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On a solvable for some systems of rational difference equations

M. M. El-Dessoky

King AbdulAziz University, Faculty of Science, Mathematics Department, P. O. Box 80203, Jeddah 21589, Saudi Arabia.

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

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Abstract

In this paper, we study the existence of solutions for a class of rational systems of difference equations of order four in four-dimensional case

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm t_n z_{n-1} y_{n-2} x_{n-3}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_n t_{n-1} z_{n-2} y_{n-3}},$$
$$z_{n+1} = \frac{z_{n-3}}{\pm 1 \pm y_n x_{n-1} t_{n-2} z_{n-3}}, \qquad t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm z_n y_{n-1} x_{n-2} t_{n-3}}$$

with the initial conditions are real numbers. Also, we study some behavior such as the periodicity and boundedness of solutions for such systems. Finally, some numerical examples are given to confirm our theoretical results and graphed by Matlab.

Keywords: Recursive sequences, difference equation system, periodic solutions.
2010 MSC: 39A10, 39A22, 39A23.

1. Introduction

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have experienced enormous growth in many areas. Many applications of discrete dynamical systems and difference equations have appeared recently in the areas of biology, economics, physics, resource management and others. The theory of difference equations occupies a central position in applicable analysis. There is no suspicion that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance

Email address: `dessokym@mans.edu.eg` (M. M. El-Dessoky)

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in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, psychology, engineering, physics, probability theory, economics, genetics, physiology and resource management. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. There are many papers deal with the difference equations system [1–33]. For example, the dynamical behavior of positive solution for the system

$$x_{n+1} = \frac{x_{n-m+1}}{A + y_n y_{n-1} \dots y_{n-m+1}}, \qquad y_{n+1} = \frac{y_{n-m+1}}{A + x_n x_{n-1} \dots x_{n-m+1}},$$

has been studied by Sroysang in [24].

In [12], El-Metwally presented solutions of the following sixteen systems of difference equations

$$x_n = \frac{x_{n-2} y_{n-1}}{\pm x_{n-2} \pm y_{n-3}}, \qquad y_n = \frac{y_{n-2} x_{n-1}}{\pm y_{n-2} \pm x_{n-3}}.$$

In [26], Stević et al. studied the solutions of rational difference equations

$$x_n = \frac{x_{n-k} y_{n-l}}{b_n x_{n-k} + a_n y_{n-l-k}}, \qquad y_n = \frac{y_{n-k} x_{n-l}}{d_n y_{n-k} + c_n x_{n-l-k}}.$$

In [4], Din has investigated the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3}}, \qquad y_{n+1} = \frac{\alpha_2 y_{n-3}}{\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3}}.$$

Yalçinkaya and Çinar [29] got the periodicity of the positive solutions of the nonlinear difference equations system

$$x_{n+1} = \frac{1}{z_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}, \qquad z_{n+1} = \frac{1}{x_{n-1}}.$$

El-Dessoky et al. [11] obtained the solutions of the difference equation systems

$$x_{n+1} = \frac{x_{n-1}}{1 + y_n x_{n-1}}, \qquad y_{n+1} = \frac{y_{n-1}}{1 + x_n y_{n-1}}, \qquad z_{n+1} = \frac{z_{n-m}}{x_n y_n}.$$

Özkan et al. [23] investigated the periodical solutions of the third order rational difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2} x_{n-1} y_n}, \qquad y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}, \qquad z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}.$$

Yazlik et al. [31] studied the behaviour of solutions of the systems of difference equations

$$x_{n+1} = \frac{y_{n-2} x_{n-3} y_{n-4}}{y_n x_{n-1} (\pm 1 \pm y_{n-2} x_{n-3} y_{n-4})}, \qquad y_{n+1} = \frac{x_{n-2} y_{n-3} x_{n-4}}{x_n y_{n-1} (\pm 1 \pm x_{n-2} y_{n-3} x_{n-4})}.$$

El-Dessoky et al. [9] investigated the form of the solution of the systems of difference equations

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_{n-2} z_{n-1} y_n}, \qquad y_{n+1} = \frac{y_{n-2}}{\pm 1 + y_{n-2} x_{n-1} z_n}, \qquad z_{n+1} = \frac{z_{n-2}}{\pm 1 + z_{n-2} y_{n-1} x_n}.$$

Also, in [19] , Kurbanli studied a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n - 1}, \qquad z_{n+1} = \frac{x_n}{z_{n-1} y_n - 1}.$$

To be motivated by the above studies, our aim in this paper is to obtain the existence of solutions for the rational systems of difference equations of order four in four-dimensional case

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm t_n z_{n-1} y_{n-2} x_{n-3}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_n t_{n-1} z_{n-2} y_{n-3}},$$
$$z_{n+1} = \frac{z_{n-3}}{\pm 1 \pm y_n x_{n-1} t_{n-2} z_{n-3}}, \qquad t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm z_n y_{n-1} x_{n-2} t_{n-3}},$$

where $n \in \mathbb{N}_0$ and the initial conditions $x_i,\ y_i,\ z_i,\ t_i$ for $i = -3,\ -2,\ -1,\ 0$ are arbitrary real numbers. We study the dynamics of theses solutions such as the periodicity and boundedness and give some numerical examples for the systems.

2. Systems and the expressions of their solutions

Here we interest to investigate the following system of some rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{-1 - t_n z_{n-1} y_{n-2} x_{n-3}}, \qquad y_{n+1} = \frac{y_{n-3}}{-1 - x_n t_{n-1} z_{n-2} y_{n-3}},$$
$$z_{n+1} = \frac{z_{n-3}}{1 + y_n x_{n-1} t_{n-2} z_{n-3}}, \qquad t_{n+1} = \frac{t_{n-3}}{1 + z_n y_{n-1} x_{n-2} t_{n-3}}, \tag{2.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions $x_i,\ y_i,\ z_i,\ t_i$ for $i = -3,\ -2,\ -1,\ 0$ are arbitrary real numbers.

Theorem 2.1. *Assume that* $\{x_n, y_n, z_n, t_n\}$ *are solutions of system* (2.1), *then for* $n = 0, 1, 2, ...,$ *we obtain*

$$x_{4n-3} = (-1)^n x_{-3} \prod_{i=0}^{n-1} \frac{(1 + (2i)x_{-3}y_{-2}z_{-1}t_0)}{(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0)}, \qquad x_{4n-2} = (-1)^n x_{-2} \prod_{i=0}^{n-1} \frac{(1 + (2i+1)t_{-3}x_{-2}y_{-1}z_0)}{(1 + (2i+2)t_{-3}x_{-2}y_{-1}z_0)},$$
$$x_{4n-1} = (-1)^n x_{-1} \prod_{i=0}^{n-1} \frac{(1 + (2i+2)z_{-3}t_{-2}x_{-1}y_0)}{(1 + (2i+3)z_{-3}t_{-2}x_{-1}y_0)}, \qquad x_{4n} = (-1)^n x_0 \prod_{i=0}^{n-1} \frac{(-1 + (2i+1)y_{-3}z_{-2}t_{-1}x_0)}{(-1 + (2i+2)y_{-3}z_{-2}t_{-1}x_0)},$$
$$y_{4n-3} = (-1)^n y_{-3} \prod_{i=0}^{n-1} \frac{(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1 + (2i-1)y_{-3}z_{-2}t_{-1}x_0)}, \qquad y_{4n-2} = (-1)^n y_{-2} \prod_{i=0}^{n-1} \frac{(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0)}{(1 + (2i)x_{-3}y_{-2}z_{-1}t_0)},$$
$$y_{4n-1} = (-1)^n y_{-1} \prod_{i=0}^{n-1} \frac{(1 + (2i+2)t_{-3}x_{-2}y_{-1}z_0)}{(1 + (2i+1)t_{-3}x_{-2}y_{-1}z_0)}, \qquad y_{4n} = (-1)^n y_0 \prod_{i=0}^{n-1} \frac{(1 + (2i+3)z_{-3}t_{-2}x_{-1}y_0)}{(1 + (2i+2)z_{-3}t_{-2}x_{-1}y_0)},$$
$$z_{4n-3} = z_{-3} \prod_{i=0}^{n-1} \frac{(1 + (2i)z_{-3}t_{-2}x_{-1}y_0)}{(1 + (2i+1)z_{-3}t_{-2}x_{-1}y_0)}, \qquad z_{4n-2} = z_{-2} \prod_{i=0}^{n-1} \frac{(-1 + (2i-1)y_{-3}z_{-2}t_{-1}x_0)}{(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0)},$$
$$z_{4n-1} = z_{-1} \prod_{i=0}^{n-1} \frac{(1 + (2i)x_{-3}y_{-2}z_{-1}t_0)}{(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0)}, \qquad z_{4n} = z_0 \prod_{i=0}^{n-1} \frac{(1 + (2i+1)t_{-3}x_{-2}y_{-1}z_0)}{(1 + (2i+2)t_{-3}x_{-2}y_{-1}z_0)},$$

and

$$t_{4n-3} = t_{-3} \prod_{i=0}^{n-1} \frac{(1 + (2i)t_{-3}x_{-2}y_{-1}z_0)}{(1 + (2i+1)t_{-3}x_{-2}y_{-1}z_0)}, \qquad t_{4n-2} = t_{-2} \prod_{i=0}^{n-1} \frac{(1 + (2i+1)z_{-3}t_{-2}x_{-1}y_0)}{(1 + (2i+2)z_{-3}t_{-2}x_{-1}y_0)},$$
$$t_{4n-1} = t_{-1} \prod_{i=0}^{n-1} \frac{(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1 + (2i+1)y_{-3}z_{-2}t_{-1}x_0)}, \qquad t_{4n} = t_0 \prod_{i=0}^{n-1} \frac{(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0)}{(1 + (2i+2)x_{-3}y_{-2}z_{-1}t_0)},$$

where $\prod_{i=0}^{-1} A_i = 1.$

Proof. For $n = 0$ the result holds. Now let $n > 1$ and that our assumption holds for $n - 1$, that is,

$$x_{4n-7} = (-1)^{n-1} x_{-3} \prod_{i=0}^{n-2} \frac{(1 + (2i)x_{-3}y_{-2}z_{-1}t_0)}{(1 + (2i+1)x_{-3}y_{-2}z_{-1}t_0)}, \qquad x_{4n-6} = (-1)^{n-1} x_{-2} \prod_{i=0}^{n-2} \frac{(1 + (2i+1)t_{-3}x_{-2}y_{-1}z_0)}{(1 + (2i+2)t_{-3}x_{-2}y_{-1}z_0)},$$
$$x_{4n-5} = (-1)^{n-1} x_{-1} \prod_{i=0}^{n-2} \frac{(1 + (2i+2)z_{-3}t_{-2}x_{-1}y_0)}{(1 + (2i+3)z_{-3}t_{-2}x_{-1}y_0)}, \qquad x_{4n-4} = (-1)^{n-1} x_0 \prod_{i=0}^{n-2} \frac{(-1+(2i+1)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i+2)y_{-3}z_{-2}t_{-1}x_0)},$$
$$y_{4n-7} = (-1)^{n-1} y_{-3} \prod_{i=0}^{n-2} \frac{(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1 + (2i-1)y_{-3}z_{-2}t_{-1}x_0)}, \qquad y_{4n-6} = (-1)^{n-1} y_{-2} \prod_{i=0}^{n-2} \frac{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)},$$
$$y_{4n-5} = (-1)^{n-1} y_{-1} \prod_{i=0}^{n-2} \frac{(1 + (2i+2)t_{-3}x_{-2}y_{-1}z_0)}{(1 + (2i+1)t_{-3}x_{-2}y_{-1}z_0)}, \qquad y_{4n-4} = (-1)^{n-1} y_0 \prod_{i=0}^{n-2} \frac{(1+(2i+3)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+2)z_{-3}t_{-2}x_{-1}y_0)},$$
$$z_{4n-7} = z_{-3} \prod_{i=0}^{n-2} \frac{(1 + (2i)z_{-3}t_{-2}x_{-1}y_0)}{(1 + (2i+1)z_{-3}t_{-2}x_{-1}y_0)}, \qquad z_{4n-6} = z_{-2} \prod_{i=0}^{n-2} \frac{(-1 + (2i-1)y_{-3}z_{-2}t_{-1}x_0)}{(-1 + (2i)y_{-3}z_{-2}t_{-1}x_0)},$$

$$z_{4n-5}=z_{-1}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)},\qquad z_{4n-4}=z_0\prod_{i=0}^{n-2}\frac{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+2)t_{-3}x_{-2}y_{-1}z_0)},$$

$$t_{4n-7}=t_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)},\qquad t_{4n-6}=t_{-2}\prod_{i=0}^{n-2}\frac{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+2)z_{-3}t_{-2}x_{-1}y_0)},$$

$$t_{4n-5}=t_{-1}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i+1)y_{-3}z_{-2}t_{-1}x_0)},\qquad t_{4n-4}=t_0\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+2)x_{-3}y_{-2}z_{-1}t_0)}.$$

From system (2.1), we deduce that

$$\begin{aligned}
x_{4n-3}=&\frac{x_{4n-7}}{-1-t_{4n-4}z_{4n-5}y_{4n-6}x_{4n-7}}\\
=&\frac{(-1)^{n-1}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}}{-1-\left[\begin{array}{c} t_0\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+2)x_{-3}y_{-2}z_{-1}t_0)}z_{-1}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}\\ (-1)^{n-1}y_{-2}\prod_{i=0}^{n-2}\frac{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}(-1)^{n-1}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}\end{array}\right]}\\
=&\frac{(-1)^{n}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}}{-1-\left[x_{-3}y_{-2}z_{-1}t_0\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+2)x_{-3}y_{-2}z_{-1}t_0)}\right]}=\frac{(-1)^{n-1}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}}{-1-\left[\frac{x_{-3}y_{-2}z_{-1}t_0}{(1+(2n-2)x_{-3}y_{-2}z_{-1}t_0)}\right]}\\
=&\frac{(-1)^{n-1}x_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)}}{-\left(\frac{1+(2n-1)x_{-3}y_{-2}z_{-1}t_0}{(1+(2n-2)x_{-3}y_{-2}z_{-1}t_0)}\right)}\\
=&(-1)^{n}x_{-3}\prod_{i=0}^{n-1}\frac{(1+(2i)x_{-3}y_{-2}z_{-1}t_0)}{(1+(2i+1)x_{-3}y_{-2}z_{-1}t_0)},
\end{aligned}$$

$$\begin{aligned}
y_{4n-3}=&\frac{y_{4n-7}}{-1-x_{4n-4}t_{4n-5}z_{4n-6}y_{4n-7}}\\
=&\frac{(-1)^{n-1}y_{-3}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}}{-1-\left[\begin{array}{c}(-1)^{n-1}x_0\prod_{i=0}^{n-2}\frac{(-1+(2i+1)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i+2)y_{-3}z_{-2}t_{-1}x_0)}t_{-1}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i+1)y_{-3}z_{-2}t_{-1}x_0)}\\ z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}(-1)^{n-1}y_{-3}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}\end{array}\right]}\\
=&\frac{(-1)^{n-1}y_{-3}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}}{-1-\left[y_{-3}z_{-2}t_{-1}x_0\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i+2)y_{-3}z_{-2}t_{-1}x_0)}\right]}=\frac{(-1)^{n-1}y_{-3}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}}{-1+\frac{y_{-3}z_{-2}t_{-1}x_0}{(-1+(2n-2)y_{-3}z_{-2}t_{-1}x_0)}}\\
=&\frac{(-1)^{n-1}y_{-3}\prod_{i=0}^{n-2}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}}{-\left(\frac{-1+(2n-3)y_{-3}z_{-2}t_{-1}x_0}{-1+(2n-2)y_{-3}z_{-2}t_{-1}x_0}\right)}\\
=&(-1)^{n}y_{-3}\prod_{i=0}^{n-1}\frac{(-1+(2i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(2i-1)y_{-3}z_{-2}t_{-1}x_0)}.
\end{aligned}$$

Also, from system (2.1), we see that

$$\begin{aligned}
z_{4n-3} &= \frac{z_{4n-7}}{1+y_{4n-4}x_{4n-5}t_{4n-6}z_{4n-7}} \\
&= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}}{1+\left[\begin{array}{c}(-1)^{n-1}y_0\prod_{i=0}^{n-2}\frac{(1+(2i+3)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+2)z_{-3}t_{-2}x_{-1}y_0)}(-1)^{n-1}x_{-1}\prod_{i=0}^{n-2}\frac{(1+(2i+2)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+3)z_{-3}t_{-2}x_{-1}y_0)} \\ t_{-2}\prod_{i=0}^{n-2}\frac{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+2)z_{-3}t_{-2}x_{-1}y_0)}z_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}\end{array}\right]} \\
&= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}}{1+z_{-3}t_{-2}x_{-1}y_0\prod_{i=0}^{n-2}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+2)z_{-3}t_{-2}x_{-1}y_0)}} = \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}}{1+\frac{z_{-3}t_{-2}x_{-1}y_0}{1+(2n-2)z_{-3}t_{-2}x_{-1}y_0}} \\
&= \frac{z_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}}{\frac{1+(2n-1)z_{-3}t_{-2}x_{-1}y_0}{1+(2n-2)z_{-3}t_{-2}x_{-1}y_0}} = z_{-3}\prod_{i=0}^{n-1}\frac{(1+(2i)z_{-3}t_{-2}x_{-1}y_0)}{(1+(2i+1)z_{-3}t_{-2}x_{-1}y_0)}.
\end{aligned}$$

Finally, from system (2.1), we see that

$$\begin{aligned}
t_{4n-3} &= \frac{t_{4n-7}}{1+z_{4n-4}y_{4n-5}x_{4n-6}t_{4n-7}} \\
&= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}}{1+\left[\begin{array}{c}z_0\prod_{i=0}^{n-2}\frac{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+2)t_{-3}x_{-2}y_{-1}z_0)}(-1)^{n-1}y_{-1}\prod_{i=0}^{n-2}\frac{(1+(2i+2)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)} \\ (-1)^{n-1}x_{-2}\prod_{i=0}^{n-2}\frac{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+2)t_{-3}x_{-2}y_{-1}z_0)}t_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}\end{array}\right]} \\
&= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}}{1+\left[t_{-3}x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+2)t_{-3}x_{-2}y_{-1}z_0)}\right]} = \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}}{1+\frac{t_{-3}x_{-2}y_{-1}z_0}{1+(2i+2)t_{-3}x_{-2}y_{-1}z_0}} \\
&= \frac{t_{-3}\prod_{i=0}^{n-2}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}}{\frac{1+(2i-1)t_{-3}x_{-2}y_{-1}z_0}{1+(2i-2)t_{-3}x_{-2}y_{-1}z_0}} = t_{-3}\prod_{i=0}^{n-1}\frac{(1+(2i)t_{-3}x_{-2}y_{-1}z_0)}{(1+(2i+1)t_{-3}x_{-2}y_{-1}z_0)}.
\end{aligned}$$

Similarly, we can prove the other relations. This completes the proof. □

Lemma 2.2. *If $x_i,\ y_i,\ z_i,\ t_i$ for $i=-3,-2,-1,0$ be arbitrary real numbers and let $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of system (2.1), then the following conditions hold:*

(i) If $x_{-3}=0$, then we have $x_{4n-3}=0$ and $y_{4n-2}=(-1)^n y_{-2},\ z_{4n-1}=z_{-1},\ t_{4n}=t_0$.
(ii) If $x_{-2}=0$, then we have $x_{4n-2}=0$ and $y_{4n-1}=(-1)^n y_{-1},\ z_{4n}=z_0,\ t_{4n-3}=t_{-3}$.
(iii) If $x_{-1}=0$, then we have $x_{4n-1}=0$ and $y_{4n}=(-1)^n y_0,\ z_{4n-3}=z_{-3},\ t_{4n-2}=t_{-2}$.
(iv) If $x_0=0$, then we have $x_{4n}=0$ and $y_{4n-3}=(-1)^n y_{-3},\ z_{4n-2}=z_{-2},\ t_{4n-1}=t_{-1}$.
(v) If $y_{-3}=0$, then we have $y_{4n-3}=0$ and $x_{4n}=(-1)^n x_0,\ z_{4n-2}=z_{-2},\ t_{4n-1}=t_{-1}$.
(vi) If $y_{-2}=0$, then we have $y_{4n-2}=0$ and $x_{4n-3}=(-1)^n x_{-3},\ z_{4n-1}=z_{-1},\ t_{4n}=t_0$.
(vii) If $y_{-1}=0$, then we have $y_{4n-1}=0$ and $x_{4n-2}=(-1)^n x_{-2},\ z_{4n}=z_0,\ t_{4n-3}=t_{-3}$.
(viii) If $y_0=0$, then we have $y_{4n}=0$ and $x_{4n-1}=(-1)^n x_{-1},\ z_{4n-3}=z_{-3},\ t_{4n-2}=t_{-2}$.

(ix) If $z_{-3}=0$, then we have $z_{4n-3}=0$ and $x_{4n-1}=(-1)^n x_{-1},\ y_{4n}=(-1)^n y_0,\ t_{4n-2}=t_{-2}$.
(x) If $z_{-2}=0$, then we have $z_{4n-2}=0$ and $x_{4n}=(-1)^n x_0,\ y_{4n-3}=(-1)^n y_{-3},\ t_{4n-1}=t_{-1}$.
(xi) If $z_{-1}=0$, then we have $z_{4n-1}=0$ and $x_{4n-3}=(-1)^n x_{-3},\ y_{4n-2}=(-1)^n y_{-2},\ t_{4n}=t_0$.
(xii) If $z_0=0$, then we have $z_{4n}=0$ and $x_{4n-2}=(-1)^n x_{-2},\ y_{4n-1}=(-1)^n y_{-1}, t_{4n-3}=t_{-3}$.
(xiii) If $t_{-3}=0$, then we have $t_{4n-3}=0$ and $x_{4n-2}=(-1)^n x_{-2}, y_{4n-1}=(-1)^n y_{-1},\ z_{4n}=z_0$.
(xiv) If $t_{-2}=0$, then we have $t_{4n-2}=0$ and $x_{4n-1}=(-1)^n x_{-1},\ y_{4n}=(-1)^n y_0,\ z_{4n-3}=z_{-3}$.
(xv) If $t_{-1}=0$, then we have $t_{4n-1}=0$ and $x_{4n}=(-1)^n x_0,\ y_{4n-3}=(-1)^n y_{-3},\ z_{4n-2}=z_{-2}$.
(xvi) If $t_0=0$, then we have $t_{4n}=0$ and $x_{4n-3}=(-1)^n x_{-3},\ y_{4n-2}=(-1)^n y_{-2},\ z_{4n-1}=z_{-1}$.

Proof. The proof follows directly from the expressions of the solutions of system (2.1). □

Theorem 2.3. *Assume that $\{x_n, y_n, z_n, t_n\}$ are solutions of the system*

$$x_{n+1}=\frac{x_{n-3}}{1+t_n z_{n-1} y_{n-2} x_{n-3}},\qquad y_{n+1}=\frac{y_{n-3}}{1-x_n t_{n-1} z_{n-2} y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{1-y_n x_{n-1} t_{n-2} z_{n-3}},\qquad t_{n+1}=\frac{t_{n-3}}{1+z_n y_{n-1} x_{n-2} t_{n-3}} \tag{2.2}$$

with $x_{-3}y_{-2}z_{-1}t_0\neq\pm1,\ t_{-3}x_{-2}y_{-1}z_0\neq-1,\ t_{-3}x_{-2}y_{-1}z_0\neq-\frac{1}{2},\ z_{-3}t_{-2}x_{-1}y_0\neq\pm1,\ y_{-3}z_{-2}t_{-1}x_0\neq 1$, $y_{-3}z_{-2}t_{-1}x_0\neq\frac{1}{2}$, takes the form

$$x_{4n-3}=\frac{x_{-3}}{(1+x_{-3}y_{-2}z_{-1}t_0)^n},\qquad x_{4n-2}=\frac{x_{-2}(1+t_{-3}x_{-2}y_{-1}z_0)^n}{(1+2t_{-3}x_{-2}y_{-1}z_0)^n},$$
$$x_{4n-1}=\frac{x_{-1}}{(1+z_{-3}t_{-2}x_{-1}y_0)^n},\qquad x_{4n}=x_0(1-y_{-3}z_{-2}t_{-1}x_0)^n,$$
$$y_{4n-3}=\frac{y_{-3}}{(1-y_{-3}z_{-2}t_{-1}x_0)^n},\qquad y_{4n-2}=y_{-2}(1+x_{-3}y_{-2}z_{-1}t_0)^n,$$
$$y_{4n-1}=\frac{y_{-1}(1+2t_{-3}x_{-2}y_{-1}z_0)^n}{(1+t_{-3}x_{-2}y_{-1}z_0)^n},\qquad y_{4n}=y_0(1+z_{-3}t_{-2}x_{-1}y_0)^n,$$
$$z_{4n-3}=\frac{z_{-3}}{(1-z_{-3}t_{-2}x_{-1}y_0)^n},\qquad z_{4n-2}=\frac{z_{-2}(-1+y_{-3}z_{-2}t_{-1}x_0)^n}{(-1+2y_{-3}z_{-2}t_{-1}x_0)^n},$$
$$z_{4n-1}=\frac{z_{-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^n},\qquad z_{4n}=z_0(1+t_{-3}x_{-2}y_{-1}z_0)^n,$$
$$t_{4n-3}=\frac{t_{-3}}{(1+t_{-3}x_{-2}y_{-1}z_0)^n},\qquad t_{4n-2}=t_{-2}(1-z_{-3}t_{-2}x_{-1}y_0)^n,$$
$$t_{4n-1}=\frac{t_{-1}(-1+2y_{-3}z_{-2}t_{-1}x_0)^n}{(-1+y_{-3}z_{-2}t_{-1}x_0)^n},\qquad t_{4n}=t_0(1-x_{-3}y_{-2}z_{-1}t_0)^n.$$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$, that is,

$$x_{4n-7}=\frac{x_{-3}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1}},\qquad x_{4n-6}=\frac{x_{-2}(1+t_{-3}x_{-2}y_{-1}z_0)^{n-1}}{(1+2t_{-3}x_{-2}y_{-1}z_0)^{n-1}}$$
$$x_{4n-5}=\frac{x_{-1}}{(1+z_{-3}t_{-2}x_{-1}y_0)^{n-1}},\qquad x_{4n-4}=x_0(1-y_{-3}z_{-2}t_{-1}x_0)^{n-1},$$
$$y_{4n-7}=\frac{y_{-3}}{(1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}},\qquad y_{4n-6}=y_{-2}(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1},$$
$$y_{4n-5}=\frac{y_{-1}(1+2t_{-3}x_{-2}y_{-1}z_0)^{n-1}}{(1+t_{-3}x_{-2}y_{-1}z_0)^{n-1}},\qquad y_{4n-4}=y_0(1+z_{-3}t_{-2}x_{-1}y_0)^{n-1},$$
$$z_{4n-7}=\frac{z_{-3}}{(1-z_{-3}t_{-2}x_{-1}y_0)^{n-1}},\qquad z_{4n-6}=\frac{z_{-2}(-1+y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1+2y_{-3}z_{-2}t_{-1}x_0)^{n-1}},$$

$$z_{4n-5}=\frac{z_{-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}, \qquad z_{4n-4}=z_0\left(1+t_{-3}x_{-2}y_{-1}z_0\right)^{n-1},$$
$$t_{4n-7}=\frac{t_{-3}}{(1+t_{-3}x_{-2}y_{-1}z_0)^{n-1}}, \qquad t_{4n-6}=t_{-2}\left(1-z_{-3}t_{-2}x_{-1}y_0\right)^{n},$$
$$t_{4n-5}=\frac{t_{-1}\left(-1+2y_{-3}z_{-2}t_{-1}x_0\right)^{n-1}}{(-1+y_{-3}z_{-2}t_{-1}x_0)^{n-1}}, \qquad t_{4n-4}=t_0\left(1-x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}.$$

It follows that, from system (2.2), we have

$$\begin{aligned}
x_{4n-3}=&\frac{x_{4n-7}}{1+t_{4n-4}z_{4n-5}y_{4n-6}x_{4n-7}}\\
=&\frac{\frac{x_{-3}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1}}}{\left[1+\frac{z_{-1}t_0(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}\frac{x_{-3}y_{-2}(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1}}\right]}\\
=&\frac{\frac{x_{-3}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1}}}{[1+x_{-3}y_{-2}z_{-1}t_0]}=\frac{x_{-3}}{(-1+x_{-3}y_{-2}z_{-1}t_0)^{n}}.\\
y_{4n-2}=&\frac{y_{4n-6}}{1-x_{4n-3}t_{4n-4}z_{4n-5}y_{4n-6}}\\
=&\frac{y_{-2}\left(1+x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}}{\left[1-\frac{x_{-3}t_0(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n}}\frac{z_{-1}y_{-2}(1+x_{-3}y_{-2}z_{-1}t_0)^{n-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}\right]}\\
=&\frac{y_{-2}\left(1+x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}}{\left[1-\frac{x_{-3}y_{-2}z_{-1}t_0}{(1+x_{-3}y_{-2}z_{-1}t_0)}\right]}=\frac{y_{-2}\left(1+x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}}{\left[\frac{1}{(1+x_{-3}y_{-2}z_{-1}t_0)}\right]}\\
=&y_{-2}\left(1+x_{-3}y_{-2}z_{-1}t_0\right)^{n}.\\
z_{4n-1}=&\frac{z_{4n-5}}{1-y_{4n-2}x_{4n-3}t_{4n-4}z_{4n-5}}\\
=&\frac{\frac{z_{-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}}{\left[1-\frac{x_{-3}y_{-2}(1+x_{-3}y_{-2}z_{-1}t_0)^{n}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n}}\frac{z_{-1}t_0(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}\right]}\\
=&\frac{\frac{z_{-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}}{[1-x_{-3}y_{-2}z_{-1}t_0]}=\frac{z_{-1}}{(-1+x_{-3}y_{-2}z_{-1}t_0)^{n}}.\\
t_{4n}=&\frac{t_{4n-4}}{1+z_{4n-1}y_{4n-2}x_{4n-3}t_{4n-4}}\\
=&\frac{t_0\left(1-x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}}{\left[1+\frac{z_{-1}y_{-2}(1+x_{-3}y_{-2}z_{-1}t_0)^{n}}{(1-x_{-3}y_{-2}z_{-1}t_0)^{n}}\frac{x_{-3}t_0(1-x_{-3}y_{-2}z_{-1}t_0)^{n-1}}{(1+x_{-3}y_{-2}z_{-1}t_0)^{n}}\right]}\\
=&\frac{t_0\left(1-x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}}{\left[1+\frac{x_{-3}y_{-2}z_{-1}t_0}{(1-x_{-3}y_{-2}z_{-1}t_0)}\right]}=\frac{t_0\left(1-x_{-3}y_{-2}z_{-1}t_0\right)^{n-1}}{\left[\frac{1}{1-x_{-3}y_{-2}z_{-1}t_0}\right]}\\
=&t_0\left(1-x_{-3}y_{-2}z_{-1}t_0\right)^{n}.
\end{aligned}$$

Also, we can prove the other relations similarly. The proof is complete. □

Theorem 2.4. *Let* $\{x_n, y_n, z_n, t_n\}$ *are solutions of the system*

$$x_{n+1}=\frac{x_{n-3}}{-1+t_nz_{n-1}y_{n-2}x_{n-3}}, \qquad y_{n+1}=\frac{y_{n-3}}{-1-x_nt_{n-1}z_{n-2}y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{1+y_nx_{n-1}t_{n-2}z_{n-3}}, \qquad t_{n+1}=\frac{t_{n-3}}{1+z_ny_{n-1}x_{n-2}t_{n-3}} \tag{2.3}$$

with the initial values are arbitrary real numbers satisfy $x_{-3}y_{-2}z_{-1}t_0 \neq 1$, $x_{-3}y_{-2}z_{-1}t_0 \neq \frac{1}{2}$, $z_{-3}t_{-2}x_{-1}y_0 \neq -1$, $z_{-3}t_{-2}x_{-1}y_0 \neq -\frac{1}{2}$, $t_{-3}x_{-2}y_{-1}z_0 \neq \pm 1$ *and* $y_{-3}z_{-2}t_{-1}x_0 \neq \pm 1$*. Then the solution is given by the following formula for* $n = 0, 1, 2, \ldots$,

$$x_{4n-3} = \frac{x_{-3}}{(-1 + x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad x_{4n-2} = (-1)^n x_{-2} (1 + t_{-3}x_{-2}y_{-1}z_0)^n,$$

$$x_{4n-1} = \frac{(-1)^n x_{-1} (1 + 2z_{-3}t_{-2}x_{-1}y_0)^n}{(1 + z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad x_{4n} = x_0 (-1 + y_{-3}z_{-2}t_{-1}x_0)^n,$$

$$y_{4n-3} = \frac{(-1)^n y_{-3}}{(1 + y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad y_{4n-2} = \frac{(-1)^n y_{-2} (-1 + x_{-3}y_{-2}z_{-1}t_0)^n}{(-1 + 2x_{-3}y_{-2}z_{-1}t_0)^n},$$

$$y_{4n-1} = \frac{y_{-1}}{(-1 + t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad y_{4n} = (-1)^n y_0 (1 + z_{-3}t_{-2}x_{-1}y_0)^n,$$

$$z_{4n-3} = \frac{z_{-3}}{(1 + z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad z_{4n-2} = z_{-2} (1 + y_{-3}z_{-2}t_{-1}x_0)^n,$$

$$z_{4n-1} = \frac{(-1)^n z_{-1} (1 - 2x_{-3}y_{-2}z_{-1}t_0)^n}{(-1 + x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad z_{4n} = (-1)^n z_0 (-1 + t_{-3}x_{-2}y_{-1}z_0)^n,$$

$$t_{4n-3} = \frac{t_{-3}}{(1 + t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad t_{4n-2} = \frac{t_{-2} (1 + z_{-3}t_{-2}x_{-1}y_0)^n}{(1 + 2z_{-3}t_{-2}x_{-1}y_0)^n},$$

$$t_{4n-1} = \frac{(-1)^n t_{-1}}{(-1 + y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad t_{4n} = (-1)^n t_0 (-1 + x_{-3}y_{-2}z_{-1}t_0)^n.$$

Proof. As the proof of Theorem 2.3. □

Here for confirming the results of this section, we consider an interesting numerical examples of the systems (2.1)–(2.2).

Example 2.5. We consider the system (2.1) with the initial conditions $x_{-3} = 0.6$, $x_{-2} = 3$, $x_{-1} = 0.9$, $x_0 = 1.3$, $y_{-3} = 2$, $y_{-2} = 1.3$, $y_{-1} = -0.5$, $y_0 = 0.1$, $z_{-3} = 1.1$, $z_{-2} = 0.6$, $z_{-1} = -0.7$, $z_0 = 1.5$, $t_{-3} = -2$, $t_{-2} = 0.9$, $t_{-1} = -3$ and $t_0 = 0.8$, see Figure 1.

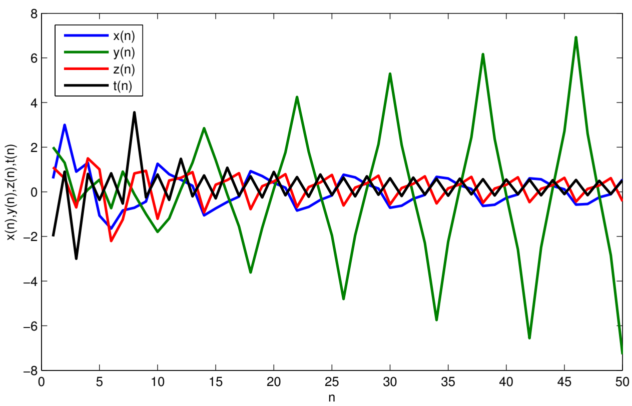

Figure 1: Sketch the behavior of the solution of system (2.1)

Example 2.6. See Figure 2 for an example for the system (2.2) with the initial values $x_{-3} = 0.46$, $x_{-2} = 0.23$, $x_{-1} = 0.29$, $x_0 = 1.16$, $y_{-3} = 0.2$, $y_{-2} = 1.3$, $y_{-1} = -0.5$, $y_0 = 0.61$, $z_{-3} = 0.21$, $z_{-2} = 0.26$, $z_{-1} = 0.27$, $z_0 = 1.85$, $t_{-3} = 0.2$, $t_{-2} = 0.09$, $t_{-1} = 0.28$ and $t_0 = 0.58$.

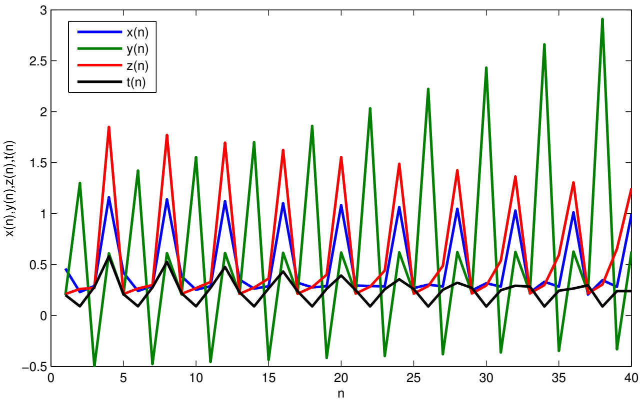

Figure 2: Draw the behavior of the solution of system (2.2).

3. Systems have a periodic solutions:

In this section, we study the solutions and periodic nature of the solutions of the following system of four nonlinear difference equations

$$x_{n+1} = \frac{x_{n-3}}{-1 - t_n z_{n-1} y_{n-2} x_{n-3}}, \qquad y_{n+1} = \frac{y_{n-3}}{-1 - x_n t_{n-1} z_{n-2} y_{n-3}},$$
$$z_{n+1} = \frac{z_{n-3}}{-1 - y_n x_{n-1} t_{n-2} z_{n-3}}, \qquad t_{n+1} = \frac{t_{n-3}}{-1 - z_n y_{n-1} x_{n-2} t_{n-3}}, \tag{3.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Theorem 3.1. *Assume that* $x_{-3}y_{-2}z_{-1}t_0 \neq -1$, $t_{-3}x_{-2}y_{-1}z_0 \neq -1$, $z_{-3}t_{-2}x_{-1}y_0 \neq -1$, $y_{-3}z_{-2}t_{-1}x_0 \neq -1$ *and* $x_{-3}y_{-2}z_{-1}t_0 \neq -2$, $t_{-3}x_{-2}y_{-1}z_0 \neq -2$, $z_{-3}t_{-2}x_{-1}y_0 \neq -2$ *and* $y_{-3}z_{-2}t_{-1}x_0 \neq -2$, *then all solutions of the system* (3.1) *are unbounded and given by the expressions*

$$x_{4n-3} = \frac{x_{-3}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad x_{4n-2} = x_{-2}(-1 - t_{-3}x_{-2}y_{-1}z_0)^n,$$
$$x_{4n-1} = \frac{x_{-1}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad x_{4n} = x_0(-1 - y_{-3}z_{-2}t_{-1}x_0)^n,$$
$$y_{4n-3} = \frac{y_{-3}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad y_{4n-2} = y_{-2}(-1 - x_{-3}y_{-2}z_{-1}t_0)^n,$$
$$y_{4n-1} = \frac{y_{-1}}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad y_{4n} = y_0(-1 - z_{-3}t_{-2}x_{-1}y_0)^n,$$
$$z_{4n-3} = \frac{z_{-3}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad z_{4n-2} = z_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^n,$$
$$z_{4n-1} = \frac{z_{-1}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad z_{4n} = z_0(-1 - t_{-3}x_{-2}y_{-1}z_0)^n,$$
$$t_{4n-3} = \frac{t_{-3}}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad t_{4n-2} = t_{-2}(-1 - z_{-3}t_{-2}x_{-1}y_0)^n,$$
$$t_{4n-1} = \frac{t_{-1}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad t_{4n} = t_0(-1 - x_{-3}y_{-2}z_{-1}t_0)^n.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$, that is,

$$x_{4n-7} = \frac{x_{-3}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^{n-1}}, \qquad x_{4n-6} = x_{-2}(-1 - t_{-3}x_{-2}y_{-1}z_0)^{n-1},$$
$$x_{4n-5} = \frac{x_{-1}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^{n-1}}, \qquad x_{4n-4} = x_0(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1},$$

$$y_{4n-7} = \frac{y_{-3}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}, \qquad y_{4n-6} = y_{-2}(-1 - x_{-3}y_{-2}z_{-1}t_0)^{n-1},$$
$$y_{4n-5} = \frac{y_{-1}}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^{n-1}}, \qquad y_{4n-4} = y_0(-1 - z_{-3}t_{-2}x_{-1}y_0)^{n-1},$$
$$z_{4n-7} = \frac{z_{-3}}{(-1 - z_{-3}t_{-2}x_{-1}y_0)^{n-1}}, \qquad z_{4n-6} = z_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1},$$
$$z_{4n-5} = \frac{z_{-1}}{(-1 - x_{-3}y_{-2}z_{-1}t_0)^{n-1}}, \qquad z_{4n-4} = z_0(-1 - t_{-3}x_{-2}y_{-1}z_0)^{n-1},$$
$$t_{4n-7} = \frac{t_{-3}}{(-1 - t_{-3}x_{-2}y_{-1}z_0)^{n-1}}, \qquad t_{4n-6} = t_{-2}(-1 - z_{-3}t_{-2}x_{-1}y_0)^{n-1},$$
$$t_{4n-5} = \frac{t_{-1}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}, \qquad t_{4n-4} = t_0(-1 - x_{-3}y_{-2}z_{-1}t_0)^{n-1}.$$

It follows that, from system (3.1), we have

$$\begin{aligned}
x_{4n} =& \frac{x_{4n-4}}{-1 - t_{4n-1}z_{4n-2}y_{4n-3}x_{4n-4}} \\
=& \frac{x_0(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{\left[-1 - \frac{t_{-1}z_{-2}(-1-y_{-3}z_{-2}t_{-1}x_0)^n}{(-1-y_{-3}z_{-2}t_{-1}x_0)^n}\frac{y_{-3}x_0(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^n}\right]} \\
=& \frac{x_0(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{\left[-1 - \frac{y_{-3}z_{-2}t_{-1}x_0}{-1-y_{-3}z_{-2}t_{-1}x_0}\right]} = \frac{x_0(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{\left[\frac{1}{-1-y_{-3}z_{-2}t_{-1}x_0}\right]} = x_0(-1 - y_{-3}z_{-2}t_{-1}x_0)^n, \\
y_{4n-3} =& \frac{y_{4n-7}}{-1 - x_{4n-4}t_{4n-5}z_{4n-6}y_{4n-7}} \\
=& \frac{\frac{y_{-3}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}}{\left[-1 - \frac{t_{-1}x_0(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}\frac{y_{-3}z_{-2}(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}\right]} \\
=& \frac{\frac{y_{-3}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)} = \frac{y_{-3}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^n}, \\
z_{4n-2} =& \frac{z_{4n-6}}{-1 - y_{4n-3}x_{4n-4}t_{4n-5}z_{4n-6}} \\
=& \frac{z_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{\left[-1 - \frac{y_{-3}x_0(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^n}\frac{t_{-1}z_{-2}(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}\right]} \\
=& \frac{z_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{\left[-1 - \frac{y_{-3}x_0z_{-2}t_{-1}}{-1-y_{-3}z_{-2}t_{-1}x_0}\right]} = \frac{z_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{\left[\frac{1}{-1-y_{-3}z_{-2}t_{-1}x_0}\right]} = z_{-2}(-1 - y_{-3}z_{-2}t_{-1}x_0)^n, \\
t_{4n-1} =& \frac{t_{4n-5}}{-1 - z_{4n-2}y_{4n-3}x_{4n-4}t_{4n-5}} \\
=& \frac{\frac{t_{-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}}{\left[-1 - \frac{y_{-3}z_{-2}(-1-y_{-3}z_{-2}t_{-1}x_0)^n}{(-1-y_{-3}z_{-2}t_{-1}x_0)^n}\frac{t_{-1}x_0(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}\right]} \\
=& \frac{\frac{t_{-1}}{(-1-y_{-3}z_{-2}t_{-1}x_0)^{n-1}}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)} = \frac{t_{-1}}{(-1 - y_{-3}z_{-2}t_{-1}x_0)^n}.
\end{aligned}$$

Also, we can prove the other relations similarly. The proof is complete. □

Theorem 3.2. *If the sequences $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of difference equation system (3.1) such that $x_{-3}y_{-2}z_{-1}t_0 = t_{-3}x_{-2}y_{-1}z_0 = z_{-3}t_{-2}x_{-1}y_0 = y_{-3}z_{-2}t_{-1}x_0 = -2$, then all solutions of the system are* *periodic with period four and takes the form*

$$\begin{aligned}
x_{4n-3} &= x_{-3},\ x_{4n-2} = x_{-2},\ x_{4n-1} = x_{-1},\ x_{4n} = x_0,\\
y_{4n-3} &= y_{-3},\ y_{4n-2} = y_{-2},\ y_{4n-1} = y_{-1},\ y_{4n} = y_0,\\
z_{4n-3} &= z_{-3},\ z_{4n-2} = z_{-2},\ z_{4n-1} = z_{-1},\ z_{4n} = z_0,\\
t_{4n-3} &= t_{-3},\ t_{4n-2} = t_{-2},\ t_{4n-1} = t_{-1},\ t_{4n} = t_0
\end{aligned}$$

or

$$\begin{aligned}
\{x_n\} &= \{x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ x_{-3},\ x_{-2}, ...\},\\
\{y_n\} &= \{y_{-3},\ y_{-2},\ y_{-1},\ y_0,\ y_{-3},\ y_{-2}, ...\},\\
\{z_n\} &= \{z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ z_{-3},\ z_{-2}, ...\},\\
\{t_n\} &= \{t_{-3},\ t_{-2},\ t_{-1},\ t_0,\ t_{-3},\ t_{-2}, ...\}.
\end{aligned}$$

Proof. The proof follows from the previous theorem and will be omitted. □

Example 3.3. We put the initial conditions as follows $x_{-3} = 0.6,\ x_{-2} = 3,\ x_{-1} = 0.9, x_0 = 1.3, y_{-3} = 0.22,$ $y_{-2} = 1.3,\ y_{-1} = -0.5,\ y_0 = 0.1,\ z_{-3} = 1.1,\ z_{-2} = 0.6,\ z_{-1} = 0.7,\ z_0 = 1.5,\ t_{-3} = 0.02,\ t_{-2} = 0.09,$ $t_{-1} = -0.3$ and $t_0 = 0.8$ for the difference system (3.1), see Figure 3.

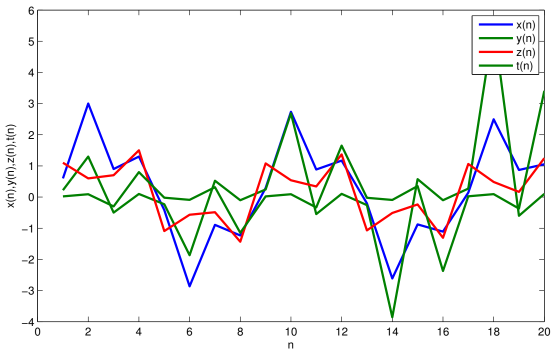

Figure 3: Draw of the behavior of the solution of system (3.1).

Example 3.4. Figure 4 shows the periodicity behavior of the solution of the difference system (3.1) with the initial conditions $x_{-3} = 2,\ x_{-2} = -0.5,\ x_{-1} = 1, x_0 = 4, y_{-3} = -5,\ y_{-2} = 10,\ y_{-1} = 2, y_0 = 0.1,\ z_{-3} =$ $15,\ z_{-2} = 0.6,\ z_{-1} = -0.7,\ z_0 = 1, t_{-3} = 2,\ t_{-2} = -4/3,\ t_{-1} = 1/6$ and $t_0 = 1/7$.

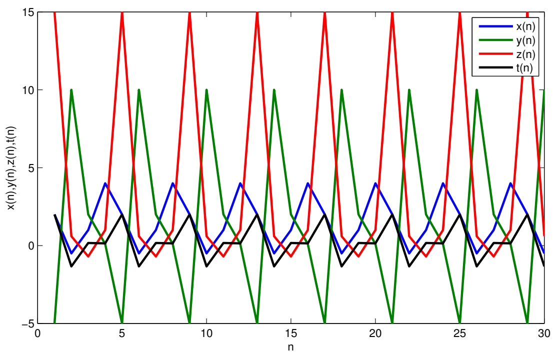

Figure 4: Plot the periodicity of the solution of system (3.1).

The following theorems can be proved similarly.

4. Other systems

In this section, we investigate the solutions of the following systems of the difference equations

$$x_{n+1}=\frac{x_{n-3}}{1-t_nz_{n-1}y_{n-2}x_{n-3}},\qquad y_{n+1}=\frac{y_{n-3}}{1-x_nt_{n-1}z_{n-2}y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{1-y_nx_{n-1}t_{n-2}z_{n-3}},\qquad t_{n+1}=\frac{t_{n-3}}{1-z_ny_{n-1}x_{n-2}t_{n-3}}. \tag{4.1}$$

$$x_{n+1}=\frac{x_{n-3}}{-1+t_nz_{n-1}y_{n-2}x_{n-3}},\qquad y_{n+1}=\frac{y_{n-3}}{-1+x_nt_{n-1}z_{n-2}y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{-1-y_nx_{n-1}t_{n-2}z_{n-3}},\qquad t_{n+1}=\frac{t_{n-3}}{-1-z_ny_{n-1}x_{n-2}t_{n-3}}. \tag{4.2}$$

$$x_{n+1}=\frac{x_{n-3}}{1+t_nz_{n-1}y_{n-2}x_{n-3}},\qquad y_{n+1}=\frac{y_{n-3}}{1-x_nt_{n-1}z_{n-2}y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{1+y_nx_{n-1}t_{n-2}z_{n-3}},\qquad t_{n+1}=\frac{t_{n-3}}{1-z_ny_{n-1}x_{n-2}t_{n-3}}. \tag{4.3}$$

$$x_{n+1}=\frac{x_{n-3}}{1-t_nz_{n-1}y_{n-2}x_{n-3}},\qquad y_{n+1}=\frac{y_{n-3}}{1+x_nt_{n-1}z_{n-2}y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{1+y_nx_{n-1}t_{n-2}z_{n-3}},\qquad t_{n+1}=\frac{t_{n-3}}{1-z_ny_{n-1}x_{n-2}t_{n-3}}. \tag{4.4}$$

$$x_{n+1}=\frac{x_{n-3}}{-1+t_nz_{n-1}y_{n-2}x_{n-3}},\qquad y_{n+1}=\frac{y_{n-3}}{-1+x_nt_{n-1}z_{n-2}y_{n-3}},$$
$$z_{n+1}=\frac{z_{n-3}}{1-y_nx_{n-1}t_{n-2}z_{n-3}},\qquad t_{n+1}=\frac{t_{n-3}}{1+z_ny_{n-1}x_{n-2}t_{n-3}}, \tag{4.5}$$

where $n\in\mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Theorem 4.1. *If $\{x_n,y_n,z_n,t_n\}$ are solutions of difference equation system (4.1), then for $n=0,1,2,...$,*

$$x_{4n-3}=x_{-3}\prod_{i=0}^{n-1}\frac{(-1+(4i)x_{-3}y_{-2}z_{-1}t_0)}{(-1+(4i+1)x_{-3}y_{-2}z_{-1}t_0)},\qquad x_{4n-2}=x_{-2}\prod_{i=0}^{n-1}\frac{(-1+(4i+1)t_{-3}x_{-2}y_{-1}z_0)}{(-1+(4i+2)t_{-3}x_{-2}y_{-1}z_0)},$$
$$x_{4n-1}=x_{-1}\prod_{i=0}^{n-1}\frac{(-1+(4i+2)z_{-3}t_{-2}x_{-1}y_0)}{(-1+(4i+3)z_{-3}t_{-2}x_{-1}y_0)},\qquad x_{4n}=x_0\prod_{i=0}^{n-1}\frac{(-1+(4i+3)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(4i+4)y_{-3}z_{-2}t_{-1}x_0)},$$
$$y_{4n-3}=y_{-3}\prod_{i=0}^{n-1}\frac{(-1+(4i)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(4i+1)y_{-3}z_{-2}t_{-1}x_0)},\qquad y_{4n-2}=y_{-2}\prod_{i=0}^{n-1}\frac{(-1+(4i+1)x_{-3}y_{-2}z_{-1}t_0)}{(-1+(4i+2)x_{-3}y_{-2}z_{-1}t_0)},$$
$$y_{4n-1}=y_{-1}\prod_{i=0}^{n-1}\frac{(-1+(4i+2)t_{-3}x_{-2}y_{-1}z_0)}{(-1+(4i+3)t_{-3}x_{-2}y_{-1}z_0)},\qquad y_{4n}=y_0\prod_{i=0}^{n-1}\frac{(-1+(4i+3)z_{-3}t_{-2}x_{-1}y_0)}{(-1+(4i+4)z_{-3}t_{-2}x_{-1}y_0)},$$
$$z_{4n-3}=z_{-3}\prod_{i=0}^{n-1}\frac{(-1+(4i)z_{-3}t_{-2}x_{-1}y_0)}{(-1+(4i+1)z_{-3}t_{-2}x_{-1}y_0)},\qquad z_{4n-2}=z_{-2}\prod_{i=0}^{n-1}\frac{(-1+(4i+1)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(4i+2)y_{-3}z_{-2}t_{-1}x_0)},$$
$$z_{4n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(-1+(4i+2)x_{-3}y_{-2}z_{-1}t_0)}{(-1+(4i+3)x_{-3}y_{-2}z_{-1}t_0)},\qquad z_{4n}=z_0\prod_{i=0}^{n-1}\frac{(-1+(4i+3)t_{-3}x_{-2}y_{-1}z_0)}{(-1+(4i+4)t_{-3}x_{-2}y_{-1}z_0)},$$
$$t_{4n-3}=t_{-3}\prod_{i=0}^{n-1}\frac{(-1+(4i)t_{-3}x_{-2}y_{-1}z_0)}{(-1+(4i+1)t_{-3}x_{-2}y_{-1}z_0)},\qquad t_{4n-2}=t_{-2}\prod_{i=0}^{n-1}\frac{(-1+(4i+1)z_{-3}t_{-2}x_{-1}y_0)}{(-1+(4i+2)z_{-3}t_{-2}x_{-1}y_0)},$$
$$t_{4n-1}=t_{-1}\prod_{i=0}^{n-1}\frac{(-1+(4i+2)y_{-3}z_{-2}t_{-1}x_0)}{(-1+(4i+3)y_{-3}z_{-2}t_{-1}x_0)},\qquad t_{4n}=t_0\prod_{i=0}^{n-1}\frac{(-1+(4i+3)x_{-3}y_{-2}z_{-1}t_0)}{(-1+(4i+4)x_{-3}y_{-2}z_{-1}t_0)},$$

where $\prod_{i=0}^{-1}A_i=1$.

Lemma 4.2. *If x_i, y_i, z_i, t_i for $i = -3, -2, -1, 0$ arbitrary real numbers and let $\{x_n,\ y_n,\ z_n,\ t_n\}$ are solutions of system* (4.1)*, then the following conditions hold:*

(i) If $x_{-3} = 0$, then we have $x_{4n-3} = 0$ and $y_{4n-2} = y_{-2},\ z_{4n-1} = z_{-1},\ t_{4n} = t_0$.
(ii) If $x_{-2} = 0$, then we have $x_{4n-2} = 0$ and $y_{4n-1} = y_{-1},\ z_{4n} = z_0,\ t_{4n-3} = t_{-3}$.
(iii) If $x_{-1} = 0$, then we have $x_{4n-1} = 0$ and $y_{4n} = y_0,\ z_{4n-3} = z_{-3},\ t_{4n-2} = t_{-2}$.
(iv) If $x_0 = 0$, then we have $x_{4n} = 0$ and $y_{4n-3} = y_{-3},\ z_{4n-2} = z_{-2},\ t_{4n-1} = t_{-1}$.
(v) If $y_{-3} = 0$, then we have $y_{4n-3} = 0$ and $x_{4n} = x_0,\ z_{4n-2} = z_{-2},\ t_{4n-1} = t_{-1}$.
(vi) If $y_{-2} = 0$, then we have $y_{4n-2} = 0$ and $x_{4n-3} = x_{-3},\ z_{4n-1} = z_{-1},\ t_{4n} = t_0$.
(vii) If $y_{-1} = 0$, then we have $y_{4n-1} = 0$ and $x_{4n-2} = x_{-2},\ z_{4n} = z_0,\ t_{4n-3} = t_{-3}$.
(viii) If $y_0 = 0$, then we have $y_{4n} = 0$ and $x_{4n-1} = x_{-1},\ z_{4n-3} = z_{-3},\ t_{4n-2} = t_{-2}$.
(ix) If $z_{-3} = 0$, then we have $z_{4n-3} = 0$ and $x_{4n-1} = x_{-1},\ y_{4n} = y_0,\ t_{4n-2} = t_{-2}$.
(x) If $z_{-2} = 0$, then we have $z_{4n-2} = 0$ and $x_{4n} = x_0,\ y_{4n-3} = y_{-3},\ t_{4n-1} = t_{-1}$.
(xi) If $z_{-1} = 0$, then we have $z_{4n-1} = 0$ and $x_{4n-3} = x_{-3},\ y_{4n-2} = y_{-2},\ t_{4n} = t_0$.
(xii) If $z_0 = 0$, then we have $z_{4n} = 0$ and $x_{4n-2} = x_{-2},\ y_{4n-1} = y_{-1}, t_{4n-3} = t_{-3}$.
(xiii) If $t_{-3} = 0$, then we have $t_{4n-3} = 0$ and $x_{4n-2} = x_{-2}, y_{4n-1} = y_{-1},\ z_{4n} = z_0$.
(xiv) If $t_{-2} = 0$, then we have $t_{4n-2} = 0$ and $x_{4n-1} = x_{-1},\ y_{4n} = y_0,\ z_{4n-3} = z_{-3}$.
(xv) If $t_{-1} = 0$, then we have $t_{4n-1} = 0$ and $x_{4n} = x_0,\ y_{4n-3} = y_{-3},\ z_{4n-2} = z_{-2}$.
(xvi) If $t_0 = 0$, then we have $t_{4n} = 0$ and $x_{4n-3} = x_{-3},\ y_{4n-2} = y_{-2},\ z_{4n-1} = z_{-1}$.

Theorem 4.3. *The form of the solutions of system* (4.2) *are given by the following formula:*

$$x_{4n-3} = \frac{x_{-3}}{\left(-1 + x_{-3}y_{-2}z_{-1}t_0\right)^n}, \qquad x_{4n-2} = \frac{(-1)^n x_{-2}\left(1 + t_{-3}x_{-2}y_{-1}z_0\right)^n}{\left(1 + 2t_{-3}x_{-2}y_{-1}z_0\right)^n},$$
$$x_{4n-1} = \frac{x_{-1}}{\left(-1 + z_{-3}t_{-2}x_{-1}y_0\right)^n}, \qquad x_{4n} = x_0\left(-1 + y_{-3}z_{-2}t_{-1}x_0\right)^n,$$
$$y_{4n-3} = \frac{y_{-3}}{\left(-1 + y_{-3}z_{-2}t_{-1}x_0\right)^n}, \qquad y_{4n-2} = y_{-2}\left(-1 + x_{-3}y_{-2}z_{-1}t_0\right)^n,$$
$$y_{4n-1} = \frac{(-1)^n y_{-1}\left(1 + 2t_{-3}x_{-2}y_{-1}z_0\right)^n}{\left(1 + t_{-3}x_{-2}y_{-1}z_0\right)^n}, \qquad y_{4n} = y_0\left(-1 + z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$z_{4n-3} = \frac{(-1)^n z_{-3}}{\left(1 + z_{-3}t_{-2}x_{-1}y_0\right)^n}, \qquad z_{4n-2} = \frac{(-1)^n z_{-2}\left(-1 + y_{-3}z_{-2}t_{-1}x_0\right)^n}{\left(-1 + 2y_{-3}z_{-2}t_{-1}x_0\right)^n},$$
$$z_{4n-1} = \frac{(-1)^n z_{-1}}{\left(1 + x_{-3}y_{-2}z_{-1}t_0\right)^n}, \qquad z_{4n} = (-1)^n z_0\left(1 + t_{-3}x_{-2}y_{-1}z_0\right)^n,$$
$$t_{4n-3} = \frac{(-1)^n t_{-3}}{\left(1 + t_{-3}x_{-2}y_{-1}z_0\right)^n}, \qquad t_{4n-2} = (-1)^n t_{-2}\left(-1 + z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$t_{4n-1} = \frac{(-1)^n t_{-1}\left(-1 + 2y_{-3}z_{-2}t_{-1}x_0\right)^n}{\left(-1 + y_{-3}z_{-2}t_{-1}x_0\right)^n}, \qquad t_{4n} = (-1)^n t_0\left(1 + x_{-3}y_{-2}z_{-1}t_0\right)^n,$$

where $x_{-3}y_{-2}z_{-1}t_0 \neq \pm1$, $z_{-3}t_{-2}x_{-1}y_0 \neq \pm1$, $t_{-3}x_{-2}y_{-1}z_0 \neq -1$, $t_{-3}x_{-2}y_{-1}z_0 \neq -\frac{1}{2}$, $y_{-3}z_{-2}t_{-1}x_0 \neq 1$ and $y_{-3}z_{-2}t_{-1}x_0 \neq \frac{1}{2}$.

Theorem 4.4. *Let $\{x_n, y_n, z_n, t_n\}$ are solutions of difference equation system* (4.3) *with $x_{-3}y_{-2}z_{-1}t_0 \neq -1$, $y_{-3}z_{-2}t_{-1}x_0 \neq 1$, $t_{-3}x_{-2}y_{-1}z_0 \neq 1$ and $z_{-3}t_{-2}x_{-1}y_0 \neq -1$, then for $n = 0, 1, 2, ...,$*

$$x_{4n-3} = \frac{x_{-3}}{\left(1 + x_{-3}y_{-2}z_{-1}t_0\right)^n}, \qquad x_{4n-2} = (-1)^n x_{-2}\left(-1 + t_{-3}x_{-2}y_{-1}z_0\right)^n,$$
$$x_{4n-1} = \frac{x_{-1}}{\left(1 + z_{-3}t_{-2}x_{-1}y_0\right)^n}, \qquad x_{4n} = (-1)^n x_0\left(-1 + y_{-3}z_{-2}t_{-1}x_0\right)^n,$$
$$y_{4n-3} = \frac{(-1)^n y_{-3}}{\left(-1 + y_{-3}z_{-2}t_{-1}x_0\right)^n}, \qquad y_{4n-2} = y_{-2}\left(1 + x_{-3}y_{-2}z_{-1}t_0\right)^n,$$

$$y_{4n-1} = \frac{(-1)^n y_{-1}}{(-1+t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad y_{4n} = y_0\left(1+z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$z_{4n-3} = \frac{z_{-3}}{(1+z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad z_{4n-2} = (-1)^n z_{-2}\left(-1+y_{-3}z_{-2}t_{-1}x_0\right)^n,$$
$$z_{4n-1} = \frac{z_{-1}}{(1+x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad z_{4n} = (-1)^n z_0\left(-1+t_{-3}x_{-2}y_{-1}z_0\right)^n,$$
$$t_{4n-3} = \frac{(-1)^n t_{-3}}{(-1+t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad t_{4n-2} = t_{-2}\left(1+z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$t_{4n-1} = \frac{(-1)^n t_{-1}}{(-1+y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad t_{4n} = t_0\left(1+x_{-3}y_{-2}z_{-1}t_0\right)^n.$$

Theorem 4.5. *Suppose that the initial conditions of the system (4.4) are arbitrary real numbers satisfies $x_{-3}y_{-2}z_{-1}t_0 \neq \pm 1$, $y_{-3}z_{-2}t_{-1}x_0 \neq -1$, $y_{-3}z_{-2}t_{-1}x_0 \neq -\frac{1}{2}$, $t_{-3}x_{-2}y_{-1}z_0 \neq 1$, $t_{-3}x_{-2}y_{-1}z_0 \neq \frac{1}{2}$, $z_{-3}t_{-2}x_{-1}y_0 \neq \pm 1$, and if $\{x_n, y_n, z_n, t_n\}$ are solutions of system (4.4), then for $n = 0, 1, 2, ...$,*

$$x_{4n-3} = \frac{(-1)^n x_{-3}}{(-1+x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad x_{4n-2} = \frac{x_{-2}\left(-1+t_{-3}x_{-2}y_{-1}z_0\right)^n}{(-1+2t_{-3}x_{-2}y_{-1}z_0)^n},$$
$$x_{4n-1} = \frac{(-1)^n x_{-1}}{(-1+z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad x_{4n} = x_0\left(1+y_{-3}z_{-2}t_{-1}x_0\right)^n,$$
$$y_{4n-3} = \frac{y_{-3}}{(1+y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad y_{4n-2} = (-1)^n y_{-2}\left(-1+x_{-3}y_{-2}z_{-1}t_0\right)^n,$$
$$y_{4n-1} = \frac{y_{-1}\left(-1+2t_{-3}x_{-2}y_{-1}z_0\right)^n}{(-1+t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad y_{4n} = (-1)^n y_0\left(-1+z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$z_{4n-3} = \frac{z_{-3}}{(1+z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad z_{4n-2} = \frac{z_{-2}\left(1+y_{-3}z_{-2}t_{-1}x_0\right)^n}{(1+2y_{-3}z_{-2}t_{-1}x_0)^n},$$
$$z_{4n-1} = \frac{z_{-1}}{(1+x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad z_{4n} = (-1)^n z_0\left(-1+t_{-3}x_{-2}y_{-1}z_0\right)^n,$$
$$t_{4n-3} = \frac{(-1)^n t_{-3}}{(-1+t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad t_{4n-2} = t_{-2}\left(1+z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$t_{4n-1} = \frac{t_{-1}\left(1+2y_{-3}z_{-2}t_{-1}x_0\right)^n}{(1+y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad t_{4n} = t_0\left(1+x_{-3}y_{-2}z_{-1}t_0\right)^n.$$

Theorem 4.6. *Assume that $\{x_n, y_n, z_n, t_n\}$ are solutions of the system (4.5), with $x_{-3}y_{-2}z_{-1}t_0 \neq 1$, $z_{-3}t_{-2}x_{-1}y_0 \neq 1$, $y_{-3}z_{-2}t_{-1}x_0 \neq 1$, $t_{-3}x_{-2}y_{-1}z_0 \neq -1$, $x_{-3}y_{-2}z_{-1}t_0 \neq 2$, $z_{-3}t_{-2}x_{-1}y_0 \neq 2$, $y_{-3}z_{-2}t_{-1}x_0 \neq 2$ and $t_{-3}x_{-2}y_{-1}z_0 \neq -2$, then for $n = 0, 1, 2, ...$*

$$x_{4n-3} = \frac{x_{-3}}{(-1+x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad x_{4n-2} = x_{-2}\left(-1-t_{-3}x_{-2}y_{-1}z_0\right)^n,$$
$$x_{4n-1} = \frac{x_{-1}}{(-1+z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad x_{4n} = x_0\left(-1+y_{-3}z_{-2}t_{-1}x_0\right)^n,$$
$$y_{4n-3} = \frac{y_{-3}}{(-1+y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad y_{4n-2} = y_{-2}\left(-1+x_{-3}y_{-2}z_{-1}t_0\right)^n,$$
$$y_{4n-1} = \frac{y_{-1}}{(-1-t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad y_{4n} = y_0\left(-1+z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$z_{4n-3} = \frac{z_{-3}}{(1-z_{-3}t_{-2}x_{-1}y_0)^n}, \qquad z_{4n-2} = z_{-2}\left(1-y_{-3}z_{-2}t_{-1}x_0\right)^n,$$
$$z_{4n-1} = \frac{z_{-1}}{(1-x_{-3}y_{-2}z_{-1}t_0)^n}, \qquad z_{4n} = z_0\left(1+t_{-3}x_{-2}y_{-1}z_0\right)^n,$$
$$t_{4n-3} = \frac{t_{-3}}{(1+t_{-3}x_{-2}y_{-1}z_0)^n}, \qquad t_{4n-2} = t_{-2}\left(1-z_{-3}t_{-2}x_{-1}y_0\right)^n,$$
$$t_{4n-1} = \frac{t_{-1}}{(1-y_{-3}z_{-2}t_{-1}x_0)^n}, \qquad t_{4n} = t_0\left(1-x_{-3}y_{-2}z_{-1}t_0\right)^n.$$

Theorem 4.7. *If the sequences* $\{x_n, y_n, z_n, t_n\}$ *are solutions of difference equation system* (4.5) *such that* $x_{-3}y_{-2}z_{-1}t_0 = z_{-3}t_{-2}x_{-1}y_0 = y_{-3}z_{-2}t_{-1}x_0 = 2$, $t_{-3}x_{-2}y_{-1}z_0 = -2$, *then* $\{x_n,\ y_n\}$ *are periodic with period four and* $\{z_n,\ t_n\}$ *are periodic with period eight and take the form*

$$\begin{array}{llll} x_{4n-3} = x_{-3}, & x_{4n-2} = x_{-2}, & x_{4n-1} = x_{-1}, & x_{4n} = x_0, \\ y_{4n-3} = y_{-3}, & y_{4n-2} = y_{-2}, & y_{4n-1} = y_{-1}, & y_{4n} = y_0, \\ z_{4n-3} = (-1)^n z_{-3}, & z_{4n-2} = (-1)^n z_{-2}, & z_{4n-1} = (-1)^n z_{-1}, & z_{4n} = (-1)^n z_0, \\ t_{4n-3} = (-1)^n t_{-3}, & t_{4n-2} = (-1)^n t_{-2}, & t_{4n-1} = (-1)^n t_{-1}, & t_{4n} = (-1)^n t_0. \end{array}$$

Example 4.8. Figure 5 shows the periodicity behavior of the solution of the difference system (4.5) with the initial conditions $x_{-3} = 2,\ x_{-2} = -0.5,\ x_{-1} = 1, x_0 = 4, y_{-3} = -5,\ y_{-2} = 3,\ y_{-1} = -2, y_0 = 0.1,\ z_{-3} = 5,\ z_{-2} = 0.6,\ z_{-1} = -0.1,\ z_0 = 1, t_{-3} = -2,\ t_{-2} = 4,\ t_{-1} = -1/6$ and $t_0 = -10/3$.

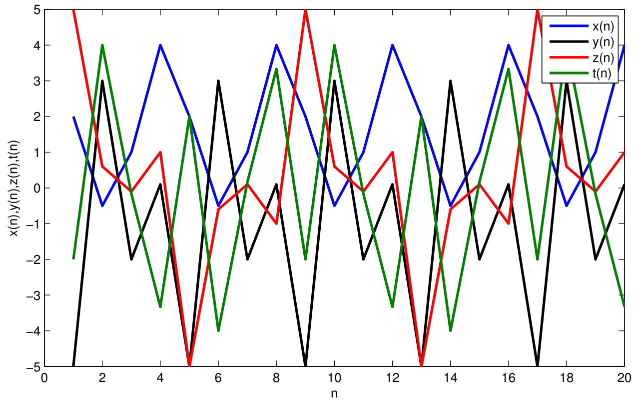

Figure 5: Plot the periodicity of the solution of system (4.5).

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