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# Global stability and stationary pattern of a diffusive prey-predator model with modified Leslie-Gower term and Holling II functional response

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**Abstract**

This paper is concerned with a diffusive prey-predator model with modified Leslie-Gower term and Holling II functional response subject to the homogeneous Neumann boundary condition. Firstly, by upper and lower solutions method, we prove the global asymptotic stability of the unique positive constant steady state solution. Secondly, introducing the cross diffusion, we obtain the existence of non-constant positive solutions. The results demonstrate that under certain conditions, even though the unique positive constant steady state is globally asymptotically stable for the model with self-diffusion, the non-constant positive steady states can exist due to the emergency of cross-diffusion, that is to say, cross-diffusion can create stationary pattern. Finally, using the bifurcation theory and treating cross diffusion as a bifurcation parameter, we obtain the existence of positive non-constant solutions. 

*Keywords:* Prey-predator model, Leslie-Gower term, upper and lower solutions method, stationary pattern, bifurcation.
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## 1. Introduction

The predator-prey relationship is one of the best common relationship in biology. In paper [1], the authors dealt with the following systems:

$$\begin{cases} \dfrac{du}{dt} = u(r_1 - b_1 u - \dfrac{a_1 v}{k_1 + u}), & t > 0, \\ \dfrac{dv}{dt} = v(r_2 - \dfrac{a_2 v}{u + k_2}), & t > 0 \end{cases} \tag{1.1}$$

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with $u(0) \geq 0$ and $v(0) \geq 0$, where $u$ and $v$ represent the prey and predator population densities at time $t$ respectively. Parameters $r_1, b_1, a_1, k_1, r_2, a_2, k_2$ are all positive. These parameters are defined as follows: $r_1$ is the growth rate of prey $u$, $b_1$ measures the strength of competition among individuals of species $u$, $a_1$ is the maximum value which per capita reduction rate of $u$ can attain, $k_1$ (respectively, $k_2$) measures the extent to which environment provides protection to prey $u$ (respectively, to predator $v$), $r_2$ describes the growth rate of $v$, and $a_2$ has a similar meaning to $a_1$. For a more detailed biological background of the model, refer to [1] and the references therein.

Given some reasonable restrictions on the model, the authors determined conditions and established results for boundedness, existence of a positively invariant and attracting set and the global stability of the coexisting interior equilibrium.

As far as (1.1) with the homogeneous Neumann boundary condition is concerned in [2, 3], the authors considered the case $k_1 = k_2 = 0$ and obtained many interesting results for positive non-constant solutions (namely, stationary patterns) in the so-called heterogeneous environment. Papers [10, 12] were mainly devoted to the study of effects of diffusion coefficients on the positive non-constant solutions to (1.1) when $k_1 > 0$ and $k_2 = 0$. For the details, please refer to these references.

In paper [11], after some simple scaling to (1.1), the authors considered the special form of (1.1)

$$\begin{cases} -d_1\Delta u = u(a - u - \dfrac{v}{1+mu}), & x \in \Omega, \\ -d_2\Delta v = v(b - \dfrac{mv}{m+u}), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $a, b$ and $m$ are positive constants. The authors mainly discussed the positive solutions in the case that the parameter m is large, and obtain a complete understanding for the existence, multiplicity and stability of positive solutions of (1.2).

In this paper, we will reconsider the model (1.2) with homogeneous Neumann boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = u(a - u - \dfrac{v}{1+mu}), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = v(b - \dfrac{mv}{m+u}), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) > 0, v(x,0) = v_0(x) > 0, & x \in \Omega. \end{cases} \tag{1.3}$$

In the above, $n$ is the outward unit normal vector of the boundary of $\partial\Omega$ which we will assume is smooth. The homogeneous Neumann boundary conditions indicate that this system is self-contained with zero population flux across the boundary. Parameters $d_1, d_2$, called self-diffusion, are positive.

It is easy to see that, if $a > b$, then system (1.3) has a unique positive constant equilibrium $(\tilde{u}, \tilde{v})$, where $\tilde{u}$ is the unique positive root of equation

$$mu^2 + (\frac{b}{m} + 1 - ma)u + b - a = 0, \tag{1.4}$$

and satisfies

$$\tilde{u} = \frac{ma - 1 - \frac{b}{m} + \sqrt{(ma - 1 - \frac{b}{m})^2 + 4m(a-b)}}{2m} < a, \quad \tilde{v} = \frac{b(m+\tilde{u})}{m}.$$

Firstly, using a comparison argument and iteration technique, we prove the global asymptotic stability of $(\tilde{u}, \tilde{v})$, which implies that the corresponding elliptic system of (1.3)

$$\begin{cases} -d_1\Delta u = u(a - u - \dfrac{v}{1+mu}), & x \in \Omega, \\ -d_2\Delta v = v(b - \dfrac{mv}{m+u}), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.5}$$

has no non-constant positive solutions. Secondly, we introduce the cross diffusion, and investigate the following equations

$$\begin{cases} -d_1\Delta u = u(a - u - \dfrac{v}{1+mu}), & x \in \Omega \\ -\Delta(d_2 v + d_3 uv) = v(b - \dfrac{mv}{m+u}), & x \in \Omega \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where parameter $d_3$ is called cross diffusion coefficient. In this model, $v$ diffuses with flux

$$J = -\nabla(d_2 v + d_3 uv) = -(d_2 + d_3 u)\nabla v - d_3 v \nabla u.$$

We observe that, the part $-d_3 v\nabla u$ of the flux is directed toward the decreasing population density of $u$, which indicate that the prey species congregate and form a huge group to protect themselves from the attack of the predator. The authors also introduced the same cross-diffusion term in the papers [4, 5, 7, 13].

In this paper, we shall adopt some of the mathematical techniques, which are used in the papers [9, 14]. For system (1.6), we have to establish a priori estimates for any positive solutions. Then based on these estimates, we will use some topological degree arguments to obtain some conclusions concerning the existence of positive non-constant steady-state solutions to (1.6).

The organization of this paper is as follows. Section 2 is devoted to the global stability of $(\tilde{u}, \tilde{v})$. In Section 3, we give a priori estimates for positive solutions to (1.6). In Section 4, we study the existence of positive non-constant solutions to system (1.6). Section 5 is concerned with the existence of non-constant positive solutions by virtue of bifurcation theory.

Throughout this paper, we denote by $0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$ the eigenvalues of $-\triangle$ in $\Omega$ with the homogeneous Neumann boundary condition. For any $k \geq 0$, we also denote the multiplicity of $\mu_k$ by $m(\mu_k)$.

## 2. Global stability of positive constant equilibrium

In this section, we will prove that the unique constant positive solution $(\tilde{u}, \tilde{v})$ is globally asymptotically stable using the technique used in the paper [15]. We first state the following lemma:

**Lemma 2.1.** *Let $f(s)$ be a positive $C^1$ function for $s \geq 0$, and let $d > 0$, $\beta \geq 0$ be constants. Further, let $T \in [0, \infty)$ and $w \in C^{2,1}(\Omega \times (T, \infty)) \cap C^{1,0}(\bar{\Omega} \times [T, \infty))$ be a positive function.*

(i) *If $w$ satisfies*

$$\begin{cases} w_t - d\Delta w \leq (\geq) w^{1+\beta} f(w)(\alpha - w), & (x,t) \in \Omega \times (T, \infty), \\ \dfrac{\partial w}{\partial n} = 0, & (x,t) \in \partial\Omega \times [T, \infty), \end{cases}$$

*and the constant $\alpha > 0$, then*

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} w(\cdot, t) \leq \alpha \quad (\liminf_{t\to\infty} \min_{\bar{\Omega}} w(\cdot, t) \geq \alpha).$$

(ii) *If $w$ satisfies*

$$\begin{cases} w_t - d\Delta w \le w^{1+\beta} f(w)(\alpha - w), & (x,t) \in \Omega \times (T, \infty), \\ \dfrac{\partial w}{\partial n} = 0, & (x,t) \in \partial\Omega \times [T, \infty), \end{cases}$$

*and the constant $\alpha \le 0$, then*

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} w(\cdot, t) \le 0.$$

**Theorem 2.2.** *For system* (1.3), *if*

$$a > \frac{b(m+a)}{m} \tag{2.1}$$

*holds, then the positive constant solution* $(\tilde{u}, \tilde{v})$ *is globally asymptotically stable.*

*Proof.* By the maximum principle of parabolic equations, for any initial values$(u_0(x), v_0(x)) > (0,0)$, solutions $(u(x,t), v(x,t))$ of system (1.3) are positive.

From the first equation of system (1.3), we have

$$\frac{\partial u}{\partial t} - d_1 \Delta u \le u(a - u).$$

By Lemma 2.1, we get

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot, t) \le a := \bar{u}_1.$$

For any given $\varepsilon > 0$, there exists $t_1^\varepsilon \gg 1$ so that

$$u(x,t) \le \bar{u}_1 + \varepsilon, \quad \forall x \in \bar{\Omega}, \quad t \ge t_1^\varepsilon.$$

From the second equation of (1.3), for $x \in \bar{\Omega}, t \ge t_1^\varepsilon$,

$$\frac{\partial v}{\partial t} - d_2 \Delta v \le v\left(b - \frac{mv}{m + \bar{u}_1 + \varepsilon}\right).$$

Lemma 2.1 tells us that

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot, t) \le \frac{b(m + \bar{u}_1 + \varepsilon)}{m} := v_1^\varepsilon.$$

In terms of the arbitrariness of $\varepsilon$, we obtain that

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot, t) \le \frac{b(m + \bar{u}_1)}{m} = v_1^0 = \bar{v}_1.$$

Then, there exists $t_2^\varepsilon \ge t_1^\varepsilon$ such that

$$v(x,t) \le \bar{v}_1 + \varepsilon, \quad \forall x \in \bar{\Omega}, \quad t \ge t_2^\varepsilon.$$

In turn, because of $a > \dfrac{b(m+a)}{m}$, from the equation $u$, we have that there exists $\varepsilon_0$ such that $a > \dfrac{b(m+a)}{m} + \varepsilon$, then $\forall 0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} \frac{\partial u}{\partial t} - d_1 \Delta u &\ge u\left(a - u - \frac{\bar{v}_1 + \varepsilon}{1 + mu}\right) \\ &= \frac{u}{1+mu}\left(-u^2 + \left(a - \frac{1}{m}\right)u + \frac{1}{m}(a - (\bar{v}_1 + \varepsilon))\right) \\ &= \frac{u}{1+mu}(u - u_2^\varepsilon)(u_1^\varepsilon - u), \end{aligned}$$

where

$$u_1^\varepsilon = \frac{a - \frac{1}{m} + \sqrt{(a - \frac{1}{m})^2 + 4\frac{1}{m}(a - (\bar{v}_1 + \varepsilon))}}{2} > 0$$

and

$$u_2^\varepsilon = \frac{a - \frac{1}{m} - \sqrt{(a - \frac{1}{m})^2 + 4\frac{1}{m}(a - (\bar{v}_1 + \varepsilon))}}{2} < 0.$$

Lemma 2.1 implies that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geq u_1^\varepsilon.$$

By virtue of the arbitrariness of $\varepsilon$, we get

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geq u_1^0 := \underline{u}_1.$$

Then for $0 < \varepsilon < \underline{u}_1$, there exists $t_3^\varepsilon$, when $t \geq t_3^\varepsilon \geq t_2^\varepsilon$,

$$u(x, t) \geq \underline{u}_1 - \varepsilon, \quad \forall x \in \bar{\Omega}$$

holds. Here

$$\underline{u}_1 = \frac{a - \frac{1}{m} + \sqrt{(a - \frac{1}{m})^2 + 4\frac{1}{m}(a - \bar{v}_1)}}{2}.$$

Equation of $v$ can be rewritten as

$$\frac{\partial v}{\partial t} - d_2\Delta v \geq v(b - \frac{mv}{m + \underline{u}_1 - \varepsilon}).$$

Thus

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot, t) \geq \frac{b(m + \underline{u}_1 - \varepsilon)}{m} := v_2^\varepsilon.$$

By the arbitrariness of $\varepsilon$, we derive that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot, t) \geq v_2^0 := \underline{v}_1 = \frac{b(m + \underline{u}_1)}{m}.$$

For $0 < \varepsilon < \underline{v}_1$, there exists $t_4^\varepsilon$ such that when $t \geq t_4^\varepsilon \geq t_3^\varepsilon$,

$$v(x, t) \geq \underline{v}_1 - \varepsilon, \quad \forall x \in \bar{\Omega}$$

holds. Therefore, equation of $u$ can be rewritten as

$$\begin{aligned}
\frac{\partial u}{\partial t} - d_1\Delta u &\leq u(a - u - \frac{\underline{v}_1 - \varepsilon}{1 + mu}) \\
&= \frac{u}{1 + mu}(-u^2 + (a - \frac{1}{m})u + \frac{1}{m}(a - (\underline{v}_1 - \varepsilon))) \\
&= \frac{u}{1 + mu}(u - u_3^\varepsilon)(u_4^\varepsilon - u).
\end{aligned}$$

Here

$$u_3^\varepsilon = \frac{a - \frac{1}{m} + \sqrt{(a - \frac{1}{m})^2 + 4\frac{1}{m}(a - (\underline{v}_1 - \varepsilon))}}{2} < 0,$$

and

$$u_4^\varepsilon = \frac{a - \frac{1}{m} - \sqrt{(a - \frac{1}{m})^2 + 4\frac{1}{m}(a - (\underline{v}_1 - \varepsilon))}}{2} > 0.$$

Similarly to the above, there exists $t_5^\varepsilon$ such that

$$u(x,t) \geq \bar{u}_2 - \varepsilon, \quad \forall x \in \bar{\Omega}, \quad t \geq t_5^\varepsilon \geq t_4^\varepsilon.$$

Here

$$\bar{u}_2 = \frac{a - \frac{1}{m} + \sqrt{(a-\frac{1}{m})^2 + 4\frac{1}{m}(a - (\underline{v}_1))}}{2} > 0.$$

Let

$$\varphi(s) = \frac{b(m+s)}{m}, \quad \psi(s) = \frac{a - \frac{1}{m} + \sqrt{(a-\frac{1}{m})^2 + 4\frac{1}{m}(a-s)}}{2}, \quad s > 0$$

then $\varphi'(s) > 0, \psi'(s) < 0$, and the constants $\bar{u}_1, \bar{v}_1, \underline{u}_1, \underline{v}_1, \bar{u}_2$ construed above satisfy

$$\underline{u}_1 \leq \liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot,t) \leq \limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot,t) \leq \bar{u}_1,$$

$$\underline{v}_1 \leq \liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot,t) \leq \limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot,t) \leq \bar{v}_1,$$

$$\underline{v}_1 = \varphi(\underline{u}_1) < \varphi(\bar{u}_1) = \bar{v}_1, \quad \underline{u}_1 = \psi(\bar{v}_1) < \psi(\underline{v}_1) = \bar{u}_2 < \bar{u}_1.$$

By the inductive method, we can construct four sequences $\{\underline{u}_i\}, \{\bar{u}_i\}, \{\underline{v}_i\}, \{\bar{v}_i\}$ by

$$\underline{v}_i = \varphi(\underline{u}_i), \quad \varphi(\bar{u}_i) = \bar{v}_i, \quad \underline{u}_i = \psi(\bar{v}_i), \quad \psi(\underline{v}_i) = \bar{u}_{i+1},$$

such that

$$\underline{u}_i \leq \liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot,t) \leq \limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot,t) \leq \bar{u}_i,$$

$$\underline{v}_i \leq \liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot,t) \leq \limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot,t) \leq \bar{v}_i.$$

By the monotonicity of $\varphi, \psi$, we have

$$\underline{v}_{i-1} < \underline{v}_i = \varphi(\underline{u}_i) < \varphi(\bar{u}_i) = \bar{v}_i < \bar{v}_{i-1},$$

$$\underline{u}_{i-1} < \underline{u}_i = \psi(\bar{v}_i) < \psi(\underline{v}_i) = \bar{u}_{i+1} < \bar{u}_i.$$

We may assume that

$$\lim_{t\to\infty} \underline{u}_i = \underline{u}, \quad \lim_{t\to\infty} \underline{v}_i = \underline{v}, \quad \lim_{t\to\infty} \bar{u}_i = \bar{u}, \quad \lim_{t\to\infty} \bar{v}_i = \bar{v}.$$

Evidently, we have $0 < \underline{u} \leq \bar{u}$ and $0 < \underline{v} \leq \bar{v}$, and

$$\underline{u} = \psi(\bar{v}), \quad \bar{u} = \psi(\underline{v}), \quad \underline{v} = \varphi(\underline{u}), \quad \bar{v} = \varphi(\bar{u}). \tag{2.2}$$

Direct computations show that (2.2) is equivalent to

$$a - \underline{u} - \frac{\bar{v}}{1+m\underline{u}} = 0, \quad a - \bar{u} - \frac{\underline{v}}{1+m\bar{u}} = 0, \tag{2.3}$$

$$\underline{v} = \frac{b(m+\underline{u})}{m}, \quad \bar{v} = \frac{b(m+\bar{u})}{m}. \tag{2.4}$$

It is deduced from (2.4) that

$$\bar{v} - \underline{v} = \frac{b}{m}(\bar{u} - \underline{u}). \tag{2.5}$$

Substituting the second (first) equation of (2.4) into the first (second) equation of (2.3) respectively, we have

$$(a - \underline{u})(1 + m\underline{u}) = \frac{b(m+\bar{u})}{m} \tag{2.6}$$

and

$$(a-\bar{u})(1+m\bar{u})=\frac{b(m+\underline{u})}{m}. \tag{2.7}$$

Subtracting (2.6) from (2.7), we have

$$(ma-1)(\bar{u}-\underline{u})-m(\bar{u}+\underline{u})(\bar{u}-\underline{u})-\frac{b}{m}(\bar{u}-\underline{u})=0. \tag{2.8}$$

Suppose, on the contrary, that $\bar{u}\neq\underline{u}$, then $\bar{u}>\underline{u}$ and it follows from (2.8) that

$$m(\bar{u}+\underline{u})=ma-1-\frac{b}{m}. \tag{2.9}$$

Noticing that $\tilde{u}$ satisfies condition (1.4), and by using of (2.8), after some calculations, we have

$$\begin{aligned}(\bar{v}-\underline{v}) &= (a-\underline{u})(1+m\underline{u})-\frac{b(m+\underline{u})}{m}\\ &= a-b+(ma-1-\frac{b}{m})\underline{u}-m\underline{u}^2\\ &= m\tilde{u}^2+(\frac{b}{m}-ma+1)\tilde{u}+(ma-1-\frac{b}{m})\underline{u}-m\underline{u}^2\\ &= (\tilde{u}-\underline{u})[m(\tilde{u}+\underline{u})+\frac{b}{m}-ma+1]\\ &= m(\tilde{u}-\underline{u})(\tilde{u}-\bar{u}).\end{aligned} \tag{2.10}$$

Similarly, we have

$$\begin{aligned}(\bar{v}-\underline{v}) &= \frac{b(m+\bar{u})}{m}-(a-\bar{u})(1+m\bar{u})\\ &= m(\bar{u}-\tilde{u})(\tilde{u}-\underline{u}).\end{aligned} \tag{2.11}$$

Combining (2.10) with (2.11), we have $\bar{u}=\underline{u}=\tilde{u}$, which contradicts with the assumption. Thus $\bar{u}=\underline{u}$, in turn, $\bar{v}=\underline{v}$.

Since system (1.3) has a unique positive constant equilibrium $(\tilde{u},\tilde{v})$, then

$$(\underline{u},\underline{v})=(\bar{u},\bar{v})=(\tilde{u},\tilde{v}).$$

Therefore, for any initial values $(\underline{u},\underline{v})<(u_0,v_0)<(\bar{u},\bar{v})$, as $t\to\infty$, the positive solution $(u(x,t),v(x,t))$ of system (1.3) uniformly converges to $(\tilde{u},\tilde{v})$. □

## 3. A priori estimates

The main purpose of this section is to give a priori positive upper and lower bounds for the positive solutions to (1.6). We first state the following lemma which is due to Lou and Ni [6].

**Lemma 3.1.** *(maximum principle) Let* $g\in C^1(\overline{\Omega}\times R)$.

(i) *If* $w\in C^2(\Omega)\cap C^1(\overline{\Omega})$ *satisfies*

$$\begin{cases}\Delta w+g(x,w)\geq 0, & x\in\Omega,\\ \dfrac{\partial w}{\partial\nu}\leq 0, & x\in\partial\Omega,\end{cases}$$

*and* $w(x_0)=\max_{\bar{\Omega}} w$, *then* $g(x_0,w(x_0))\geq 0$.

(ii) *If* $w\in C^2(\Omega)\cap C^1(\overline{\Omega})$ *satisfies*

$$\begin{cases}\Delta w+g(x,w)\leq 0, & x\in\Omega,\\ \dfrac{\partial w}{\partial\nu}\geq 0, & x\in\partial\Omega,\end{cases}$$

*and* $w(x_0)=\min_{\bar{\Omega}} w$, *then* $g(x_0,w(x_0))\leq 0$.

**Lemma 3.2** (Harnark inequality)**.** *Let $z \in C^2(\Omega) \times C^1(\bar{\Omega})$ be a positive solution to $\Delta z(x) + c(x)z(x) = 0$, where $c \in C(\bar{\Omega})$ satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant $C$ which depends only on $B$ where $\|c\|_\infty \leq B$ such that $\max_{\bar{\Omega}} z \leq C \min_{\bar{\Omega}} z$.*

**Theorem 3.3.** *Let $a, b, m, d, d^* > 0$ be fixed. Then there exist positive constants $C(a, b, m, d, d^*)$ and $\bar{C}(a, b, m, d, d^*)$ such that for all*

$$d_1, d_2 \geq d, \quad 0 < d_3 \leq d^*,$$

*any positive solution $(u, v)$ of (1.6) satisfies*

$$\max_{\bar{\Omega}} u \leq \bar{C}(a, b, m, d, d^*) \min_{\bar{\Omega}} u, \quad \max_{\bar{\Omega}} v \leq \bar{C}(a, b, m, d, d^*) \min_{\bar{\Omega}} v, \tag{3.1}$$

*and*

$$\|u, v\|_{C^{2+\alpha}(\bar{\Omega})} \leq C(a, b, m, d, d^*). \tag{3.2}$$

*Proof.* We will prove that (3.1) and

$$\max_{\bar{\Omega}} u \leq C(a, b, m, d, d^*), \quad \max_{\bar{\Omega}} v \leq C(a, b, m, d, d^*). \tag{3.3}$$

Applying the maximum principle to the equation of $u$ of (1.6), we find that

$$\max_{\bar{\Omega}} u \leq a. \tag{3.4}$$

Let $\varphi(x) = d_2 v + d_3 uv$, and $\varphi(x_0) = \max_{\bar{\Omega}} \varphi$. Thus

$$v(x_0)(b - \frac{mv(x_0)}{m + u(x_0)}) \geq 0.$$

It follows from above that

$$v(x_0) \leq \frac{(m+a)b}{m}.$$

Then we have

$$d_2 \max_{\bar{\Omega}} v \leq \varphi(x_0) = d_2 v(x_0) + d_3 v(x_0)u(x_0) \leq (d_2 + d_3 \max_{\bar{\Omega}} u)v(x_0) \leq (d_2 + d_3 a)\frac{(m+a)b}{m},$$

which implies that

$$\max_{\bar{\Omega}} v \leq (1 + \frac{d^* a}{d})\frac{(m+a)b}{m}. \tag{3.5}$$

Applying the Harnack inequality to the equations of $u$, we see that

$$\max_{\bar{\Omega}} u \leq C_1 \min_{\bar{\Omega}} u, \tag{3.6}$$

where $C_1$ is a positive constant.

We rewrite the equation of $v$ as follows:

$$-\Delta\varphi = \varphi(b - \frac{mv}{m+u})(d_2 + d_3 u)^{-1} = p\varphi, \tag{3.7}$$

where

$$\|p\|_\infty = \|(b - \frac{mv}{m+u})(d_2 + d_3 u)^{-1}\|_\infty \leq \frac{1}{d_2}(b + (1 + \frac{d^* a}{d})\frac{(m+a)b}{m}).$$

Applying the Harnack inequality to (3.7), we follow that there exists a positive constant $C_2$ such that

$$\max_{\bar{\Omega}} \varphi \leq C_2 \min_{\bar{\Omega}} \varphi. \tag{3.8}$$

From the formula of $\varphi$, we have $v = \dfrac{\varphi}{d_2 + d_3 u}$, and

$$\frac{\max_{\bar\Omega} v}{\min_{\bar\Omega} v} \le \frac{\dfrac{\max_{\bar\Omega}\varphi}{d_2 + d_3 \min_{\bar\Omega} u}}{\dfrac{\min_{\bar\Omega}\varphi}{d_2 + d_3 \max_{\bar\Omega} u}} \le \frac{\max_{\bar\Omega}\varphi}{\min_{\bar\Omega}\varphi}\frac{\max_{\bar\Omega} u}{\min_{\bar\Omega} u} \le C_1 C_2.$$

It is deduced from above that

$$\max_{\bar\Omega} v \le C_3 \min_{\bar\Omega} w. \tag{3.9}$$

From (3.6)-(3.9) and (3.4)-(3.5), we prove (3.1) and (3.3), respectively.

Now, we prove the estimate (3.2). Due to (3.3), by the regularity for elliptic equations we have that $u$ and $v(d_2 + d_3 u)$ belong to $C^{1+\alpha}(\bar\Omega)$, and the $C^{1+\alpha}(\bar\Omega)$ norms of them depend only on the parameters $d, d^*$ and the parameters $a, m, b$. It follows that $v \in C^{1+\alpha}(\bar\Omega)$ and $\|v\|_{C^{1+\alpha}(\bar\Omega)}$ depends only on the parameters $d, d^*$ and $a, m, b$. Using the regularity of elliptic equations again, the estimate (3.2) follows. □

In the following we give the positive lower bound of positive solutions. We first state a lemma whose proof is omitted.

**Lemma 3.4.** *Let $d_{ij} \in (0, \infty)$, $i = 1, 2, 3$ and $(u_j, v_j)$ be the corresponding positive solution of (1.6) with $d_i = d_{ij}$. Assume that $d_{ij} \to d_i \in [0, \infty]$ and $(u_j, v_j) \to (u^*, v^*)$ uniformly on $\bar\Omega$. If $u^*, v^*$ are constants, then $(u^*, v^*)$ must satisfy*

$$a - u^* - \frac{v^*}{1 + mu^*} = 0, \qquad b - \frac{mv^*}{m + u^*} = 0.$$

*In particular, if $u^*, v^*$ are positive constants, then $(u^*, v^*) = (\tilde u, \tilde v)$.*

**Theorem 3.5** (lower bound)**.** *Let $d$ and $d^*$ be two fixed positive constants. Then there exists a positive constant $C(d, d^*)$ such that, for any $d_1, d_2 \ge d$, and $0 < d_3 \le d^*$, every positive solution $(u, v, w)$ of (1.6) satisfies*

$$\min_{\bar\Omega} u, \min_{\bar\Omega} v \ge \frac{1}{C(d, d^*)}. \tag{3.10}$$

*Proof.* If the conclusion does not hold, then there exist a sequence $\{d_{1j}, d_{2j}, d_{3j}\}_{j=1}^{\infty}$ satisfying $d_{1j}, d_{2j} \ge d$, and $0 < d_{3j} \le d^*$ and a sequence of corresponding positive solutions $(u_j, v_j)$ of (1.6) with $d_i = d_{ij}$, such that $\min_{\bar\Omega}\{\min_{\bar\Omega} u_j, \min_{\bar\Omega} v_j\} \to 0$. As $d_{1j}, d_{2j} \ge d$, subject to a subsequence, we may assume that $d_{ij} \to d_i \in [d, \infty]$ for $i = 1, 2$ and $d_{3j} \to d_3 \in [0, d^*]$. By (3.2), we may also assume that $(u_j, v_j)$ converges to some nonnegative functions $(u, v)$ in $[C^{2+\alpha}(\bar\Omega)]^2$. It is easy to see that $(u, v)$ also satisfies the estimate (3.2), and $\min_{\bar\Omega} u = 0$ or $\min_{\bar\Omega} v = 0$. Moreover, we observe that, if $d_1, d_2 < \infty$, then $(u, v)$ satisfies (1.6); If $d_1 = \infty$, as $(u_j, v_j)$ satisfies (1.6), then $u$ satisfies $\Delta u = 0$ in $\Omega$ and $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Hence $u$ is a nonnegative constant. Analogous conclusions hold for $d_2$.

Next we derive a contradiction for every possible case.

(1) The case of $d_1, d_2 < \infty$.
If $\min_{\bar\Omega} u = 0$ holds, then $u = 0$ on $\bar\Omega$ by Harnack inequality, and $v$ satisfies

$$\begin{cases} -d_2 \Delta v = v(b - v), & x \in \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.11}$$

Then we obtain that $v = b$ or $v = 0$ on $\bar\Omega$. Thus $(u_j, v_j) \to (0, b)$ or $(u_j, v_j) \to (0, 0)$, which contradict with Lemma 3.4, so $\min_{\bar\Omega} u > 0$.

If $\min_{\bar{\Omega}} v = 0$ holds, then Harnack inequality tells us $v = 0$ on $\bar{\Omega}$, and $u$ satisfies

$$\begin{cases} -d_1\Delta u = u(a-u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.12}$$

It follows from above that $u = a$ on $\bar{\Omega}$. Thus $(u_j, v_j) \to (a, 0)$, which contradicts with Lemma 3.4, so $\min_{\bar{\Omega}} v > 0$.

(2) If $d_1 = \infty$, then $u = \check{u}$ is a nonnegative constant. If $\min\limits_{\bar{\Omega}} u = 0$ holds, then $\check{u} = 0$ on $\bar{\Omega}$, and in turn $v = 0$ or $v = b$ on $\bar{\Omega}$. Then $(u_j, v_j) \to (0, 0)$ or $(u_j, v_j) \to (0, b)$, from Step 1, we know that this is impossible. Hence $\check{u} > 0$.

(2a) When $d_2 < \infty$, if $\min\limits_{\bar{\Omega}} v = 0$ holds, then we have $v = 0$ on $\bar{\Omega}$. Thus $(u_j, v_j) \to (\check{u}, 0)$, and this is a contradiction with Lemma 3.4. Hence, $\min_{\bar{\Omega}} v > 0$.

(2b) When $d_2 = \infty$, then $v = \check{v}$ is a nonnegative constant. If $\min\limits_{\bar{\Omega}} v = 0$ holds, then we have $\check{v} = 0$ on $\bar{\Omega}$. Thus $(u_j, v_j) \to (\check{u}, 0)$, and this is a contradiction with Lemma 3.4. Hence, $\min_{\bar{\Omega}} v > 0$.

(3) If $d_2 = \infty$, then $v = \check{v}$ is a nonnegative constant. If $\min\limits_{\bar{\Omega}} v = 0$ holds, then we have $\check{v} = 0$ on $\bar{\Omega}$. In turn, we have $u = 0$ or $u = a$ on $\bar{\Omega}$. Thus $(u_j, v_j) \to (0, 0)$ or $(u_j, v_j) \to (a, 0)$, and there is a contradiction with Lemma 3.4. Hence, $\check{v} > 0$.

When $d_1 < \infty$, if $\min\limits_{\bar{\Omega}} u = 0$ holds, then we have $u = 0$ on $\bar{\Omega}$. Thus $(u_j, v_j) \to (0, \check{v})$, which contradicts with Lemma 3.4. The proof is complete. □

## 4. Existence results

In this Section, we shall give the existence of non-constant positive solutions to (1.6) due to the emergence of the cross-diffusion.

Let $U = (u, v)^T$, $U_0 = (\tilde{u}, \tilde{v})$, $\Phi(U) = (d_1 u, d_2 v + d_3 uv)^T$, then system (1.6) is translated into

$$-\Delta\Phi(U) = G(U), \quad \text{with } \; G(U) = \begin{pmatrix} u(a - u - \dfrac{v}{1+mu}) \\ v(b - \dfrac{mv}{m+u}) \end{pmatrix}.$$

Obviously, $U$ is a positive solution of (1.6) if and only if

$$\mathcal{F}(U) = U - (I-\Delta)^{-1}\big\{\Phi_{\mathrm{U}}^{-1}(U)[G(U) + \nabla U \Phi_{\mathrm{UU}}(U)\nabla U^T] + U\big\} = 0. \tag{4.1}$$

In particular,

$$\mathcal{F}_U(U_0) = I - (I-\Delta)^{-1}\big\{\Phi_{\mathrm{U}}^{-1}(U_0)G_U(U_0) + I\big\}.$$

Employing the formula for the index of fixed point [8], by the same argument as in [9], we know that in order to facilitate our computation of index$(I - \mathcal{F}, U_0)$, we need to determine the sign of $H(\mu)$, where $H(\mu)$ is defined by

$$H(\mu) = \det\{\Phi_{\mathrm{U}}^{-1}(U_0)\}\det\{\mu\Phi_{\mathrm{U}}(U_0) - G_{\mathrm{U}}(U_0)\}. \tag{4.2}$$

Directly computing, we have $\det\{\Phi_{\mathrm{U}}^{-1}(U_0)\}$ is positive, and

$$\det\{\mu\Phi_{\mathrm{U}}(U_0) - G_{\mathrm{U}}(U_0)\} = C_2(d_3)\mu^2 + C_1(d_3)\mu + C_0(d_3) = C(d_3;\mu), \tag{4.3}$$

where

$$\begin{aligned} C_2(d_3) &= d_1(d_2 + d_3\tilde{u}) > 0, \\ C_1(d_3) &= \frac{d_1 m\tilde{v}}{m+\tilde{u}} - \tilde{u}(d_2 + d_3\tilde{u})(-1 + \frac{m\tilde{v}}{(1+m\tilde{u})^2}) - \frac{d_3\tilde{u}\tilde{v}}{(1+m\tilde{u})}, \\ C_0(d_3) &= \frac{m\tilde{u}\tilde{v}^2}{(1+m\tilde{u})(m+\tilde{u})^2} - \frac{m\tilde{u}\tilde{v}}{m+\tilde{u}}(-1 + \frac{m\tilde{v}}{(1+m\tilde{u})^2}). \end{aligned}$$

In the sequel, we will consider the dependence of $C(\varepsilon, d_3; \mu)$ on parameter $d_3$. Let $\tilde{\mu}_1, \tilde{\mu}_2$ be the roots of $C(d_3; \mu) = 0$ satisfying $Re\tilde{\mu}_1 \le Re\tilde{\mu}_2$, then

$$\tilde{\mu}_1\tilde{\mu}_2 = \frac{\det\{G_U(U_0)\}}{C_2}.$$

It is easy to see that if

$$-1 + \frac{m\tilde{v}}{(1+m\tilde{u})^2} < 0 \tag{4.4}$$

holds, then $\det\{G_U(U_0)\} > 0$. Combining with $C_2 > 0$, we get $\tilde{\mu}_1\tilde{\mu}_2 > 0$.

After some computations,we have the following limitation

$$\begin{aligned}
&\lim_{d_3\to\infty} \frac{C_2(d_3)}{d_3} = d_1\tilde{u} = a_2 > 0,\\
&\lim_{d_3\to\infty} \frac{C_1(d_3)}{d_3} = -\tilde{u}^2(-1 + \frac{m\tilde{v}}{(1+m\tilde{u})^2}) - \frac{\tilde{u}\tilde{v}}{(1+m\tilde{u})} = a_1,\\
&\lim_{d_3\to\infty} \frac{C_0(d_3)}{d_3} = 0,
\end{aligned}$$

that is,

$$\lim_{d_3\to\infty} \frac{C(d_3;\mu)}{d_3} \quad = \quad a_2\mu^2 + a_1\mu = \mu[a_2\mu + a_1].$$

When

$$-\tilde{u}^2(-1 + \frac{m\tilde{v}}{(1+m\tilde{u})^2}) - \frac{\tilde{u}\tilde{v}}{(1+m\tilde{u})} < 0, \tag{4.5}$$

then $a_1 < 0$. We only consider the case $a_1 < 0$. By continuity, we have that when $d_3$ is large enough, $\tilde{\mu}_1 > 0$ and $\tilde{\mu}_2 > 0$, and $\tilde{\mu}_1, \tilde{\mu}_2$ satisfy

$$\lim_{d_3\to\infty} \tilde{\mu}_1 = 0, \qquad \lim_{d_3\to\infty} \tilde{\mu}_2 = -\frac{a_1}{a_2} := \mu^+. \tag{4.6}$$

From the above analysis, we have the following conclusion.

**Theorem 4.1.** *Assume that conditions* (2.1) *and*

$$0 < -\tilde{u}(-1 + \frac{m\tilde{v}}{(1+m\tilde{u})^2}) < \frac{\tilde{v}}{(1+m\tilde{u})} \tag{4.7}$$

*hold, if $\mu^+$ determined by* (4.6) *satisfies $\mu^+ \in (\mu_n, \mu_{n+1})$ for some $n \ge 2$, and $\sum_{k=2}^{n} m(\mu_k)$ is odd, then there exists a $\hat{d}_3 > 0$ such that system* (1.6) *has at least one non-constant positive solution when $d_3 \ge \hat{d}_3$.*

*Proof.* We argue by contradiction. Assume that for some $d_3 = \bar{d}_3 \ge \hat{d}_3$, system (1.6) has no positive non-constant solutions. In the following, we fix $d_3 = \bar{d}_3 \ge \hat{d}_3$.

For $t \in [0,1]$, define

$$\Phi(t;U) = (d_1u, d_2v + td_3uv)^T,$$

and considering the following system

$$\begin{cases} -\Delta\Phi(t;U) = G(U), & x \in \Omega,\\ \dfrac{\partial U}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

Obviously for $0 \le t \le 1$, $U_0$ is the only positive constant solution of system (4.8) and $U$ is a positive solution of system (1.6) if and only if $U$ is a positive solution of system (4.8) when $t = 1$. $U$ is a positive solution of (4.8) if and only if

$$\mathcal{F}(t;U) = U - (I-\Delta)^{-1}\{\Phi_{\mathrm{U}}^{-1}(t;U)[G(U) + \nabla U\Phi_{\mathrm{UU}}(t;U)\nabla U^T] + U\} = 0.$$

By a priori estimates, it is easy to see that any positive solution of system (4.8) must lie in $B$, where

$$B = \{(u, v) \in C(\Omega) \times C(\Omega) \times C(\Omega) : C^{-1} < u, v < C\}.$$

and $\mathcal{F}(t; U) \neq 0$ on $\partial B$. So $\deg(\mathcal{F}(t; U), B, 0)$ is well defined. By the homotopy invariance of topology degree, we have

$$\deg(\mathcal{F}(1, \cdot), B, 0) = \deg(\mathcal{F}(0, \cdot), B, 0). \tag{4.9}$$

Note that

$$H(t; \mu) = \det\{\Phi_{\mathrm{U}}^{-1}(t; U_0)\} \det\{\mu\Phi_{\mathrm{U}}(t; U_0) - G_{\mathrm{U}}(U_0)\},$$

and when $t = 0$, under the condition (4.7), we have that $H(0; \mu) > 0$ for all $i \geq 1$. By virtue of the formula

$$\operatorname{index}(I - \mathcal{F}(t, \cdot), U_0) = (-1)^{\sum_{i \geq 1,\ H(\mu_i) < 0} m(\mu_i)}, \tag{4.10}$$

we have

$$\operatorname{index}(I - \mathcal{F}(0, \cdot), U_0) = (-1)^0 = 1. \tag{4.11}$$

Since we assume that system (1.6) has no positive non-constant solutions, and $\sum\limits_{k=2}^{n} m(\mu_k)$ is odd, then we have

$$\operatorname{index}(I - \mathcal{F}(1, \cdot), U_0) = (-1)^{\sum\limits_{k=2}^{n} m(\mu_k)} = -1. \tag{4.12}$$

On the other hand, from the assumption, we know that $\mathcal{F}(1; U) = 0$ and $\mathcal{F}(0; U) = 0$ have a unique positive solution $U_0$ on $B$. So

$$\deg(\mathcal{F}(0, \cdot), B, 0) = \operatorname{index}(I - \mathcal{F}(0, \cdot), U_0) = 1, \tag{4.13}$$

$$\deg(\mathcal{F}(1, \cdot), B, 0) = \operatorname{index}(I - \mathcal{F}(1, \cdot), U_0) = -1, \tag{4.14}$$

and (4.13)-(4.14) contradict with (4.9). So the proof is complete. □

## 5. Bifurcation

In this section, we will discuss the existence of non-constant positive solutions to system (1.6) by using of bifurcation. In the following, we fix the parameters $a, b, m, d_1, d_2$ and treat $d_3$ as a bifurcation parameter.

**Definition 5.1.** $(\hat{d}_3, U_0)$ is said to be a bifurcation point of system (1.6) if for any $\delta \in (0, \hat{d}_3)$, there exists $d_3 \in [\hat{d}_3 - \delta, \hat{d}_3 + \delta]$ such that system (1.6) has a non-constant positive solution. Otherwise, $(\hat{d}_3, U_0)$ is a regular point.

Define $N = \{\mu > 0 | H(\mu) = 0\}$, and $S_p = \{\mu_2, \mu_3, \mu_4, \cdots\}$, where $H(\mu)$ is defined by (4.2) and $S_p$ consists of the positive spectrum of $-\Delta$ on $\Omega$ with the homogeneous Neumann boundary condition. To emphasize the dependence of $H(\mu)$, $\mathcal{F}(U)$ and $N$ on $d_3$, we write $H(d_3; \mu)$, $\mathcal{F}(d_3; U)$ and $N(d_3)$, respectively, where $\mathcal{F}(U)$ is defined by (4.1).

**Theorem 5.2** (Local bifurcation with respect to $d_3$)**.** *Let* $\hat{d}_3 > 0$.

(1) *If* $S_p \cap N(\hat{d}_3) = \emptyset$, *then* $(\hat{d}_3, U_0)$ *is a regular point of system* (1.6).

(2) *Suppose* $S_p \cap N(\hat{d}_3) \neq \emptyset$, *and the positive roots of* $H(\hat{d}_3; \mu) = 0$ *are all simple. If* $\sum_{\mu_j \in N(\hat{d}_3)} m(\mu_j)$ *is odd, then* $(\hat{d}_3, U_0)$ *is a bifurcation point of system* (1.6).

*Proof.* (1) If $S_p \cap N(\hat{d}_3) = \emptyset$, then $H(\hat{d}_3; \mu_i) \neq 0$ for $i \in \{1, 2, \cdots\}$. So it is evident that 0 is not an eigenvalue of $\mathcal{F}_U(\hat{d}_3, U_0)$, which implies that $\mathcal{F}_U(\hat{d}_3, U_0)$ is a homeomorphism from $X$ to itself. By virtue of the implicit function theorem, it follows that for all $d_3$ close to $\hat{d}_3$, $U = U_0$ is the only solution to $\mathcal{F}(d_3; U) = 0$ in a small neighborhood $B_{U_0,\delta}$ of $U_0$, i.e., $(\hat{d}_3, U_0)$ is a regular point of system (1.6).

(2) Suppose $S_p \cap N(\hat{d}_3) \neq \emptyset$. After a series of calculus, the characteristic polynomial of $\mu_i I - \Phi_U^{-1}(U_0)G_U(U_0)$ is given by

$$\lambda^2 + A_1\lambda + A_2 = 0, \tag{5.1}$$

where

$$A_2 = 3\mu_i^2 + 2\mu_i \frac{C_2}{C_3} + \frac{C_1}{C_3},$$

$$A_3 = -\mu_i^3 - \frac{C_2}{C_3}\mu_i^2 - \mu_i \frac{C_1}{C_3} - \frac{C_0}{C_3}.$$

It is easy to verify that 0 is a simple eigenvalue of $\mu_i I - \Phi_U^{-1}(U_0)G_U(U_0)$ for $\mu_i \in S_p \cap N(\hat{d}_3)$, i.e., 0 is a simple eigenvalue of $\mathcal{F}_U(\hat{d}_3, U_0)$ for $\mu_i \in S_p \cap N(\hat{d}_3)$. On the contrary, we assume that there exists a $\hat{d}_3 > 0$ such that the following are true:

(a) $S_p \cap N(\hat{d}_3) \neq \emptyset$ and $\sum_{\mu_j \in N(\hat{d}_3)} m(\mu_j)$ is odd;

(b) there exists a $\delta \in (0, \hat{d}_3)$ such that for every $d_3 \in [\hat{d}_3 - \delta, \hat{d}_3 + \delta]$, $U = U_0$ is the only solution of $\mathcal{F}(d_3; U) = 0$ in $B_{U_0,\delta}$.

Since $\mathcal{F}(d_3; \cdot)$ is a compact perturbation of the identity function, in view of (b), the Leray–Schauder degree $\deg(\mathcal{F}(d_3; \cdot), B_{U_0,\delta}, 0)$ is well defined and does not depend on $d_3 \in [\hat{d}_3 - \delta, \hat{d}_3 + \delta]$. If $\mathcal{F}_U(d_3; U_0)$ is invertible, then

$$\begin{aligned} \deg(\mathcal{F}(d_3; \cdot), B_{U_0,\delta}, 0) &= \operatorname{index}(I - \mathcal{F}(d_3; \cdot), U_0) \\ &= (-1)^{\nu(d_3)} = (-1)^{\sum_{j \geq 1; H(d_3;\mu_j) < 0} m(\mu_j)} \end{aligned} \tag{5.2}$$

for $d_3 \in [\hat{d}_3 - \delta, \hat{d}_3 + \delta]$.

Let $\tilde{H}(d_3; \mu) = \det\{\Phi_U(U_0)\} H(d_3; \mu)$. For $\mu_j \in S_p \cap N(\hat{d}_3)$, we have $\tilde{H}(\hat{d}_3; \mu_j) = 0$. Direct computations show that

$$\frac{\partial \tilde{H}(\hat{d}_3; \mu_j)}{\partial d_3} = a_2 \mu_j^2 + a_1 \mu_j \neq 0.$$

Then we may select $\delta \ll 1$ such that

$$\frac{\partial \tilde{H}(d_3; \mu_j)}{\partial d_3} \neq 0, \quad \mu_j \in S_p \cap N(\hat{d}_3), \quad d_3 \in [\hat{d}_3 - \delta, \hat{d}_3 + \delta].$$

Therefore,

$$\tilde{H}(\hat{d}_3 + \delta; \mu_j)\tilde{H}(\hat{d}_3 - \delta; \mu_j) < 0,$$

in turn,

$$H(\hat{d}_3 + \delta; \mu_j) H(\hat{d}_3 - \delta; \mu_j) < 0, \quad \mu_j \in S_p \cap N(\hat{d}_3). \tag{5.3}$$

Since $S_p$ does not have any accumulation point, by taking $\delta$ sufficiently small, we may assume that $S_p \cap N(d_3) = \emptyset$ for all $d_3 \in [\hat{d}_3 - \delta, \hat{d}_3) \cup (\hat{d}_3, \hat{d}_3 + \delta]$. Therefore, $\mathcal{F}_U(d_3; U_0)$ is invertible for all $d_3 \in [\hat{d}_3 - \delta, \hat{d}_3) \cup (\hat{d}_3, \hat{d}_3 + \delta]$.

In view of (5.2), we have

$$\begin{aligned} \deg(\mathcal{F}(\hat{d}_3 + \delta; \cdot), B_{U_0,\delta}, 0) &= (-1)^{\nu(\hat{d}_3 + \delta)}, \\ \deg(\mathcal{F}(\hat{d}_3 - \delta; \cdot), B_{U_0,\delta}, 0) &= (-1)^{\nu(\hat{d}_3 - \delta)}. \end{aligned} \tag{5.4}$$

If $\mu_j \in N(\hat{d}_3)$, combining (5.3) with the assumptions, we have that

$$|\nu(\hat{d}_3 - \delta) - \nu(\hat{d}_3 + \delta)| = \sum_{\mu_j \in N(\hat{d}_3)} m(\mu_j) \quad \text{is} \quad \text{odd}. \tag{5.5}$$

By virtue of (5.4), it follows that

$$\deg(\mathcal{F}(\hat{d}_3 + \delta; \cdot), B_{U_0,\delta}, 0) \neq \deg(\mathcal{F}(\hat{d}_3 - \delta; \cdot), B_{U_0,\delta}, 0),$$

which is a contradiction to the homotopy invariance of degree. The proof is complete. □

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