Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5772–5779
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On iteration invariants for $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity and weak $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity of non-autonomous discrete systems

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Communicated by M. Eslamian

Abstract

In this paper, let $(X, f_{1,\infty})$ be a non-autonomous discrete system on a compact metric space X. For a positive k, the properties $\widehat{P}(k)$ and $\widehat{Q}(k)$ of Furstenberg families are introduced for any integer $k > 0$. Based on the two properties, we prove that $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity and weak $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity are inherited under iterations.

Keywords: Non-autonomous discrete system, Furstenberg family, $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity, weak $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity.
2010 MSC: 54H20, 54B20, 37B99, 37D45.

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1. Introduction

A classical discrete dynamical system is a pair (X, f), where X is a nontrivial metric space with a metric d and $f : X \longrightarrow X$ is a continuous map. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

Let $(X, f_{1,\infty})$ be a non-autonomous discrete system. That is, $f_{1,\infty} = (f_n)_{n=1}^{\infty}$ is a sequence of continuous maps on a metric space (X, d). It is clear that if $f_n = f$ for any integer $n \geq 1$, then the non-autonomous discrete system $(X, f_{1,\infty})$ is just a classical discrete dynamical system. For any positive integers i and n, we set $f_i^n = f_{i+(n-1)} \circ \cdots \circ f_i$ and $f_i^0 = id_X$. The orbit of any point $x \in X$ is the set

$$\{f_1^n(x) : n \in \mathbb{Z}^+\} = orb(x, f_{1,\infty}).$$

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Received 2016-07-02

We denote by $(X, f_{1,\infty}^{[k]})$ the k-th iterate of $(X, f_{1,\infty})$, where $f_{1,\infty}^{[k]} = (f_{k(n-1)+1}^{k})_{n=1}^{\infty}$. Non-autonomous discrete systems were mentioned and studied in [7, 8]. They also were relevant to non-autonomous difference equations (see [3, 4]). Let $\mathcal{W}$ denote one of the following eight properties: Li-Yorke chaos, dense chaos, dense δ-chaos, generic chaos, generic δ-chaos, Li-Yorke sensitivity, sensitivity and spatio-temporal chaos. In [27], Wu and Zhu proved that for a non-autonomous discrete system $(X, f_{1,\infty})$ on a compact metric space which converges uniformly to a map, the $\mathcal{W}$-chaoticity of sequences with the form $(f_n \circ \cdots \circ f_1)(x)$ was inherited under iterations. In 2015, Huang et al. presented some sufficient conditions of sensitivity and cofinitely sensitivity for non-autonomous systems on nontrivial metric spaces (see [6]).

Over the last ten years or so, many research works have been devoted to the sensitivity of discrete dynamical systems (see [5, 9–29]). One of the most significant features is the introduction of some stronger forms of sensitivity for discrete dynamical systems in [15]. In [18], Tan and Zhang defined sensitive pairs via Furstenberg families and considered the relation of the following three notions: sensitivity, $\mathscr{F}$-sensitivity and $\mathscr{F}$-sensitive pairs, where $\mathscr{F}$ is a Furstenberg family. They also gave some sufficient conditions for transitive systems to have $\mathscr{F}$-sensitive pairs and gave some examples showing that $\mathscr{F}$-sensitivity cannot imply the existence of $\mathscr{F}$-sensitive pairs, and that there is no immediate relation between the existence of sensitive pairs and Li-Yorke chaos. In particular, Tan and Zhang proved that if the system (X, f) is $\mathscr{F}_s$-transitive, then there exists $\delta > 0$ such that $\{n \in \mathbb{Z}^+ : diam f^n(U) > \delta\} \in \mathscr{F}_s$ for any non-empty open subset $U \subset X$ (see [18]). In 2009, Tan and Xiong introduced the notion of $(\mathscr{F}_1, \mathscr{F}_2)$-chaos via Furstenberg family couple $\mathscr{F}_1, \mathscr{F}_2$ and obtained some sufficient conditions for a discrete dynamical system to be $(\mathscr{F}_1, \mathscr{F}_2)$-chaos (see [17]), and they pointed out that for a discrete dynamical system, Li-Yorke chaos and distributional chaos can be treated as chaos in Furstenberg families sense. In [9], Li proved that $(\mathscr{F}_1, \mathscr{F}_2)$-chaos and $(\mathscr{F}_1, \mathscr{F}_2)$-$\delta$-chaos are topological conjugacy invariant. In [26], Wu and Zhu gave the concepts of dense $(\mathscr{F}_1, \mathscr{F}_2)$-$\delta$-chaos, general $(\mathscr{F}_1, \mathscr{F}_2)$-chaos, general strong $(\mathscr{F}_1, \mathscr{F}_2)$-chaos and $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity. At the same time, they presented some equivalent conditions between $\mathscr{F}$-sensitivity and $(\mathscr{F}_1, \mathscr{F}_2)$-chaos. In [21, 23], Wu et al. proved that $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity of a discrete dynamical system is inherited in its inverse limit dynamical system.

In this paper, we introduce the weak $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity for discrete systems and study the problems on iteration invariants for $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity and weak $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity of non-autonomous discrete systems.

2. Preliminaries

Let $\mathbb{Z}^+$ be the set of non-negative integers and $\mathcal{P}$ be the collection of all subsets of $\mathbb{Z}^+$. For a subset $\mathscr{F}$ of $\mathcal{P}$, it is called a Furstenberg family, if it is hereditary upwards, i.e., $F_1 \subset F_2$ and $F_1 \in \mathscr{F}$ imply $F_2 \in \mathscr{F}$ (see [1]). For a family $\mathscr{F}$, the dual family (see [1]) is

$$\kappa\mathscr{F} = \left\{F \in \mathcal{P} : \mathbb{Z}^+ \setminus F \notin \mathscr{F}\right\}.$$

For $i \in \mathbb{Z}^+$ and $F \in \mathcal{P}$, set $F + i = \{j + i : j \in F\} \cap \mathbb{Z}^+$ and $F - i = \{j - i : j \in F\} \cap \mathbb{Z}^+$. A Furstenberg family $\mathscr{F}$ is called positively translation-invariant, if for any $F \in \mathscr{F}$ and any $i \in \mathbb{Z}^+$, $F + i \in \mathscr{F}$. A Furstenberg family $\mathscr{F}$ is called negatively translation-invariant, if for any $F \in \mathscr{F}$ and any $i \in \mathbb{Z}^+$, $F - i \in \mathscr{F}$. Let $\mathscr{F}_{inf}$ be the collection of all infinite subsets of $\mathbb{Z}^+$.

For $A \subset \mathbb{Z}^+$, define

$$\overline{\text{dens}}(A) = \limsup_{n\to+\infty} \frac{1}{n} \left|A \cap [0, n-1]\right|,$$

and

$$\underline{\text{dens}}(A) = \liminf_{n\to+\infty} \frac{1}{n} \left|A \cap [0, n-1]\right|.$$

Then, $\overline{\text{dens}}(A)$ and $\underline{\text{dens}}(A)$ are the *upper density* and the *lower density* of A, respectively. Fix any $\alpha \in [0, 1]$ and denote by $\widehat{\mathscr{M}_\alpha}$ (resp. $\widehat{\mathscr{M}^\alpha}$) the family consisting of sets $A \subset \mathbb{Z}^+$ with $\underline{\text{dens}}(A) \geq \alpha$ (resp. $\overline{\text{dens}}(A) \geq \alpha$).

Definition 2.1 ([26])**.** Let (X, f) be a discrete dynamical system on a metric space (X, d) and $\mathscr{F}_i$ be a Furstenberg family for every $i \in \{1, 2\}$. (X, f) is said to be

(1) $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, if there exists some $\delta > 0$ such that for any $x \in X$ and any $\varepsilon' > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon'$ such that the following hold:

   (a) for any $\varepsilon > 0$, $\{n \in \mathbb{Z}^+ : d(f^n(x), f^n(y)) < \varepsilon\} \in \mathscr{F}_1$;
   (b) $\{n \in \mathbb{Z}^+ : d(f^n(x), f^n(y)) > \delta\} \in \mathscr{F}_2$.

(2) Li-Yorke sensitive, if there exists $\delta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$, there exists $y \in X$ with $d(x, y) < \varepsilon$ such that $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ and $\limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta$.

Definition 2.2. Let (X, f) be a discrete dynamical system on a metric space (X, d) and $\mathscr{F}_i$ be a Furstenberg family for every $i \in \{1, 2\}$. (X, f) is said to be weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, if there exists some $\delta > 0$ such that for any nonempty open set $\mathcal{U} \subset X$, there exist $x, y \in \mathcal{U}$ such that

(1) for any $\varepsilon > 0$, $\{n \in \mathbb{Z}^+ : d(f^n(x), f^n(y)) < \varepsilon\} \in \mathscr{F}_1$;

(2) $\{n \in \mathbb{Z}^+ : diam f^n(\mathcal{U}) > \delta\} \in \mathscr{F}_2$, where

$$diam f^n(\mathcal{U}) = \sup\{d(f^n(x), f^n(x)) : x, y \in \mathcal{U}\}.$$

Similarly, for non-autonomous discrete systems one can give the following definitions.

Definition 2.3. Let $(X, f_{1,\infty})$ be a non-autonomous discrete system on a metric space (X, d) and $\mathscr{F}_i$ be a Furstenberg family for every $i \in \{1, 2\}$. $(X, f_{1,\infty})$ is said to be

(1) $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, if there exists some $\delta > 0$ such that for any $x \in X$ and any $\varepsilon' > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon'$ such that

   (a) for any $\varepsilon > 0$, $\{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) < \varepsilon\} \in \mathscr{F}_1$;
   (b) $\{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) > \delta\} \in \mathscr{F}_2$.

(2) Li-Yorke sensitive, if there exists $\delta > 0$ such that for any $x \in X$ and any $\varepsilon > 0$, there exists $y \in X$ with $d(x, y) < \varepsilon$ such that $\liminf_{n\to\infty} d(f_1^n(x), f_1^n(y)) = 0$ and $\limsup_{n\to\infty} d(f_1^n(x), f_1^n(y)) \geq \delta$.

Clearly, $(X, f_{1,\infty})$ is Li-Yorke sensitive, if and only if it is $(\mathscr{F}_{inf}, \mathscr{F}_{inf})$-sensitive.

Definition 2.4. Let $(X, f_{1,\infty})$ be a non-autonomous discrete system on a metric space (X, d) and $\mathscr{F}_i$ be a Furstenberg family for every $i \in \{1, 2\}$. $(X, f_{1,\infty})$ is said to be weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, if there exists some $\delta > 0$ such that for any nonempty open set $\mathcal{U} \subset X$, there exist $x, y \in \mathcal{U}$ such that

(1) for any $\varepsilon > 0$, $\{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) < \varepsilon\} \in \mathscr{F}_1$;

(2) $\{n \in \mathbb{Z}^+ : diam f_1^n(\mathcal{U}) > \delta\} \in \mathscr{F}_2$, where

$$diam f_1^n(\mathcal{U}) = \sup\{d(f_1^n(x), f_1^n(x)) : x, y \in \mathcal{U}\}.$$

In [13], the properties $P(k)$ and $Q(k)$ of Furstenberg families are proposed for studying the problem on iteration invariants for $(\mathscr{F}_1, \mathscr{F}_2)$-scrambled set. Inspired by [13], we define the properties $\widehat{P}(k)$ and $\widehat{Q}(k)$ of the Furstenberg family.

Definition 2.5. Let k be a positive integer and $\mathscr{F}$ be a Furstenberg family.

(1) $\mathscr{F}$ is said to have the property $\widehat{P}(k)$, if for any $F \in \mathscr{F}$, there exists $j \in \{0, 1, \ldots, k-1\}$ such that for each $m \in \mathbb{N}$,

$$F_{k,j,m} := \{i \in \mathbb{Z}^+ : ki + j \in F, i \geq m\} \in \mathscr{F}.$$

(2) $\mathscr{F}$ is said to have the property $\widehat{Q}(k)$, if for any $F \in \mathscr{F}$ and any $m \in \mathbb{N}$,

$$F_{k,m} := \{ki + j \in \mathbb{Z}^+ : j \in \{0, 1, \ldots, k-1\}, i \in F \cap [m, \infty)\} \in \mathscr{F}.$$

Remark 2.6. It is not difficult to verify that both $\mathscr{F}_{inf}$ and $\widehat{\mathscr{M}^\alpha}$ ($\alpha \in [0,1]$) have the properties $\widehat{P}(k)$ and $\widehat{Q}(k)$ for any $k \in \mathbb{N}$.

Let $f_{1,\infty} = (f_n)_{n=1}^\infty$ be a sequence of continuous maps on a metric space (X, d). We say that $(X, f_{1,\infty})$ is a non-autonomous discrete system (see [8]). Also, the following lemma will be applied to the main results.

Lemma 2.7 ([27]). *Suppose that non-autonomous discrete system $(X, f_{1,\infty})$ converges uniformly to a map f. Then for any $\varepsilon > 0$ and any $k \in \mathbb{N}$, there are $\xi(\varepsilon) > 0$ and $N(k) \in \mathbb{N}$ such that for any $x, y \in X$ with $d(x, y) < \xi(\varepsilon)$ and any $n \geq N(k)$, $d(f_n^k(x), f_n^k(y)) < \frac{\varepsilon}{2}$.*

3. Main results

In this section, inspired by [27] we study the problems on iteration invariants for $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity and weak $(\mathscr{F}_1, \mathscr{F}_2)$-sensitivity for non-autonomous discrete systems.

Theorem 3.1. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system which converges uniformly to a map f and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Furstenberg families such that $\mathscr{F}_1$ and $\mathscr{F}_2$ have the property $\widehat{P}(k)$, and that $\mathscr{F}_1$ is positively translation-invariant. If $f_{1,\infty}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, then so is $f_{1,\infty}^{[k]}$ for any integer $k \geq 2$.*

Proof.

(1) Since $\{f_{1,\infty}\}$ converges uniformly to f, by Lemma 2.7, for any $\varepsilon > 0$, there are $\xi(\varepsilon) > 0$ and $N(k) \in \mathbb{N}$ such that for any $x, y \in X$ with $d(x, y) < \xi(\varepsilon)$, any $n \geq N(k)$ and any $j \in \{0, 1, \ldots, k-1\}$, one has $d(f_n^{k-j}(x), f_n^{k-j}(y)) < \frac{\varepsilon}{2}$. Since $f_{1,\infty}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, by the definition, for the above $\xi(\varepsilon) > 0$, any $x \in X$ and any $\overline{\delta} > 0$, there exists $y \in X$ with $d(x, y) < \overline{\delta}$ and

$$F := \{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) < \xi(\varepsilon)\} \in \mathscr{F}_1.$$

As the family $\mathscr{F}_1$ has the property $\widehat{P}(k)$, there exists $j \in \{0, 1 \ldots, k-1\}$ such that for each $m \in \mathbb{N}$,

$$F_{k,j,m} := \{i \in \mathbb{Z}^+ : ki + j \in F, i \geq m\} \in \mathscr{F}_1,$$

i.e.,

$$F_{k,j,m} := \{i \in \mathbb{Z}^+ : d(f_1^{ki+j}(x), f_1^{ki+j}(y)) < \xi(\varepsilon), i \geq m\} \in \mathscr{F}_1.$$

It is clear that for any $i \in F_{k,j,N(k)}$, $ki + j \in F$ and $ki + j + 1 > N(k)$. This implies that

$$d(f_1^{ki+j+k-j}(x), f_1^{ki+j+k-j}(y)) = d(f_{ki+j+1}^{k-j}(f_1^{ki+j}(x)), f_{ki+j+1}^{k-j}(f_1^{ki+j}(y))) < \frac{\varepsilon}{2}.$$

As

$$F_{k,j,N(k)} \subset \{i \in \mathbb{Z}^+ : d(f_1^{k(i+1)}(x), f_1^{k(i+1)}(y)) < \varepsilon\},$$

and $\mathscr{F}_1$ is positively translation-invariant, $F_{k,j,N(k)} + 1 \in \mathscr{F}$. Clearly,

$$F_{k,j,N(k)} + 1 \subset \{i \in \mathbb{Z}^+ : d(f_1^{ki}(x), f_1^{ki}(y)) < \varepsilon\},$$

where $F_{k,j,N(k)} + 1 = \{i + 1 : i \in F_{k,j,N(k)}\}$. By the above argument one has

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki}(x), f_1^{ki}(y)) < \varepsilon\} \in \mathscr{F}_1,$$

where $f_1^{ki} = f_{k(i-1)+1}^k \circ \cdots \circ f_1^k$.

(2) By the definition, there is a $\delta > 0$ such that for the above pair $x, y \in X$,

$$E = \{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) > \delta\} \in \mathscr{F}_2.$$

As $\mathscr{F}_2$ has the property $\widehat{P}(k)$, there exists a $j \in \{0, 1, \dots, k-1\}$ such that for each $m \in \mathbb{N}$,

$$E_{k,j,m} = \{i \in \mathbb{Z}^+ : ki + j \in E, i \geq m\} \in \mathscr{F}_2,$$

i.e.,

$$E_{k,j,m} = \{i \in \mathbb{Z}^+ : d(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \delta, i \geq m\} \in \mathscr{F}_2.$$

Since $\{f_{1,\infty}\}$ converges uniformly to f, by Lemma 2.7, for $\delta > 0$, there exist $\delta(k) > 0$ and $N(k) \in \mathbb{N}$ such that for any pair $x, y \in X$ with $d(x, y) < \delta(k)$ and any $n \geq N(k)$, for each $j \in \{0, 1, \dots, k-1\}$, $d(f_n^j(x), f_n^j(y)) \leq \delta$.

Now, we assert that

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki}(x), f_1^{ki}(y)) > \delta(k), i \geq N(k)\} \in \mathscr{F}_2.$$

If

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki}(x), f_1^{ki}(y)) > \delta(k), i \geq N(k)\} \notin \mathscr{F}_2,$$

then we have

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki}(x), f_1^{ki}(y)) \leq \delta(k), i \geq N(k)\} \in \kappa\mathscr{F}_2.$$

It is easy to see that

$$\{i \geq N(k) : d(f_1^{ki}(x), f_1^{ki}(y)) \leq \delta(k)\} \subset \{i \geq N(k) : d(f_{ki+1}^j[f_1^{ki}(x)], f_{ki+1}^j[f_1^{ki}(y)] \leq \delta\}$$

for any $j \in \{0, 1, \dots, k-1\}$. Therefore,

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \delta, i \geq N(k)\} \in \kappa\mathscr{F}_2$$

for any $j \in \{0, 1, \dots, k-1\}$. That is,

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \delta, i \geq N(k)\} \notin \mathscr{F}_2$$

for any $j \in \{0, 1, \dots, k-1\}$. This is a contradiction, since

$$\{i \in \mathbb{Z}^+ : d(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \delta(k), i \geq N(k)\} \subset \{i \in \mathbb{Z}^+ : d(f_1^{ki+j}(x), f_1^{ki+j}(y)) > \delta(k)\} \in \mathscr{F}_2.$$

Thus, by the definition, $f_{1,\infty}^{[k]}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive. □

Theorem 3.2. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system which converges uniformly to a map f and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Furstenberg families such that $\mathscr{F}_1$ and $\mathscr{F}_2$ have the property $\widehat{Q}(k)$, and that $\mathscr{F}_2$ is negatively translation-invariant. If $f_{1,\infty}^{[k]}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive for some integer $k \geq 2$, then so is $f_{1,\infty}$.*

Proof.

(1) Since $\{f_{1,\infty}\}$ converges uniformly to f, by Lemma 2.7, for any $\varepsilon > 0$, there are $\xi(\varepsilon) > 0$ and $N(k) \in \mathbb{N}$ such that for any $x, y \in X$ with $d(x, y) < \xi(\varepsilon)$ and any $n \geq N(k)$, one has $d(f_n^j(x), f_n^j(y)) < \frac{\varepsilon}{2}$ for each $j \in \{0, 1, \dots, k-1\}$.

Since $f_{1,\infty}^{[k]}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, by the definition, for the above $\xi(\varepsilon) > 0$, any $x \in X$ and $\overline{\delta} > 0$, there exists $y \in X$ with $d(x, y) < \overline{\delta}$ and

$$F = \{n \in \mathbb{Z}^+ : d(f_1^{kn}(x), f_1^{kn}(y)) < \xi(\varepsilon)\} \in \mathscr{F}_1.$$

By Lemma 2.7, we have that

$$d(f_1^{kn+j}(x), f_1^{kn+j}(y)) = d(f_{kn+1}^{j}[f_1^{kn}(x)], f_{kn+1}^{j}[f_1^{kn}(y)]) < \varepsilon$$

for any integer $n \geq N(k)$. As the family $\mathscr{F}_1$ has the property $\widehat{Q}(k)$, there exists a $j \in \{0, 1, \ldots, k-1\}$ such that

$$F_{k,m} = \{kn + j \in \mathbb{Z}^+ : n \in F, n \geq m\} \in \mathscr{F}_1$$

for each $m \in \mathbb{N}$. So, $F_{k,N(k)} \in \mathscr{F}_1$. Clearly, $F_{k,N(k)} \subset \{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) < \varepsilon\}$. Consequently,

$$\{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) < \varepsilon\} \in \mathcal{F}_1.$$

(2) By the definition, there is a $\delta(k) > 0$ such that for the above pair $x, y \in X$,

$$E = \{n \in \mathbb{Z}^+ : d(f_1^{kn}(x), f_1^{kn}(y)) > \delta(k)\} \in \mathscr{F}_2.$$

Since $\{f_{1,\infty}\}$ converges uniformly to f, by Lemma 2.7, for the above $\delta(k) > 0$, there are $\delta > 0$ and $N(k) \in \mathbb{N}$ such that for any $p, q \in X$ with $d(p, q) \leq \delta$ and any $n \geq N(k)$, $d(f_n^j(p), f_n^j(q)) \leq \frac{\delta(k)}{2}$ for each $j \in \{0, 1, \ldots, k-1\}$. Without loss of generality, we can assume that there exists h such that $N(k) = hk$ and $k(i-1) + j > N(k)$ for any integer $i > h$ and any $j \in \{0, 1, \ldots, k-1\}$. Clearly, for any $i \in E$,

$$d(f_1^{ki}(x), f_1^{ki}(y)) = d(f_{k(i-1)+j+1}^{k-j}[f_1^{ki+j}(x)], f_{k(i-1)+j+1}^{k-j}[f_1^{ki+j}(y)]) > \delta(k).$$

By Lemma 2.7, we have that

$$d(f_1^{k(i-1)+j}(x), f_1^{k(i-1)+j}(y)) > \delta$$

for any integer $i > h$ and any $j \in \{0, 1, \ldots, k-1\}$. If

$$d(f_1^{k(i-1)+j}(x), f_1^{k(i-1)+j}(y)) \leq \delta,$$

by Lemma 2.7, we can deduce a contradiction. As

$$\bigcup_{i \in E, i > h} \{(i-1)k, (i-1)k + 1, \ldots, (i-1)k + k - 1\} \subset \{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) > \delta\},$$

and the family $\mathscr{F}_2$ is negatively translation-invariant,

$$\{n \in \mathbb{Z}^+ : d(f_1^n(x), f_1^n(y)) > \delta\} \in \mathscr{F}_2.$$

Thus, $f_{1,\infty}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive. □

By Theorems 3.1 and 3.2, we have the following corollary.

Corollary 3.3. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system which converges uniformly to a map f and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Furstenberg families such that $\mathscr{F}_1$ and $\mathscr{F}_2$ have the properties $\widehat{P}(k)$ and $\widehat{Q}(k)$, and that $\mathscr{F}_1$ and $\mathscr{F}_2$ are translation-invariant. Then the following three results are equivalent:*

(1) *$f_{1,\infty}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive.*

(2) *$f_{1,\infty}^{[k]}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive for some integer $k \geq 2$.*

(3) *$f_{1,\infty}^{[k]}$ is $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive for any integer $k \geq 2$.*

Careful readers can check that some slight changes in the proof of Theorems 3.1 and 3.2 lead to the following theorems.

Theorem 3.4. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system which converges uniformly to a map f and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Furstenberg families such that $\mathscr{F}_1$ and $\mathscr{F}_2$ have the property $\widehat{P}(k)$, and that $\mathscr{F}_1$ is positively translation-invariant. If $f_{1,\infty}$ is weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive, then so is $f_{1,\infty}^{[k]}$ for any integer $k \geq 2$.*

Theorem 3.5. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system which converges uniformly to a map f and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Furstenberg families such that $\mathscr{F}_1$ and $\mathscr{F}_2$ have the property $\widehat{Q}(k)$, and that $\mathscr{F}_2$ is negatively translation-invariant. If $f_{1,\infty}^{[k]}$ is weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive for some integer $k \geq 2$, then so is $f_{1,\infty}$.*

By Theorems 3.4 and 3.5, we have the following corollary.

Corollary 3.6. *Let $(X, f_{1,\infty})$ be a non-autonomous discrete system which converges uniformly to a map f and $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Furstenberg families such that $\mathscr{F}_1$ and $\mathscr{F}_2$ have the properties $\widehat{P}(k)$ and $\widehat{Q}(k)$, and that $\mathscr{F}_1$ and $\mathscr{F}_2$ are translation-invariant. Then the following three results are equivalent:*

(1) *$f_{1,\infty}$ is weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive.*

(2) *$f_{1,\infty}^{[k]}$ is weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive for some integer $k \geq 2$.*

(3) *$f_{1,\infty}^{[k]}$ is weakly $(\mathscr{F}_1, \mathscr{F}_2)$-sensitive for any integer $k \geq 2$.*

At the end of this paper, some examples are given to illustrate some applications of our main results.

Example 3.7. Let (X, f) be a weakly mixing system. Applying [2, Corollary 3.9] implies that (X, f) is Li-Yorke sensitive, i.e., $(\mathscr{F}_{inf}, \mathscr{F}_{inf})$-sensitive. Let $f_{1,\infty} = (f_n = f)_{n=1}^{\infty}$. As $\mathscr{F}_{inf}$ is positively translation-invariant and has the property $\widehat{P}(k)$ for any $k \in \mathbb{N}$, it follows from Theorem 3.1 that $(X, f_{1,\infty}^{[k]})$ is $(\mathscr{F}_{inf}, \mathscr{F}_{inf})$-sensitive for any $k \in \mathbb{N}$.

Example 3.8. Let $\Sigma_2 = \{0,1\}^{\mathbb{Z}} = \{(\ldots, x_{-2}, x_{-1}; x_0, x_1, x_2, \ldots) : x_n \in \{0,1\}, \forall n \in \mathbb{Z}\}$ with the product metric

$$d(x, y) = \sum_{n=-\infty}^{+\infty} \frac{|x_n - y_n|}{2^{|n|}}$$

for any pair $x = (\ldots, x_{-2}, x_{-1}; x_0, x_1, x_2, \ldots)$, $y = (\ldots, y_{-2}, y_{-1}; y_0, y_1, y_2, \ldots) \in \Sigma_2$. The space (Σ_2, d) is called the two-side symbolic space (with two symbols).

Define the map $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ by

$$\sigma(\ldots, x_{-2}, x_{-1}; x_0, x_1, x_2, \ldots) = (\ldots, x_{-2}, x_{-1}, x_0; x_1, x_2, \ldots)$$

for any $(\ldots, x_{-2}, x_{-1}; x_0, x_1, x_2, \ldots) \in \Sigma_2$. Clearly, σ is a homeomorphism and is called the shift map on Σ_2. Define a non-autonomous discrete system $f_{1,\infty} = (f_n)_{n=1}^{\infty}$ with

$$f_n = \begin{cases} \sigma, & n \in \bigcup_{i=0}^{+\infty}\{4i+1, 4i+2, 4i+3\}, \\ \sigma^{-1}, & n \in \{4i : i \in \mathbb{N}\}. \end{cases}$$

It is not difficult to verify that $f_{1,\infty}^{[4]} = (f_{4(n-1)+1}^{4} = f_{4n} \circ \cdots \circ f_{4n-3} = \sigma^2)_{n=1}^{\infty}$. This implies that $f_{1,\infty}^{[4]}$ is $(\widehat{\mathscr{M}^1}, \widehat{\mathscr{M}^1})$-sensitive, as σ^2 is $(\widehat{\mathscr{M}^1}, \widehat{\mathscr{M}^1})$-sensitive. This, together with Remark 2.6, Theorem 3.1, and Theorem 3.2, implies that $(X, f_{1,\infty}^{[k]})$ is $(\widehat{\mathscr{M}^1}, \widehat{\mathscr{M}^1})$-sensitive for any $k \in \mathbb{N}$. In particular, $(X, f_{1,\infty}^{[k]})$ is Li-Yorke sensitive. Clearly, $f_{1,\infty}$ does not converge uniformly. Therefore, this example also shows that there exists a non-autonomous discrete system which does not converge uniformly satisfying the conclusions of Theorems 3.1 and 3.2.

Acknowledgment

We thank Xinxing Wu for very useful discussions and his valuable suggestion related to Example 3.7 and Example 3.8. This research was supported by the National Natural Science Foundation of China (Grant NO. 11501391) and Project of Enhancing School With Innovation of Guangdong Ocean University (Grant No. GDOU2016050207).

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