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Stochastic Hopf Bifurcation of a novel finance chaotic system

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Abstract

The paper investigated the existence and stability of the Stochastic Hopf Bifurcation for a novel finance chaotic system with noise by the orthogonal polynomial approximation method, which reduces the stochastic nonlinear dynamical system into its equal deterministic nonlinear dynamical system. And according to the Gegenbauer polynomial approximation in Hilbert space, the financial system with random parameter can be reduced into the deterministic equivalent system. The parameter condition to ensure the appearance of Hopf bifurcation in this novel finance chaotic system is obtained by the Hopf bifurcation theorem. We show that a supercritical Hopf bifurcation occurs at systems' unique equilibriums s_0. In addition, the stability and direction of the Hopf bifurcation is investigated by the calculation of the first Lyapunov coefficient. And the critical value of stochastic Hopf bifurcation is determined by deterministic parameters and the intensity of random parameter in stochastic system. Finally, the simulation results are presented to support the analysis.

Keywords: Stochastic chaos, stability, stochastic Hopf bifurcation, Gegenbauer polynomial approximation.
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1. Introduction

The applications of dynamical systems and chaos involve mathematical biology, financial systems, chaos control, synchronization, electronic circuits, secure communications, image encryption, cryptography and

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neuroscience research [5, 6, 7, 9, 12, 13, 14, 17, 28, 32, 33]. And the stochastic bifurcation and chaos are a hot topic in the area of nonlinear dynamics in the past few decades. The study of dynamical systems is a useful tool to help achieving model, analyze, and understand these phenomena. The chaotic behavior in economic system was first studied in 1985 [11]. The occurrence of this behavior in economics means that the economic system has an inherent indefiniteness. So the study of the finance chaotic system has important value for the stable economic growth. Alexander pointed out that some complicated processes in financial markets needed more in-depth analysis [2]. Liao studied Hopf bifurcation of a chaotic macroeconomic model [20]. In literature [4], the dynamical behavior and slow manifold of a nonlinear finance chaotic system were investigated. In real life, the inevitable indefinitely changes in the installation, measure, material and production as well as with the work environment (such as temperature, humidity, vibration, pressure, time etc.), and these uncertainties can be usually described with a particular statistical characteristics of random variables, leading to the random system widely exists in the nature.

The stochastic systems are widespread in nature, and the demand to the veracity and accuracy of the actual model is become higher and higher. So, more and more random systems are used to depict the dynamic relationship among things, especially stochastic system of with random parameter. There are several methods to analysis the stochastic dynamical systems with random parameters. The first one is the Monte Carlo method [27], which is simple and popular but takes longer time. The second method is the stochastic finite element method [10, 16], which consumes a little time but the random variables are required to be a small amount. Gassert [1]has provided a complete description of these graphs, and then uses these graphs to determine the decomposition of primes in the Chebyshev radical extensions. The third method is the orthogonal polynomial approximation that is based on the theory of orthogonal polynomial expansion [15, 18, 26, 30]. This method is out the limitation of the mentioned above two methods, it has been widely applied in studying the evolutionary random responses of stochastic structure system [8, 29] and the stochastic bifurcation and chaos in some typical dynamical models were successfully analyzed by the Chebyshev polynomial approximation [19, 23, 24, 31]. Ma [21] discussed the stochastic Hopf bifurcation in Brusselator system with random parameter and discovered that different from the deterministic system, the critical value of stochastic Hopf bifurcation is determined not only by deterministic parameters in stochastic system, but also by the intensity of random parameter. For the financial model, the most important is not the absolute value of parameters in the model, but the relationship between the parameters and how relative changes of them affect the system behavior. By choosing the appropriate coordinate system and setting an appropriate dimension to every state variable [25], the further simplified financial model is written as the following system [22]:

$$\begin{cases} \dot{X} = Z + (Y - a)X, \\ \dot{Y} = 1 - bY - X^2, \\ \dot{Z} = -X - cZ, \end{cases} \tag{1.1}$$

where a is the saving; b is the per-investment cost; c is the elasticity of demands of commercials. a, b, c are positive real constants. X is the interest rate, Y is the investment demand, Z is the price exponent.

They investigated the existence of both Hopf bifurcation and topological horseshoe for a novel finance chaotic system. And through rigorous mathematical analysis a Hopf bifurcation occurs at systems' three equilibriums $S_{0,1,2}$ and Hopf bifurcation at equilibrium S_0 is non-degenerate and supercritical. However, they haven't analyzed the stochastic Hopf bifurcation of the system. The deterministic models assume that parameters in the systems are all deterministic irrespective environmental fluctuations. Hence they have some limitations in mathematical modeling of ecological systems, besides they are quite difficult to fitting data perfectly and to predict the future dynamics of the system accurately [3]. In this paper, the Gegenbauer polynomial approximations used to study the stability and Hopf bifurcation of stochastic financial system with random parameters. The rest of this paper is organized as follows. We first transform the original stochastic finance chaotic system into its equivalent deterministic one by orthogonal polynomial approximation in Section 2. Section 3 is devoted to studying existence, direction and stability of Hopf bifurcation of stochastic finance chaotic system. The numerical simulations about the stochastic finance chaotic system are given in Section 4. Section 5 concludes the paper.

2. Gegenbauer polynomial approximations for financial system

The premise of Gegenbauer polynomial approximation is to use the random variables following $\lambda - PDF$ or their derivative $PDFs$ to approximate original random variables. When $\lambda = 0$, $\lambda - PDF$ is the concave probability density function

$$p_{\xi}^{0}(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in [-1, 1].$$

When $\lambda = \frac{1}{2}$, $\lambda - PDF$ is the uniform distribution probability density function

$$p_{\xi}^{\frac{1}{2}}(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in [-1, 1].$$

When $\lambda = 1$, $\lambda - PDF$ is the uniform distribution probability density function

$$p_{\xi}^{1}(x) = \frac{2}{\pi\sqrt{1-x^2}}, \qquad x \in [-1, 1].$$

The $\lambda - PDF$ is a family of bounded $PDFs$ symmetrically distributed in the interval $[-1, 1]$ with a mono-peak or mono-valley, which can be defined with the random variable ξ in the following form:

$$p_{\xi}^{0}(x) = \begin{cases} \rho_\lambda (1-x^2)^{\lambda - \frac{1}{2}}, & |\xi| \leq 1, \\ 0, & |\xi| > 1. \end{cases} \tag{2.1}$$

In which $\lambda \geq 0$ is a parameter and the normalizing coefficient ρ_λ can be expressed as:

$$\rho_\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}.$$

We selected the orthogonal polynomial basis according to $\lambda - PDF$ of the random variable in the equation. As the orthogonal polynomial basis for the $\lambda - PDF$, choose the Gegenbauer polynomials which could be put as follows:

$$G_n(\xi) = \sum_{i=1}^{n} \frac{1}{k!(n-k)!} \frac{(2\lambda)_n + (2\lambda+n)_k}{(\lambda+\frac{1}{2})_k} (\frac{\xi-1}{2})^k, \qquad n = 0, 1, 2. \tag{2.2}$$

While the recurrent formulas for the Gegenbauer Polynomials is

$$\xi G_n^{\lambda}(\xi) = \frac{2\lambda+n-1}{2(\lambda+n)} G_{n-1}^{\lambda} + \frac{n+1}{2(\lambda+n)} G_{n+1}^{\lambda}. \tag{2.3}$$

The orthogonal relationships for the Gegenbauer Polynomials can be derived as

$$\int_{-1}^{1} \rho_{\xi}^{\lambda} G_i^{\lambda}(\xi) G_j^{\lambda}(\xi) d\xi = \begin{cases} b_n^{\lambda}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.4}$$

It's easy to know that the Eq. (1.1) has a unique equilibrium $(0, 1/b, 0)$. Applying the translation

$$\begin{cases} x = X, \\ y = Y - \dfrac{1}{b}, \\ z = Z. \end{cases} \tag{2.5}$$

Then we can obtain the following equation with the unique equilibrium $(0, 0, 0)$

$$\begin{cases} \dot{x} = (\frac{1}{b} - a)x + z + xy, \\ \dot{y} = -by - x^2, \\ \dot{z} = -x - \bar{c}z. \end{cases} \tag{2.6}$$

If a, b is a deterministic parameter, c is a random parameter, and then Eq. (2.6) is a stochastic financial model. Suppose that $\bar{c}$ can be expressed as

$$\bar{c} = c + \delta\xi. \tag{2.7}$$

Respectively, which are all positive constants satisfying inequalities: $a > 1$. Where c is the mean value of $\bar{c}$, ξ is a bounded random variable defined on $[-1, 1]$ with a given arch-like PDF, and δ is the intensity of ξ. Thus, the responses of (2.4) should be a function of time t and the random variable ξ, namely

$$\begin{cases} x = x(t, \xi), \\ y = y(t, \xi), \\ z = z(t, \xi). \end{cases} \tag{2.8}$$

It follows from the orthogonal polynomial approximation that the responses of system (2.6) can be expressed approximately by the following series under condition of the convergence in mean square

$$\begin{cases} x(t, \xi) = \sum_{i=0}^{N} x_i(t) G_i^{\lambda}(\xi), \\ y(t, \xi) = \sum_{i=0}^{N} y_i(t) G_i^{\lambda}(\xi), \\ z(t, \xi) = \sum_{i=0}^{N} z_i(t) G_i^{\lambda}(\xi), \end{cases} \tag{2.9}$$

where

$$x_i(t) = \int_{-1}^{1} \rho_{\xi}^{\lambda} x(t, \xi) G_i^{\lambda}(\xi) d\xi, \;\; y_i(t) = \int_{-1}^{1} \rho_{\xi}^{\lambda} y(t, \xi) G_i^{\lambda}(\xi) d\xi, \;\; z_i(t) = \int_{-1}^{1} \rho_{\xi}^{\lambda} z(t, \xi) G_i^{\lambda}(\xi) d\xi,$$

and $G_i^{\lambda}(\xi)$ represents the i-th orthogonal and N represents the largest order of the polynomials we have taken.

In this paper, we take $N = 1$, then

$$\begin{cases} x(t, \xi) = \sum_{i=0}^{1} x_i(t) G_i^{\lambda}(\xi), \\ y(t, \xi) = \sum_{i=0}^{1} y_i(t) G_i^{\lambda}(\xi), \\ z(t, \xi) = \sum_{i=0}^{1} z_i(t) G_i^{\lambda}(\xi), \end{cases} \tag{2.10}$$

which are approximate solutions with a minimal mean square residual error.

Substituting (2.7) and (2.10) into (2.6), we have

$$\begin{cases} \sum_{i=0}^{1} \dot{x}_i(t)G_i^\lambda(\xi) = (\frac{1}{b} - a)\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi) + \sum_{i=0}^{1} z_i(t)G_i^\lambda(\xi) + \sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi)\sum_{i=0}^{1} y_i(t)G_i^\lambda(\xi), \\ \sum_{i=0}^{1} \dot{y}_i(t)G_i^\lambda(\xi) = -b\sum_{i=0}^{1} y_i(t)G_i^\lambda(\xi) - (\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi))^2, \\ \sum_{i=0}^{1} \dot{y}_i(t)G_i^\lambda(\xi) = -\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi) - c\sum_{i=0}^{1} z_i(t)G_i^\lambda(\xi) - \delta\xi\sum_{i=0}^{1} z_i(t)G_i^\lambda(\xi). \end{cases} \tag{2.11}$$

Since any product of two Gegenbauer polynomials can be reduced into a linear combination of individual Gegenbauer polynomials, the nonlinear terms

$$\sum_{i=0}^{1} y_i(t)G_i^\lambda(\xi)\sum_{i=0}^{1} y_i(t)G_i^\lambda(\xi),\ \ \sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi)\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi),$$

on the right side of Eq. (2.11) can be expanded into

$$\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi)\sum_{i=0}^{1} y_i(t)G_i^\lambda(\xi) = \sum_{i=0}^{2} M_i(t)G_i^\lambda(\xi),\quad (\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi))^2 = \sum_{i=0}^{2} K_i(t)G_i^\lambda(\xi). \tag{2.12}$$

By the recurrent formulas (2.2) of Chebyshev polynomials, the term of the third equation of (2.10) can be reduced to

$$\delta\xi\sum_{i=0}^{1} z_i(t)G_i^\lambda(\xi) = \delta\sum_{i=0}^{1}(z_{i-1}(t) + z_{i+1}(t))\alpha_i^\lambda G_i^\lambda(\xi), \tag{2.13}$$

where x_{-1} and x_2 are supposed to be zero. Substituting (2.12) and (2.13) into (2.11), we have

$$\begin{cases} \sum_{i=0}^{1} \dot{x}_i(t)G_i^\lambda(\xi) = (\frac{1}{b} - a)\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi) + \sum_{i=0}^{1} z_i(t)G_i^\lambda(\xi) + \sum_{i=0}^{2} M_i(t)G_i^\lambda(\xi), \\ \sum_{i=0}^{1} \dot{y}_i(t)G_i^\lambda(\xi) = -b\sum_{i=0}^{1} y_i(t)G_i^\lambda(\xi) - \sum_{i=0}^{2} K_i(t)G_i^\lambda(\xi), \\ \sum_{i=0}^{1} \dot{y}_i(t)G_i^\lambda(\xi) = -\sum_{i=0}^{1} x_i(t)G_i^\lambda(\xi) - c\sum_{i=0}^{1} z_i(t)G_i^\lambda(\xi) - \frac{\delta}{2}\sum_{i=0}^{1}(z_{i-1}(t) + z_{i+1}(t))\alpha_i^\lambda G_i^\lambda(\xi). \end{cases} \tag{2.14}$$

Multiplying $G_i^\lambda(\xi), (i = 1, 2, 3, 4)$ to both sides of (2.14) in sequence and then taking expectation with respect to ξ, owing to the orthogonal relationship of Gegenbauer polynomials, we finally have

$$\begin{cases} \dot{x}_0 = (\frac{1}{b} - a)x_0 + z_0 + M_0, \\ \dot{y}_0 = -by_0 - K_0, \\ \dot{z}_0 = -x_0 - cz_0 - \frac{\delta}{2}z_1, \\ \dot{x}_1 = (\frac{1}{b} - a)x_1 + z_1 + M_1, \\ \dot{y}_1 = -by_1 - K_1, \\ \dot{z}_1 = -x_1 - cz_1 - \frac{\delta}{2}z_1. \end{cases} \tag{2.15}$$

3. The stochastic Hopf bifurcation analysis

3.1. Existence of Hopf bifurcation

The Jacobian matrix J of the system (2.15) at the equilibrium $S=(0,0,0,0,0,0)$ is

$$J=\begin{pmatrix} \frac{1}{b}-a & 0 & 1 & 0 & 0 & 0\\ 0 & -b & 0 & 0 & 0 & 0\\ -1 & 0 & -c & 0 & 0 & -\frac{\delta}{2}\\ 0 & 0 & 0 & \frac{1}{b}-a & 0 & 1\\ 0 & 0 & 0 & 0 & -b & 0\\ 0 & 0 & -\frac{\delta}{2} & -1 & 0 & -c \end{pmatrix}. \tag{3.1}$$

With aid of Maple, we obtained the characteristic equation as follows:

$$f(\lambda)=a_0\lambda^6+a_1\lambda^5+a_2\lambda^4+a_3\lambda^3+a_4\lambda^2+a_5\lambda+a_6, \tag{3.2}$$

where

$$\begin{aligned}
a_0&=1,\ a_1=\frac{2(cb+b^2+ab-1)}{b},\\
a_2&=\frac{-8b^2-\delta^2b^2+16cb^3-16cb+16ab^2c+4c^2b^2+16ab^3+4b^4-8ab+4a^2b^2+4}{4b^2},\\
a_3&=\frac{1}{2b^2}(4c+4b^3-\delta^2b^3+b\delta^2+4c^2b^3-4c^2b+4ab^4+4cb^4-4ab^2-12cb^2+4a^2b^3\\
&\quad-\delta^2ab^2+4ac^2b^2+16cab^3+4a^2b^2c-8abc),\\
a_4&=\frac{1}{4b^2}(4c^2b^4+4a^2b^4+8b^4-\delta^2-b^4\delta^2+4c^2+8cb+8ab^3-8b^2+2ab\delta^2-a^2b^2\delta^2+16acb^4\\
&\quad-8ac^2b+4a^2b^2c^2+4\delta^2b^2-16c^2b^2-4a\delta^2b^3+16ca^2b^3+16ac^2b^3-24acb^2),\\
a_5&=\frac{1}{2b}(-4c^2b^2+4ab^3+4cb^3+\delta^2b^2-4cb-a\delta^2b^3+4ac^2b^3+4a^2c^2b^2+4a^2cb^3\\
&\quad-\delta^2+4c^2+2a\delta^2b-8ac^2b-a^2\delta^2b^2),\\
a_6&=c^2-2cb+b^2-\frac{1}{4}\delta^2-\frac{1}{4}\delta^2a^2b^2+a^2b^2c^2+\frac{1}{2}\delta^2-2abc^2+2cab^2.
\end{aligned}$$

Lemma 3.1. *By the definition of Hopf bifurcation, we know that if* (3.2) *has a pair of conjugate complex roots* $\lambda_{1,2}=\alpha(c)\pm i\omega(c)$ *and other real root* $\lambda_{3,4,5,6}$*, Hopf bifurcation occurs when the bifurcation parameter* $c=c_0$*, meet the conditions*
(i) $\alpha(c_0)=0$*,*
(ii) $\omega(c_0)>0$*,*
(iii) $\dot{\alpha}(c_0)\neq 0$*,*
then Hopf bifurcation will occur in the system, where the Hopf bifurcation value, that is $c=(2a+\delta)/2$*,* (3.2) *has a pair of conjugate pure virtual roots* $\lambda_{1,2}=\pm\frac{1}{2}c\sqrt{(\frac{2a-2}{c}-6)(\frac{2a-2}{c}-2)}i$ *and* $\lambda_{3,4,5,6}$ *are less than zero.*

Lemma 3.2. *The stochastic system* (2.6) *undergoes the Hopf bifurcation at the equilibrium* $(0,0,0)$ *when* c *passes through the critical value* $c_0=(2a+\delta)/2$*.*

Proof. According to Lemma 3.1, we let $\frac{2-2bc-2ab-b\delta}{4b}=0$ and get the Hopf bifurcation critical value $c_0=(2a+\delta)/2$, then substitute it into the eigenvalues:

$$\begin{aligned}
&\lambda_1=-\frac{2a+\delta}{4}\sqrt{(\frac{4a-4}{2a+\delta}-6)(\frac{4a-4}{2a+\delta}-2)}i,\ \lambda_2=\frac{2a+\delta}{4}\sqrt{(\frac{4a-4}{2a+\delta}-6)(\frac{4a-4}{2a+\delta}-2)}i,\\
&\lambda_3=-\frac{\delta-a}{4}+\frac{1}{2}<0,\ \lambda_4=-\frac{\delta-a}{4}-\frac{1}{2}<0,\ \lambda_5=-\frac{1}{2a+\delta},\ \lambda_6=-\frac{1}{2a+\delta},\ ,
\end{aligned}$$

$$
\begin{aligned}
\frac{d\lambda}{dc} &= \frac{df(\lambda)/dc}{df(\lambda)/d\lambda} = \frac{2\lambda^5 + 2(2b - 2/b + 3a + \delta/2)\lambda^4 + Q\lambda^3 + R\lambda^2 + Y\lambda + X}{6\lambda^5 + \frac{10(cb+b^2+ab-1)}{b}\lambda^4 + M\lambda^3 + N\lambda^2 + O\lambda + P},\\
M &= \frac{1}{b^2}(-8b^2 - \delta^2 b^2 + 16cb^3 - 16cb + 16ab^2c + 4c^2b^2 + 16ab^3 + 4b^4 - 8ab + 4a^2b^2 + 4),\\
N &= \frac{3}{2b^2}(4c + 4b^3 - \delta^2 b^3 + b\delta^2 + 4c^2b^3 - 4c^2b + 4ab^4 + 4cb^4 - 4ab^2 - 12cb^2\\
&\quad + 4a^2b^3 - \delta^2 ab^2 + 4ac^2b^2 + 16cab^3 + 4a^2b^2c - 8cab),\\
O &= \frac{1}{2b^2}(4c^2b^4 + 4a^2b^4 + 8b^4 - \delta^2 - b^4\delta^2 + 4c^2 + 8cb + 8ab^3 - 8b^2 + 2ab\delta^2 - \delta^2 a^2b^2\\
&\quad + 16acb^4 - 8ac^2b + 4a^2c^2b^2 + 4b^2\delta^2 - 16c^2b^2 - 4ab^3\delta^2 + 16ca^2b^3 + 16ac^2b^3 - 24acb^2),\\
P &= \frac{1}{2b}(-4c^2b^2 + 4ab^3 + 4cb^3 + b^2\delta^2 - 4bc - a\delta^2 b^3 + 4ac^2b^3 + 4a^2c^2b^2\\
&\quad + 4a^2cb^3 - \delta^2 + 4c^2 + 2ab\delta^2 - 8ac^2b - a^2b^2\delta^2),\\
Q &= \frac{1}{b^2}(2 + 4b^3c + 4bc + 2b^4 - 6b^2 + 4ab^2c + 2a^2b^2 - 4ab),\\
R &= \frac{1}{b^2}(2b^4c + 2c + 2b + 4ab^4 - 4abc + 2a^2b^2c - 8b^2c + 4a^2b^3 + 8ab^3c - 6ab^2),\\
Y &= \frac{1}{b}(-4b^2c + 2b^3 - 2b + 4ab^3c + 4a^2b^2c + 2a^2b^3 + 4c - 8abc),\\
X &= 2c - 2b + 2a^2b^2c - 4abc + 2ab^2,\\
\frac{dRe\lambda}{dc}|_{c=c_0} &= \frac{S - U(\frac{2a+\delta}{4})^2(\frac{4a-4}{2a+\delta} - 6)(\frac{4a-4}{2a+\delta} - 2) + V}{ZW(\frac{2a+\delta}{4})^2(\frac{4a-4}{2a+\delta} - 6)(\frac{4a-4}{2a+\delta} - 2)} \neq 0,\\
U &= \frac{1}{b^2}(2b^4a + b^4\delta + 2a + \delta + 2b + 4ab^4 - 4a^2b - 2ab\delta + 2a^3b^2\\
&\quad + a^2b^2\delta - 8ab^2 - 4b^2\delta + 12a^2b^3 + 4ab^3\delta - 6ab^2),\\
V &= 2a + \delta - 2b + 2a^3b^2 + a^2b^2\delta - 2a^2b + 2ab\delta + 2ab^2,\\
W &= 4a + 2\delta + 4b^3 - b^3\delta^2 + b\delta^2 + (2a+\delta)^2b^3 - (2a+\delta)^2b + 4ab^4\\
&\quad + 2(2a+\delta)b^4 - 4ab^2 - 6(2a+\delta)b^2 + 4a^2b^3 - ab^2\delta^2 + a(2a+\delta)^2b^2\\
&\quad + 8a(2a+\delta)^2b^3 + 2a^2(2a+\delta)b^2 - 4(2a+\delta)ab,\\
S &= 2(2b - 2/b + 3a + \delta^2)(\frac{2a+\delta}{4})^4(\frac{4a-4}{2a+\delta} - 6)^2(\frac{4a-4}{2a+\delta} - 2)^2,\\
Z &= (11ab + 5b\delta + b^2 - 1)(\frac{2a+\delta}{4})^4(\frac{4a-4}{2a+\delta} - 6)^2(\frac{4a-4}{2a+\delta} - 2)^2\frac{3}{2b^2}.
\end{aligned}
$$

According to the Hopf bifurcation theory, c_0 is the system Hopf bifurcation critical value. When the parameter c pass through the critical value, the system (2.15) occurs the Hopf bifurcation in the equilibrium $S = (0, 0, 0, 0, 0, 0)$ if $\delta > 2 + a$. □

3.2. Direction and stability of the Hopf bifurcation

In this section, we further investigate the Hopf bifurcation of the system (2.15) by the calculation of the fist Lyapunov coefficient [29]. Let C^n

(1) $< x, y >= \overline{< x, y >}, < x, y >= \overline{x}^T y = \sum_{i=0}^{n} \overline{x}_i y_i$;
(2) $\alpha, \beta \in C, < x, \alpha y + \beta z >= \alpha < x, y > +\beta < x, y >, x, y, z \in C^n$;
(3) $< x, y >\geq 0$, if and only if $x = 0, < x, x >= 0$.

Consider the continuous-time nonlinear dynamical system

$$\dot{x} = f(x, v), \quad (x \in R^n), \tag{3.3}$$

where $v \in R^n$ is considered as the bifurcation parameter. As $v = v_c$, the Eq. (2.9) has the equilibrium $x = x_0$, and the right of the Eq. (2.9) can be expressed as

$$\begin{aligned} &F(x) = Jx + N(x), \\ &N(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + o(||x||)^4, \end{aligned} \tag{3.4}$$

where J is the Jacobian matrix of the Eq. (2.9), $B(x, x)$ and $C(x, x, x)$ are bilinear and trilinear functions respectively which can be written as

$$\begin{aligned} B_i(x, y) &= \sum_{j,k=1}^{n} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}|_{\xi=0} x_j y_k, i = 1, ..., n, \\ C_i(x, y, z) &= \sum_{j,k,l=1}^{n} \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}|_{\xi=0} x_j y_k z_l, i = 1, ..., n. \end{aligned} \tag{3.5}$$

Suppose that the Jacobian matrix J has a pair of complex eigenvalues on the imaginary axis $\lambda_{1,2} = \pm i\omega(\omega > 0)$, and these eigenvalues are the only eigenvalues with $Re\lambda = 0$. Let $p \in C^n$ be a complex eigenvector corresponding to λ_1 and $q \in C^n$ be an adjoint eigenvector which satisfy the following properties

$$Jq = i\omega q, J\overline{q} = -i\omega\overline{q}, J^T p = -i\omega p, J^T\overline{p} = -i\omega\overline{p}, < p, q >= \textstyle\sum_{i=1}^{n} \overline{p}_i q_i = 1. \tag{3.6}$$

We also define the following coefficients

$$\begin{aligned} &G_{20} =< p, B(q, q) >, G_{11} =< p, B(q, \overline{q}) >, G_{02} =< p, B(\overline{q}, \overline{q}) >, \\ &G_{21} =< p, C(q, q, \overline{q}) > -2 < p, B(q, J^{-1}B(q, \overline{q})) > + < p, B(\overline{q}, (2i\omega E - J)^{-1}B(q, q)) > \\ &\qquad + \frac{1}{i\omega} < p, B(q, q) >< p, B(q, \overline{q}) > -\frac{2}{i\omega}| < p, B(q, \overline{q}) > |^2 - \frac{1}{3i\omega}| < p, B(\overline{q}, \overline{q}) > |^2, \end{aligned}$$

then the first Lyapunov coefficient at the origin is defined by

$$L_1(0) = \tfrac{1}{2\omega^2} Re(iG_{20}G_{11} + \omega G_{21}). \tag{3.7}$$

The $N(x)$ in Eq. (2.9) can be expressed as

$$N(x) = \begin{cases} N_1(x) = x_0 y_0 + y_1 x_1, \\ N_2(x) = x_0^2 + x_1^2, \\ N_3(x) = 0, \\ N_4(x) = x_0 y_1 + y_0 x_1, \\ N_5(x) = 2x_0 x_1, \\ N_6(x) = 0, \end{cases} \tag{3.8}$$

where $x = (x_0, y_0, z_0, x_1, y_2, z_3)^T$. Then for Eq. (2.9) can be obtained

$$B_i(\xi, \eta) = \textstyle\sum_{j,k=1}^{n} (\frac{\partial^2 N_i(x)}{\partial x_j \partial y_k})|_{x=0} \xi_j \eta_k, i = 0, ..., n, C_i(\xi, \eta, \gamma) = 0.$$

The linear combination of B is

$$
\begin{aligned}
x=(x_0,y_0,z_0,x_1,y_2,z_3)^T B(\xi,\eta)&=(B_1,B_2,B_3,B_4,B_5,B_6),\\
B_1(\xi,\eta)=\xi_1\eta_2+\xi_4\eta_5,\ B_2(\xi,\eta)&=0,\ B_3(\xi,\eta)=0,\\
B_4(\xi,\eta)=\xi_1\eta_5+\xi_4\eta_2,\ B_5(\xi,\eta)&=0,\ B_6(\xi,\eta)=0.
\end{aligned}
$$

As $\lambda_1=-\frac{2a+\delta}{2}\sqrt{(\frac{4a-4}{2a+\delta}-6)(\frac{4a-4}{2a+\delta}-2)}i$, with the aid of Maple, the $p,q\in C^n$ can be computed

$$q=(1,1,0,0,1,1), \tag{3.9}$$

$$
\begin{aligned}
p=(&-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}},4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}},\\
&\frac{4(2a+\delta)+2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2+2},\frac{4(2a+\delta)+2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2+2},0,0).
\end{aligned}
\tag{3.10}
$$

Using the Maple, the following results are obtained

$$
J^{-1}=\begin{pmatrix}
\frac{(a-2)(2a+\delta)(\delta^2-a)}{E} & 0 & \frac{2(2a+\delta)(\delta+a-2)}{E} & \frac{-2\delta}{E} & 0 & -\frac{2\delta(\delta+a)}{E}\\
0 & -1 & 0 & 0 & 0 & 0\\
\frac{1}{a-1} & 0 & \frac{\delta-2}{2(\delta-1)(1-a)} & 0 & 0 & \frac{-\delta}{2(\delta-1)(1-a)}\\
0 & 0 & \frac{-a\delta}{2(\delta-1)(1-a)} & \frac{1}{a-1} & 0 & \frac{a(2-\delta)}{2(\delta-1)(1-a)}\\
0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & \frac{-\delta}{2(\delta-1)(1-a)} & \frac{1}{a-1} & 0 & \frac{\delta-2}{2(\delta-1)(1-a)}
\end{pmatrix}
\tag{3.11}
$$

$$
\begin{aligned}
&E=12a^2\delta^2+12a^3\delta+4a\delta^3+4a^4-8a^2-12a\delta-4\delta^2+4,\\
&J^{-1}B(q,\overline{q})=(\frac{(a-2)(2a+\delta)(\delta^2-a)-4\delta}{E},0,\frac{-2(a+\delta-2)(\delta+2a)+4\delta(\delta+a)}{E},\\
&\qquad\frac{-2\delta+2(2a+\delta)^2(\delta+a)+2\delta^2(a-1)-4}{E},0,\frac{2\delta(\delta+a)-4(a+\delta-2)(2a+\delta)}{E})^T,\\
&(2i\omega E-J)^{-1}B(q,q)=(Hi-\frac{1}{b}+a,0,0,2Hi-\frac{1}{b},0,-2)^T,
\end{aligned}
$$

$$H=\frac{2a+\delta}{2}\sqrt{\frac{4a(\frac{4a-4}{2a+\delta}-2)-4}{2a(\frac{4a-4}{2a+\delta}-2)+\delta(\frac{4a-4}{2a+\delta}-2)}}, \tag{3.12}$$

$$B(\overline{q},(2i\omega E-J)^{-1}B(q,q))=(3Hi+2a,0,0,2Hi-\frac{1}{b},0,0)^T,$$

$$
\begin{aligned}
<p,B(\overline{q},(2i\omega E-J)^{-1}B(q,q))>=&(3Hi+2a)(-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}})\\
&+\frac{4(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2},
\end{aligned}
\tag{3.13}
$$

$$B(q,J^{-1}B(q,\overline{q}))=(\frac{4\delta(a+\delta)-2(2a+\delta)(\delta-2+a)-}{E},0,0,\frac{2\delta^2(a-1)-2\delta+2(a+\delta)(\delta+2a)^2}{E},0,0),$$

$$
\begin{aligned}
<p, B(q, J^{-1}B(q,\overline{q}))>=&(-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}})(\frac{4\delta(a+\delta)-2(a+\delta-2)(2a+\delta)}{E}) \\
&+(\frac{4(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2})(\frac{2\delta^2(a-1)-2\delta+2(a+\delta)(\delta+2a)^2}{E}),
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
&<p, C(q,q,\overline{q})>=0, \quad \frac{1}{i\omega}G_{20}G_{11}=A+B,\\
&A=\frac{64\sqrt{(4a+3\delta+2)(2+\delta)}(a+\delta+\sqrt{(a+\delta)^2-\frac{1}{4}})(4a+2\delta-\sqrt{2(2a+3\delta)(2a-\delta)-4})}{(4a+3\delta+2)(2+\delta)(4a^2-\delta^2-2)},\\
&B=\frac{\sqrt{(4a+3\delta+2)(2+\delta)}}{4a+3\delta+2}[32(a+4a+3\delta+2)^2+32(a+4a+3\delta+2)\sqrt{(a+4a+3\delta+2)^2-\frac{1}{4}}-4\\
&\quad-\frac{384a^2+384a4a^2-\delta^2-2-324a^2-\delta^2-2^2-64}{(4a^2-\delta^2-2)^2}\\
&\quad-\frac{64(a+4a^2-\delta^2-2)\sqrt{2(2a+34a^2-\delta^2-2)(2a-4a^2-\delta^2-2)-4}}{(4a^2-\delta^2-2)^2}],\\
&<p, B(q,\overline{q})>=(-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}})+\frac{8(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}i,\\
&\frac{2}{i\omega}|<p, B(q,\overline{q})>|^2=\frac{2}{i\omega}K, \quad \frac{1}{3i\omega}|<p, B(\overline{q},\overline{q})>|^2=\frac{1}{3i\omega}K,\\
&K=32(a+\delta)^2+\frac{64(2a+\delta)^2+32(2a+3\delta)(2a-\delta)-64}{(4a^2-\delta^2-2)^2}+32(a+\delta)\sqrt{(a+4a+3\delta+2)^2-\frac{1}{4}}\\
&\quad-\frac{-64(2a+\delta)\sqrt{2(2a+34a^2-\delta^2-2)(2a-4a^2-\delta^2-2)-4}}{(4a^2-\delta^2-2)^2}-4,
\end{aligned}
$$

$$
G_{20}=<p, B(q,q)>=-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}}+\frac{8(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}i, \tag{3.15}
$$

$$
G_{11}=<p, B(q,\overline{q})>=-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}}+\frac{8(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}i, \tag{3.16}
$$

$$
G_{02}=<p, B(\overline{q},\overline{q})>=-4(a+\delta)-4\sqrt{(a+\delta)^2-\frac{1}{4}}+\frac{8(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}i, \tag{3.17}
$$

$$
\begin{aligned}
G_{21}=&<p, C(q,q,\overline{q})>-2<p, B(q, J^{-1}B(q,\overline{q}))>+<p, B(\overline{q},(2i\omega E-J)^{-1}B(q,q))>\\
&+\frac{1}{i\omega}<p, B(q,q)><p, B(q,\overline{q})>-\frac{2}{i\omega}|<p, B(q,\overline{q})>|^2-\frac{1}{3i\omega}|<p, B(\overline{q},\overline{q})>|^2\\
=&(8a+8\delta+8\sqrt{(a+\delta)^2-\frac{1}{4}})(\frac{-2(a+\delta-2)(2a+\delta)+4\delta(a+\delta)}{E})\\
&+\frac{8(2a+\delta)+4\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}\frac{-2\delta+2(a+\delta)(2a+\delta)^2+2\delta^2(a-1)-4}{E}
\end{aligned}
$$

$$
\begin{aligned}
&-(\frac{1}{b}-a)\frac{4(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}+A\\
&+[-12H\sqrt{(a+\delta)^2-\frac{1}{4}}+2H\frac{4(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}\\
&+B+\frac{14\sqrt{(4a+3\delta+2)(\delta+2)}}{3(4a+3\delta+2)(\delta+2)}K]i.
\end{aligned}
$$

Substitute the Eqs. (3.13-3.15) into (3.4), we can get the first Lyapunov coefficient as follows:

$$
\begin{aligned}
L_1(0) &= \frac{1}{2\omega^2}Re(iG_{20}G_{11}+\omega G_{21})\\
&= \frac{(64a+64\delta+64\sqrt{(a+\delta)^2-1/4})(4a+2\delta-\sqrt{2(2a+3\delta)(2a-\delta)-4})}{(4a+3\delta+2)(\delta+2)(4a^2-\delta^2-2)}\\
&-\frac{\sqrt{(4a+3\delta+2)(\delta+2)}}{(4a+3\delta+2)(\delta+2)}[(8a+8\delta+8\sqrt{(a+\delta)^2-\frac{1}{4}})(\frac{4\delta(a+\delta)}{E})\\
&+\frac{4(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}\frac{-2\delta+2(a+\delta)(2a+\delta)^2+2\delta^2(a-1)-4}{E}\\
&-(\frac{1}{b}-a)\frac{4(2a+\delta)-2\sqrt{2(2a+3\delta)(2a-\delta)-4}}{4a^2-\delta^2-2}+A]\neq 0.
\end{aligned}
\tag{3.18}
$$

As we choose the parameters $0.5<\delta<2, a>1$, the first Lyapunov coefficient $L_1(0)<0$ corresponding to different random intensity. It is to say that as there is a supercritical Hopf bifurcation at the point $(x,y,z)=(0,0,0)$ for stochastic financial system in Eq. (2.7). Otherwise, the Hopf bifurcation is subcritical. In next section, we will verify the theoretical analysis by numerical simulation.

4. The numerical simulation example

We fixed $a=2.949$, and chose the initial condition $X_0=(2.321,0.821,0.75851,0.652,0.516,0.634)$. The Eq. (2.7) is a deterministic Financial system when the random intensity $\delta=0, c=\bar{c}$. We know that when $c=c_0$, the deterministic financial system undergo the supercritical Hopf bifurcation at the equilibrium. When the parameter $a=2.949$, we can get the critical value $\frac{2a+\delta}{2}$, and the deterministic chaotic system undergo the supercritical Hopf bifurcation at the equilibrium.

When $c=10,\delta$ is chosen as $0.0, 0.025, 0.05, 0.1$ respectively, the phase trajectories of DRM and EMR all converge at zero, as shown in Fig. 1(a). When the parameter $c=2.9875$, the phase trajectories of DRM and EMR all converged at their limit cycles respectively, as shown in Fig. 1(b). The Fig. 1(c) is local amplification figure of Fig. 1(b).

From the Fig.1 we know that the bifurcation parameter is far from the critical value, the phase trajectories of the deterministic system accord with the phase trajectories of stochastic financial system. The supercritical Hopf bifurcation occurs in both two systems.

5. Conclusions

In the paper, we studied the stability and Hopf bifurcation of stochastic financial system with noise. We have considered the case with the noise and the Hopf bifurcation which can occur when the value of c increases. The significance of the result in the realistic problem can be explained as follows. As the stochastic disturbance is inevitable, it is reasonable to study the Hopf bifurcation of the stochastic dynamical system at the equilibrium point more than the stability of the deterministic system at the equilibrium point, and the most possibility with which the trajectory will stay (occur) in the neighborhood of the limit ring. The position where stochastic Hopf bifurcation occurs will become bigger as $c=c_0$ increases, and when it reaches

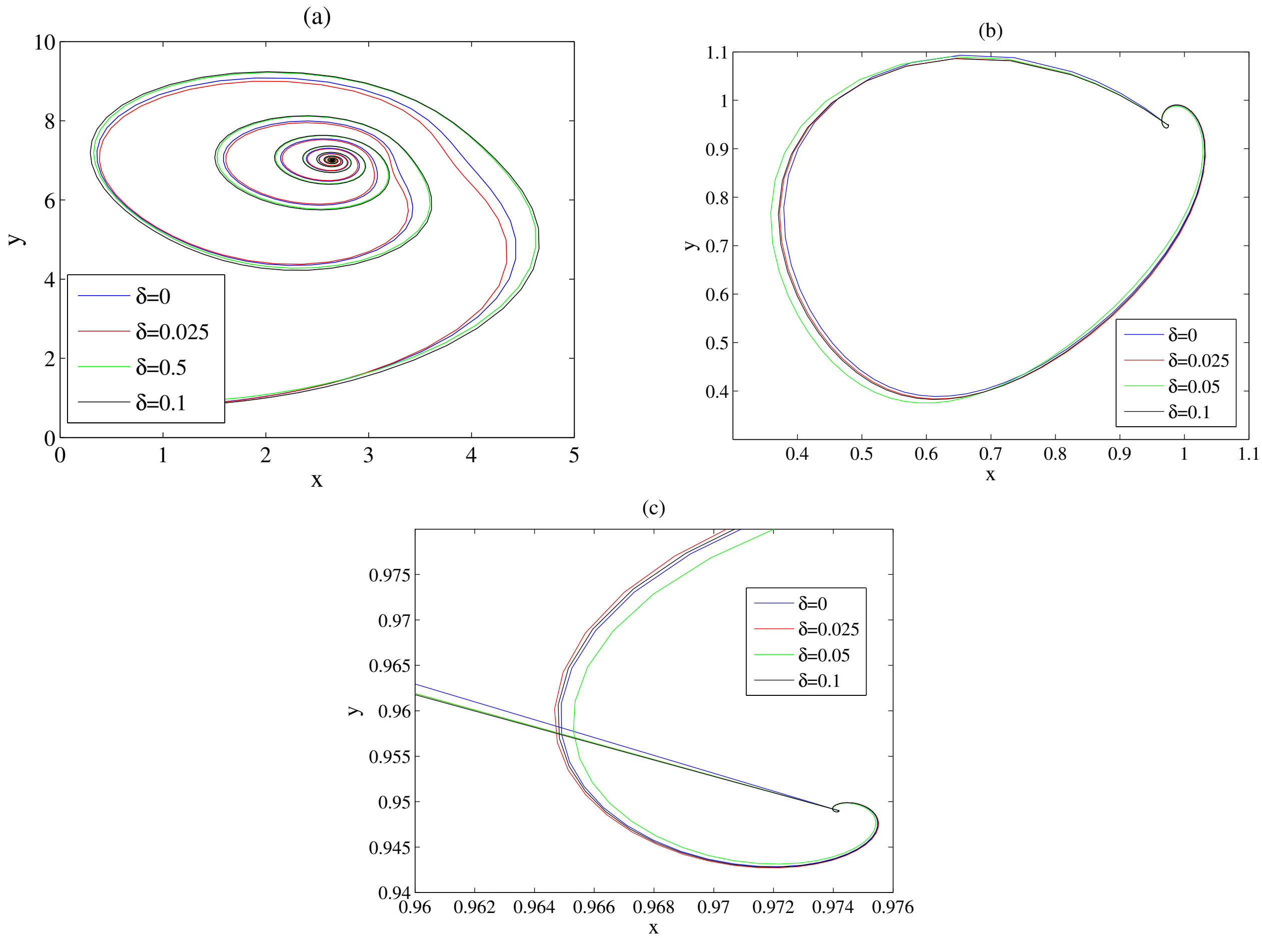

Figure 1: Phase portraits for Hop bifurcation (a) $c = 10$; (b) $c = 2.9875$; (c) Local amplification figure of (b).

the threshold value of financial system, it can cause financial system in the meaning of probability in which the Hopf bifurcation happened. According to the expression of $c = c_0$, it is obvious that as the intensities of the random effect increase becomes bigger. For instance, the stochastic Hopf bifurcation can result from the variation of intensity of the random parameter alone. We also find that the direction and stability of bifurcation in stochastic financial system are not changed, as well as the random intensity is small. In conclusion, the theoretical results are verified by numerical simulations.

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References

[1] T. Alden Gassert, *Chebyshev action on finite fields*, Discrete Math., **315** (2014), 83–94. 1
[2] L. L. Alexander, *Predictability and unpredictability in financial markets*, Phys. D, **113** (1999), 321–347. 1
[3] M. Bandyopadhyay, J. Chattopadhyay, *Ratio-dependent predator-prey model: effect of environmental fluctuation and stability*, Nonlinearity, **18** (2005), 913–936. 1
[4] G. L. Cai, J. Huang, *A new finance chaotic attractor*, Int. J. Nonlinear Sci., **3** (2007), 213–220. 1
[5] E. M. Elabbasy, A. A. Elsadany, Y. Zhang, *Bifurcation analysis and chaos in a discrete reduced Lorenz system*, Appl. Math. Comput., **228** (2014), 184–194. 1
[6] A. A. Elsadany, *Competition analysis of a triopoly game with bounded rationality*, Chaos Solitons Fract., **45** (2012), 1343–1348. 1

[7] A. M. A. El-Sayed, A. Elsaid, H. M. Nour, A. Elsonbaty, *Dynamical behavior, chaos control and synchronization of a memristor-based ADVP circuit*, Commun. Nonlinear Sci. Numer. Simul., **18** (2013), 148–170. 1

[8] T. Fang, X. L. Leng, C. Q. Song, *Chebyshev polynomial approximation for dynamical response problem of random system*, J. Sound Vibration, **226** (2003), 198–206. 1

[9] G. Gambino, V. Sciacca, *Intermittent and passivity based control strategies for a hyperchaotic system*, Appl. Math. Comput., **221** (2013), 367–382. 1

[10] R. Ghanem, P. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Springer, Berlin, (1991). 1

[11] J. Grandmont , *On endogenous competitive business cycles*, Econometrica, **53** (1985), 995–1045. 1

[12] A. S. Hegazi, A. E. Matouk, *Chaos synchronization of the modified autonomous Van der Pol-Duffing circuits via active control*, Appl. Chaos Nonlinear Dyn. Sci. Eng. Underst. Complex Syst., **3** (2013), 185–202. 1

[13] Z. T. Huang, Q. G. Yang, J. F. Cao, *The stochastic stability and bifurcation behavior of an Internet congestion control model*, Math. Comput. Model., **54** (2011), 1954–1965. 1

[14] B. Jiang, Y. Lu, J. Zhang, Q. Bi, *Bifurcations and some new traveling wave solutions for the CH-γ equation*, Appl. Math. Comput., **228** (2014), 220–233. 1

[15] H. Kim, Y. Kim, D. Yoon, *Dependence of polynomial chaos on random types of forces of KdV equations*, Appl. Math. Model., **36** (2012), 3080–3093. 1

[16] M. Kleiberand, T. D. Hien, *The Stochastic Finite Element Method: Basic Perturbation Technique and Computer Implementation*, Wiley Press, Chichester, (1992). 1

[17] T. H. Lee, J. H. Parka, S. M. Lee, O. M. Kwon, *Robust synchronization of chaotic systems with randomly occurring uncertainties via stochastic sampled-data control*, Int. J. Control, **86** (2013), 107–119. 1

[18] O. P. Le Maitre, H. N. Najm, R. G. Ghanem, O. M. Knio, *Multi-resolution analysis of Wiener-type uncertainty propagation schemes*, J. Comput. Phys., **197** (2004), 502–531. 1

[19] X. L. Leng, C. L. Wu, X. P. Ma, G. Meng, T. Fang, *Bifurcation and chaos analysis of stochastic Duffing system under harmonic excitations*, Nonlinear Dynam., **42** (2005), 185–198. 1

[20] X. Liao, C. Li, S. Zhou, *Hopf bifurcation and chaos in macroeconomic models with policy lag*, Chaos Solitons Fract., **25** (2005), 91–108. 1

[21] S. J. Ma, *The stochastic Hopf bifurcation analysis in Brusselator system with random parameter*, Appl. Math. Comput., **219** (2012), 306–319. 1

[22] C. Ma, X. Y. Wang, *Hopf bifurcation and topological horseshoe of a novel finance chaotic system*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 721–730. 1

[23] S. J. Ma, W. Xu, *Period-doubling bifurcation in an extended van der Pol system with bounded random parameter*, Commun. Nonlinear Sci. Numer. Simul., **13** (2008), 2256–2265. 1

[24] S. J. Ma, W. Xu, T. Fang, *Analysis of period-doubling bifurcation in double-well stochastic Duffing system via Laguerre polynomial approximation*, Nonlinear Dynam., **52** (2008), 289–299. 1

[25] G. Marchuk, *Mathematical Models in Immunology*, Nauka, Moscow, (1983). 1

[26] C. L. Pettit, P. S. Beran, *Spectral and multiresolution Wiener expansions of oscillatory stochastic processes*, J. Sound Vibration, **294** (2006), 752–779. 1

[27] M. Shinozuka, *Probabilistic modeling of concrete structure*, J. Eng. Mech. Div. proc., **98** (1972), 1433–1451. 1

[28] A. Talukdar, A. G. Radwan, K. N. Salama, *Nonlinear dynamics of memristor based 3rd order oscillatory system*, Microelectron. J., **43** (2012), 169–175. 1

[29] C. L. Wu, X. P. Ma, T. Fang, *A complementary note on Gegenbauer polynomial approximation for random response problem of stochastic structure*, Prob. Engin. Mech., **21** (2006), 410–419. 1, 3.2

[30] D. B. Xiu, G. E. Karniadakis, *Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos*, Comput. Methods Appl. Mech. Engrg., **191** (2002), 4927–4948. 1

[31] Y. Zhang, W. Xu, T. Fang, *Stochastic Hopf bifurcation and chaos of stochastic Bonhoeffer-van der Pol system via Chebyshev polynomial approximation*, Appl. Math. Comput., **190** (2007), 1225–1236. 1

[32] J. Zhao, Y. Wu, Q. Liu, *Chaos synchronization between the coupled systems on network with unknown parameters*, Appl. Math. Comput., **229** (2014), 254–259. 1

[33] C. Zhu, *A novel image encryption scheme based on improved hyperchaotic sequences*, Opt. Commun., **285** (2012), 29–37. 1