. From (2.10), we deduce that

$$\begin{aligned}d(Sv, z) &\preceq K[d(Sv, Tx_{2n+1}) + d(Tx_{2n+1}, z)]\\ &\preceq K\{\lambda_1 d(Fv, Gx_{2n+1}) + \lambda_2 d(Fv, Sv) + \lambda_3 d(Gx_{2n+1}, Tx_{2n+1})\\ &\qquad + \lambda_4[d(Fv, Tx_{2n+1}) + d(Gx_{2n+1}, Sv)] + d(Tx_{2n+1}, z)\}\\ &\preceq K\{\lambda_1 d(z, z_{2n}) + \lambda_2 d(z, Sv) + \lambda_3 d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z)\end{aligned}$$

$$+ \lambda_4[d(z, z_{2n+1}) + Kd(z_{2n}, z) + Kd(z, Sv)]\} + K\lambda_1 d(Sv, z),$$

which follows that

$$\begin{aligned}(e - K\lambda_1 - K\lambda_2 - K^2\lambda_4)d(Sv, z) \preceq K\{&\lambda_1 d(z, z_{2n}) + \lambda_3 d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z)\\ &+ \lambda_4[d(z, z_{2n+1}) + Kd(z_{2n}, z)]\}.\end{aligned} \tag{2.15}$$

Now that

$$\begin{aligned}\rho(K\lambda_1 + K\lambda_2 + K^2\lambda_4) &= K\rho(\lambda_1 + \lambda_2 + K\lambda_4)\\ &\le \rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4)\\ &< 1,\end{aligned}$$

makes clear that $e - K\lambda_1 - K\lambda_2 - K^2\lambda_4$ is invertible, then from (2.15) we obtain that

$$\begin{aligned}d(Sv, z) \preceq (e - K\lambda_1 - K\lambda_2 - K^2\lambda_4)^{-1}K\{&\lambda_1 d(z, z_{2n}) + \lambda_3 d(z_{2n}, z_{2n+1})\\ &+ d(z_{2n+1}, z) + \lambda_4[d(z, z_{2n+1}) + Kd(z_{2n}, z)]\}.\end{aligned} \tag{2.16}$$

By Lemma 1.13, we know that the right side of the inequality (2.16) is a $c$-sequence, this means $z = Sv$. Accordingly, $Sv = Fv = z$. That is to say, $v$ is a coincidence point and $z$ is a point of coincidence of $S$ and $F$.

In the following, we prove that $z$ is the unique point of coincidence of pairs $\{S, F\}$ and $\{T, G\}$. To this end, we assume that there exists another point of coincidence $z^*$ of these four mappings. That is, $Sv^* = Fv^* = Tu^* = Gu^* = z^*$ (say). From (2.10), we acquire that

$$\begin{aligned}d(z, z^*) &= d(Sv, Tu^*)\\ &\preceq \lambda_1 d(Fv, Gu^*) + \lambda_2 d(Fv, Sv) + \lambda_3 d(Gu^*, Tu^*)\\ &\quad + \lambda_4[d(Fv, Tu^*) + d(Gu^*, Sv)]\\ &= (\lambda_1 + 2\lambda_4)d(z, z^*).\end{aligned} \tag{2.17}$$

In view of $K \ge 1$, it follows that

$$\begin{aligned}\lambda_1 + 2\lambda_4 &\preceq K\lambda_1 + K\lambda_4 + K^2\lambda_4\\ &\preceq \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 + K\lambda_1 + K\lambda_4 + \frac{1}{2}K\lambda_2 + \frac{1}{2}K\lambda_3 + K^2\lambda_4.\end{aligned} \tag{2.18}$$

Using Lemma 1.10, we arrive at

$$\begin{aligned}&\rho(\lambda_2 + \lambda_3 + 2K\lambda_1 + 2K\lambda_4 + K\lambda_2 + K\lambda_3 + 2K^2\lambda_4)\\ &\quad = \rho\{[(\lambda_3 + K\lambda_4) + K(\lambda_1 + \lambda_2 + K\lambda_4)]\\ &\qquad + [(\lambda_2 + K\lambda_4) + K(\lambda_1 + \lambda_3 + K\lambda_4)]\}\\ &\quad \le [\rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4)]\\ &\qquad + [\rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4)]\\ &\quad < 1 + 1 = 2,\end{aligned}$$

which establishes that

$$\rho(\frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 + K\lambda_1 + K\lambda_4 + \frac{1}{2}K\lambda_2 + \frac{1}{2}K\lambda_3 + K^2\lambda_4) < 1. \tag{2.19}$$

Making full use of (2.17)–(2.19), and Lemma 2.1, we get $d(z, z^*) = \theta$, that is, $z = z^*$.

Finally, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then by Lemma 1.14, we claim that $F, G, S$, and $T$ have a unique common fixed point.

Similarly, we can prove the case when $Fx$, $Gx$ or $Tx$ is complete. □

**Corollary 2.13.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $F, S, T : X \to X$ satisfy $SX \cup TX \subseteq FX$, and that one of $SX, TX$ or $FX$ is a complete subspace of $X$. Suppose that*

$$d(Sx, Ty) \preceq \lambda_1 d(Fx, Fy) + \lambda_2 d(Fx, Sx) + \lambda_3 d(Fy, Ty) + \lambda_4 [d(Fx, Ty) + d(Fy, Sx)]$$

*for all $x, y \in X$, where $\lambda_i \in P$ are some vectors with $\lambda_i\lambda_j = \lambda_j\lambda_i$ $(i = 1, 2, 3, 4)$. If $\rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4) < 1$ and $\rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4) < 1$, then $\{S, F\}$ and $\{T, F\}$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ and $\{T, F\}$ are weakly compatible pairs, then $F, S$, and $T$ have a unique common fixed point.*

**Corollary 2.14.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $S, T : X \to X$ satisfy that one of $SX$ or $TX$ is a complete subspace of $X$. Suppose that*

$$d(Sx, Ty) \preceq \lambda_1 d(x, y) + \lambda_2 d(x, Sx) + \lambda_3 d(y, Ty) + \lambda_4 [d(x, Ty) + d(y, Sx)]$$

*for all $x, y \in X$, where $\lambda_i \in P$ are some vectors with $\lambda_i\lambda_j = \lambda_j\lambda_i$ $(i = 1, 2, 3, 4)$. If $\rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4) < 1$ and $\rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4) < 1$, then $S$ and $T$ have a unique common fixed point.*

**Corollary 2.15.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mappings $F, S : X \to X$ satisfy that $SX \subseteq FX$, and that one of $SX$ or $FX$ is a complete subspace of $X$. Suppose that*

$$d(Sx, Sy) \preceq \lambda_1 d(Fx, Fy) + \lambda_2 d(Fx, Sx) + \lambda_3 d(Fy, Sy) + \lambda_4 [d(Fx, Sy) + d(Fy, Sx)]$$

*for all $x, y \in X$, where $\lambda_i \in P$ are some vectors with $\lambda_i\lambda_j = \lambda_j\lambda_i$ $(i = 1, 2, 3, 4)$. If $\rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4) < 1$ and $\rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4) < 1$, then $S$ and $F$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ is a weakly compatible pair, then $S$ and $F$ have a unique common fixed point.*

**Corollary 2.16.** *Let $(X, d)$ be a cone b-metric space over Banach algebra $\mathcal{A}$ with the coefficient $K \geq 1$ and $P$ be a solid cone in $\mathcal{A}$. Suppose that self-mapping $S : X \to X$ satisfies $SX$ is a complete subspace of $X$. Suppose that*

$$d(Sx, Sy) \preceq \lambda_1 d(x, y) + \lambda_2 d(x, Sx) + \lambda_3 d(y, Sy) + \lambda_4 [d(x, Sy) + d(y, Sx)]$$

*for all $x, y \in X$, where $\lambda_i \in P$ are some vectors with $\lambda_i\lambda_j = \lambda_j\lambda_i$ $(i = 1, 2, 3, 4)$. If $\rho(\lambda_3 + K\lambda_4) + K\rho(\lambda_1 + \lambda_2 + K\lambda_4) < 1$ and $\rho(\lambda_2 + K\lambda_4) + K\rho(\lambda_1 + \lambda_3 + K\lambda_4) < 1$, then $S$ has a unique fixed point.*

**Corollary 2.17.** *Let $(X, d)$ be a cone b-metric space with the coefficient $K \geq 1$. Suppose that self-mappings $F, G, S, T : X \to X$ satisfy $SX \subseteq GX$, $TX \subseteq FX$, and that one of $SX, TX, FX$ or $GX$ is a complete subspace of $X$. Suppose that*

$$d(Sx, Ty) \preceq \lambda_1 d(Fx, Gy) + \lambda_2 d(Fx, Sx) + \lambda_3 d(Gy, Ty) + \lambda_4 [d(Fx, Ty) + d(Gy, Sx)]$$

*for all $x, y \in X$, where $\lambda_i \geq 0$ $(i = 1, 2, 3, 4)$ are some real constants. If $K\lambda_1 + K\lambda_2 + \lambda_3 + K\lambda_4 + K^2\lambda_4 < 1$ and $K\lambda_1 + \lambda_2 + K\lambda_3 + K\lambda_4 + K^2\lambda_4 < 1$, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then $F, G, S$, and $T$ have a unique common fixed point.*

**Corollary 2.18.** *Let $(X, d)$ be a metric space. Suppose that self-mappings $F, G, S, T : X \to X$ satisfy $SX \subseteq GX$, $TX \subseteq FX$, and that one of $SX, TX, FX$ or $GX$ is a complete subspace of $X$. Suppose that*

$$\begin{aligned} d(Sx, Ty) \leq{} & \lambda_1 d(Fx, Gy) + \lambda_2 d(Fx, Sx) + \lambda_3 d(Gy, Ty) \\ & + \lambda_4 [d(Fx, Ty) + d(Gy, Sx)] \end{aligned} \tag{2.20}$$

*for all $x, y \in X$, where $\lambda_i \geq 0$ $(i = 1, 2, 3, 4)$ are some real constants. If $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$, then $\{S, F\}$ and $\{T, G\}$ have a unique point of coincidence in $X$. Moreover, if $\{S, F\}$ and $\{T, G\}$ are weakly compatible pairs, then $F, G, S$, and $T$ have a unique common fixed point.*

*Remark* 2.19. Corollary 2.17 greatly generalizes Theorem 2.8 of [2] since our cone $b$-metric space is much larger than cone metric space.

*Remark* 2.20. In all of the above theorems and corollaries, if the coefficients of the generalized contractions are real or complex constants, then by utilizing the similar ways, we can get the same assertions in the setting of *tvs*-cone $b$-metric spaces over locally convex Hausdorff algebras.

*Remark* 2.21. The condition (2.9) of Corollary 2.8 or (2.20) of Corollary 2.18 cannot be replaced by the following condition:

$$d(Sx,Ty) \leq \lambda \max\{d(Fx,Gy), d(Fx,Sx), d(Gy,Ty), d(Fx,Ty), d(Gy,Sx)\}, \tag{2.21}$$

where $0 < \lambda < 1$. The following example illustrates this assertion.

**Example 2.22.** Let $X = \{x,y,u,v\} = \{(0,0,0), (4,0,0), (2,2,0), (2,-2,1)\} \subset \mathbb{R}^3$ with usual metric. Then

$$\begin{aligned} &d(x,u) = 2\sqrt{2}, \quad d(x,y) = 4, \quad d(x,v) = 3, \\ &d(y,u) = 2\sqrt{2}, \quad d(y,v) = 3, \quad d(u,v) = \sqrt{17}. \end{aligned}$$

Define $S,T : X \to X$ by

$$\begin{aligned} &Sx = u, \quad Sy = v, \quad Su = v, \quad Sv = u, \\ &Tx = y, \quad Ty = x, \quad Tu = y, \quad Tv = x. \end{aligned}$$

We have

$$S(X) = \{u,v\}, \qquad T(X) = \{x,y\}.$$

Further, we have

$$\begin{aligned} d(Sx,Tx) &= d(u,y) = 2\sqrt{2} < d(x,Tx) = 4, \\ d(Sx,Ty) &= d(u,x) = 2\sqrt{2} < d(y,Ty) = 4, \\ d(Sx,Tu) &= d(u,y) = 2\sqrt{2} < d(x,Tu) = 4, \\ d(Sx,Tv) &= d(u,x) = 2\sqrt{2} < d(v,Sx) = \sqrt{17}. \end{aligned}$$

Similarly, for all $a,b \in X$ we get $d(Sa,Tb) = 2\sqrt{2}$ or $d(Sa,Tb) = 3$.

Thus we have

$$d(Sa,Tb) \leq \frac{3}{4} \max\{d(a,b), d(a,Sa), d(b,Tb), d(a,Tb), d(b,Sa)\}$$

for all $a,b \in X$, that is, (2.21) is satisfied, where $F = G = I$ (identity mapping). But $S$ and $T$ have not a common fixed point.

## 3. Application

In this section, we shall apply the obtained assertions to cope with the existence and uniqueness of solution for some equations.

We consider the following nonlinear integral equations:

$$\begin{cases} \phi(x) = \int_0^x k(x,t,\phi(t))\mathrm{d}t, \\ \phi(x) = \int_0^x \phi(t)\mathrm{d}t, \end{cases} \tag{3.1}$$

where $x \in [0,T]$.

**Theorem 3.1.** *Let* $L_p[0,T] = \{x = x(t) : \int_0^T |x(t)|^p \mathrm{d}t < \infty\}$ $(0 < p < 1)$. *For* (3.1), *assume that the following conditions hold:*

*(i)* *If* $k(x,t,\phi(t)) = \phi(t)$ *for all* $0 \le t \le x \le T$, *then*

$$k(x,t,\int_0^t \phi(s)\mathrm{d}s) = \int_0^t k(t,s,\phi(s))\mathrm{d}s,$$

*for all* $0 \le t \le x \le T$.

*(ii)* *There is a constant* $L \in (0, 2^{1-\frac{1}{p}})$ *such that the partial derivative* $k_u$ *of* $k$ *with respect to* $u$ *exists and* $|k_u(x,t,u)| \le L$ *for all* $0 \le t \le x \le T$, *and* $-\infty < u < \infty$.

*Then the integral equation* (3.1) *has a unique common solution in* $L_p[0,T]$.

*Proof.* Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(u_1,u_2)\| = |u_1| + |u_2|$ and the multiplication by

$$uv = (u_1,u_2)(v_1,v_2) = (u_1v_1, u_1v_2 + u_2v_1).$$

Let $P = \{u = (u_1,u_2) \in \mathcal{A} : u_1,u_2 \ge 0\}$. It is clear that $P$ is a cone and $\mathcal{A}$ is a Banach algebra with a unit $e = (1,0)$. Let $X = L_p[0,T]$. We endow $X$ with the cone $b$-metric

$$d(\phi,\varphi) = \left(\left\{\int_0^T |\phi(x) - \varphi(x)|^p \mathrm{d}x\right\}^{\frac{1}{p}}, \left\{\int_0^T |\phi(x) - \varphi(x)|^p \mathrm{d}x\right\}^{\frac{1}{p}}\right)$$

for all $x,y \in X$. It is clear that $(X,d)$ is a complete cone $b$-metric space over Banach algebra $\mathcal{A}$ with the coefficient $s = 2^{\frac{1}{p}-1}$. Define the mappings $S,F : X \to X$ by

$$S\phi(x) = \int_0^x k(x,t,\phi(t))\mathrm{d}t, \qquad F\phi(x) = \int_0^x \phi(t)\mathrm{d}t$$

for all $x \in [0,T]$. Then

$$\begin{aligned}
d(S\phi(x), S\varphi(x)) &= \left(\left\{\int_0^T \left|\int_0^x k(x,t,\phi(t))\mathrm{d}t - \int_0^x k(x,t,\varphi(t))\mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}},\right. \\
&\qquad \left.\left\{\int_0^T \left|\int_0^x k(x,t,\phi(t))\mathrm{d}t - \int_0^x k(x,t,\varphi(t))\mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}}\right) \\
&= \left(\left\{\int_0^T \left|\int_0^x [k(x,t,\phi(t)) - k(x,t,\varphi(t))]\mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}},\right. \\
&\qquad \left.\left\{\int_0^T \left|\int_0^x [k(x,t,\phi(t)) - k(x,t,\varphi(t))]\mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}}\right) \\
&\preceq \left(L\left\{\int_0^T \left|\int_0^x [\phi(t) - \varphi(t)]\mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}},\right. \\
&\qquad \left.L\left\{\int_0^T \left|\int_0^x [\phi(t) - \varphi(t)]\mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}}\right) \\
&= \left(L\left\{\int_0^T |F\phi(x) - F\varphi(x)|^p \,\mathrm{d}x\right\}^{\frac{1}{p}},\right.
\end{aligned}$$

$$L \left\{ \int_0^T |F\phi(x) - F\varphi(x)|^p \,\mathrm{d}x \right\}^{\frac{1}{p}} \Bigg)$$
$$\preceq (L,1) d(F\phi(x), F\varphi(x)).$$

Because
$$\|(L,1)^n\|^{\frac{1}{n}} = \|(L^n, nL^{n-1})\|^{\frac{1}{n}} \to L < 2^{1-\frac{1}{p}} = \frac{1}{s} \ \ (n \to \infty),$$
which implies that $\rho((L,1)) < \frac{1}{s}$. Now choose $\lambda_1 = (L,1)$ and $\lambda_2 = \lambda_3 = \lambda_4 = \theta$. Owing to (i), it is easy to see that the mappings $S$ and $F$ are weakly compatible. Therefore, all conditions of Corollary 2.15 are satisfied. Or, all conditions of Corollary 2.6 are satisfied, where $\lambda = (L,1)$. As a result, $S$ and $F$ have a unique common fixed point $x^* \in X$. That is, $x^*$ is the unique common solution of the system of integral equation (3.1). $\square$

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