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Existence and multiplicity of solutions for nonlinear fractional differential equations

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Abstract

In this paper, we consider the following fractional initial value problems:

$$\begin{aligned} &D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)), \qquad t \in (0, 1], \\ &u^{(k)}(0) = \eta_k, \qquad k = 0, 1, ..., n-1, \end{aligned}$$

where $n-1 < \beta < \alpha < n$, $(n \in \mathbf{N})$ are real numbers, D^{α} and D^{β} are the Caputo fractional derivatives and $f \in C([0, 1] \times \mathbf{R})$. Using the fixed point index theory, we study the existence and multiplicity of positive solutions and obtain some new results.

Keywords: Fractional differential equation, positive solution, index fixed point theorem.
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1. Introduction

Fractional differential equations have received much attention recently. They arise in many engineering and scientific disciplines as the modeling of systems and processes in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (See [6, 10, 11]). A significant feature of a fractional order differential operator, in contrast to its counterpart in classical calculus, is its non local behavior. It means that the

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future state of a dynamical system or process based on the fractional differential operator depends on its current state as well its past states. Therefore, many papers and books on fractional calculus, fractional differential equations and fractional integral equations have appeared. Qualitative theory of differential equations is very useful in applications. So, recently much attention has been focused on the study of the existence and multiplicity of solutions of positive solutions for boundary and initial value problems of fractional differential equations. There are many techniques to deal with the existence of solutions of fractional differential equations such as fixed point theorems [5, 9, 13], upper and lower solutions method [8], fixed point index [4, 12, 13], coincidence theory [1], etc. In [2, 3, 7], the authors considered the existence of solutions of the following initial value problems

$$D^{\alpha}u(t) = f(t, D^{\beta}u(t)), \qquad t \in (0,1],$$
$$u^{(k)}(0) = \eta_k, \qquad k = 0, 1, ..., m-1,$$

but the assumptions are almost strong. In this paper, our object is to improve the situation. We consider

$$D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)), \qquad t \in (0,1], \tag{1.1}$$
$$u^{(k)}(0) = \eta_k, \qquad k = 0, 1, ..., n-1, \tag{1.2}$$

where $n-1 < \beta < \alpha < n$, $(n \in \mathbf{N})$, D^{α} , D^{β} are the Caputo fractional derivatives and $f \in C([0,1] \times \mathbf{R})$. By using the properties of Green function and index fixed point theorem, some new existence results for positive solutions are obtained. Moreover, the existence of two positive solutions on the initial value problem (1.1)–(1.2) is also considered.

The rest of the paper is organized as follows: In Section 2, we present some known results and introduce conditions to be used in the next section. The main results are formulated and proved in Section 3. An example is also presented to demonstrate the applications of the main results.

2. Background materials

For the convenience of the reader, in this section we shall state some necessary definitions and preliminary results. Let $E = C[0,1]$ be the Banach space with the maximum norm $||u|| = \max_{t\in[0,1]} u(t)$.

Definition 2.1 ([6, 11])**.** The fractional integral of order α for the function $u \in C[0,\infty) \bigcap L^1_{loc}[0,\infty)$ is defined as

$$I^{\alpha}u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds, \qquad 0 \le t \le 1,$$

where $m-1 < \alpha < m$ and $m \in \mathbf{N}$.

Definition 2.2 ([6, 11])**.** The α-th Caputo derivative of u is defined by

$${}^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds, \qquad 0 \le t \le 1,$$

where $n-1 < \alpha < n, n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

For the Caputo fractional derivative, the following equality holds

$${}^{c}D^{\alpha}(a_0 t^r + a_1 t^{r-1} + ... + a_1) = 0, \qquad m-1 < \alpha \le m,$$

where the degree of the polynomial is no more than $m-1$, i.e., $r \le m-1$. Moreover, the α-order integral of the α-order Caputo fractional derivative requires the knowledge of the initial values of the function and its integer order derivatives just as in the case of the integer order,

$$I^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{k=0}^{m-1} f^k(0^+)\frac{t^k}{k!}, \qquad m-1 < \alpha \le m. \tag{2.1}$$

Property 2.3 ([11])**.** *Let $\alpha \geq 0$ and let $n = [\alpha]+1$. If $u(t) \in AC^n[0,1]$, then the Caputo fractional derivative exists almost everywhere on $[0,1]$ and is represented by*

$$^cD^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^n(s)}{(t-s)^{\alpha-n+1}} ds. \tag{2.2}$$

Definition 2.4. $u \in C^m[0,1]$ is called a solution of (1.1)–(1.2) if it satisfies (1.1) and (1.2).

Definition 2.5. The beta function is defined by

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds, \quad x,y > 0.$$

It is known that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We can reduce problem (1.1)–(1.2) to an integral equation in E [3, 7].

Lemma 2.6. *Let $n \in \boldsymbol{N}$, $n-1 < \beta < \alpha < n$ and assume*

(i) *: $f : [0,1] \times \boldsymbol{R} \to \boldsymbol{R}$ is a continuously differentiable function;*

(ii) *: $f(0,0) = 0$ and $f(t,0) \neq 0$ on a compact subinterval of $[0,1]$.*

Then $u \in C^n[0,1]$ is a solution of (1.1)–(1.2) if and only if

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds, \qquad 0 \leq t \leq 1,$$

where $v \in C[0,1]$ is a solution of the equation

$$v(t) = \int_0^1 G(t,s) f(s, v(s)) ds, \qquad 0 \leq t \leq 1,$$

with

$$G(t,s) = \frac{1}{\Gamma(\alpha-\beta)} (t-s)_+^{\alpha-\beta-1}, \tag{2.3}$$

such that

$$(t-s)_+^{\alpha-\beta-1} = \begin{cases} (t-s)^{\alpha-\beta-1}, & 0 \leq s \leq t \leq 1, \\ 0, & s \geq t. \end{cases} \tag{2.4}$$

By making use of (2.3), we can prove that $G(t,s)$ has the following properties.

Proposition 2.7. $G(t,s) \leq t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} \leq s^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, \qquad 0 \leq s \leq t \leq 1.$

Proposition 2.8. $G(t,s) \geq \frac{1}{\Gamma(\alpha-\beta)}[s(1-s)]^{\alpha-\beta-1}, \qquad 0 \leq s \leq t \leq 1.$

Definition 2.9. An operator $T : D \to E$ is said to be completely continuous if it is continuous and compact.

Lemma 2.10 (The Schauder Fixed Point Theorem)**.** *Let $\boldsymbol{X}$ be a normed linear space, $\boldsymbol{X}_0 \subset \boldsymbol{X}$ be a convex closed set and $T : \boldsymbol{X}_0 \to \boldsymbol{X}_0$ be a completely continuous mapping. Then the mapping T has a fixed point in $\boldsymbol{X}_0$.*

Let

$$P = \{u \in E | u(t) \geq \frac{1}{\Gamma(\alpha-\beta)} ||u||, \quad t \in [0,1]\}.$$

Obviously, P is a cone in the Banach space E.

Define the operators T and L as follows

$$(Tu)(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s,u(s))ds, \tag{2.5}$$

$$(Lu)(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} u(s)ds. \tag{2.6}$$

To establish the existence of multiple positive solutions in E of problem (1.1)–(1.2), let us list the following assumptions, which will stand throughout the paper:

(H1) : $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous;

(H2) : $\liminf_{u\to o} \frac{f(t,u)}{u} > m$ uniformly with respect to $t \in [0,1]$;

(H3) : $\limsup_{u\to\infty} \frac{f(t,u)}{u} < k$ uniformly with respect to $t \in [0,1]$;

(H4) : $\limsup_{u\to o} \frac{f(t,u)}{u} < k$ uniformly with respect to $t \in [0,1]$;

(H5) : $\liminf_{u\to\infty} \frac{f(t,u)}{u} > m$ uniformly with respect to $t \in [0,1]$;

(H6) : There exists $r_0 > 0$ such that

$$f(t,u) < \left\{ \int_0^1 [s(1-s)]^{\alpha-\beta-1} \right\}^{-1} r_0 ds = \frac{r_0}{B(\alpha-\beta,\alpha-\beta)}, \quad 0 \le u \le r_0, \quad 0 \le t \le 1; \tag{2.7}$$

(H7) : There exist $\bar{r}_0 > 0$ such that

$$f(t,u) > \Gamma(\alpha-\beta)\{ \int_0^1 [s(1-s)]^{\alpha-\beta-1} \}^{-1} \bar{r}_0 ds \tag{2.8}$$

$$= \frac{\Gamma(\alpha-\beta)\bar{r}_0}{B(\alpha-\beta,\alpha-\beta)} \quad 0 \le u \le \bar{r}_0, \qquad 0 \le t \le 1, \tag{2.9}$$

where

$$m = \max\{1, \frac{B(\alpha-\beta,\alpha-\beta)}{\Gamma(\alpha-\beta)}\}, \tag{2.10}$$

$$k = \frac{1}{B(\alpha-\beta,\alpha-\beta)}, \tag{2.11}$$

and $B(x,y)$ is the known beta function.

Lemma 2.11. *Suppose* (H1) holds, then the operator $T : P \to P$ is completely continuous.

Proof. Firstly, we show that $T(P) \subset P$. By Proposition 2.7, we have

$$||Tu|| \le \int_0^t s^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} f(s,u(s))ds,$$

and then by Proposition 2.8, we obtain

$$(Tu)(t) \ge \frac{1}{\Gamma(\alpha-\beta)} \int_0^t [s(1-s)]^{\alpha-\beta-1} f(s,u(s))ds \ge \frac{1}{\Gamma(\alpha-\beta)} ||Tu||.$$

Hence, $T(p) \subset P$. By a standard argument, we can show that T is continuous and compact. Therefore is completely continuous. □

Lemma 2.12 ([4]). *Let $\Omega \subset E$ be a bounded open set, $T : \bar{\Omega} \cap P \to P$ be completely continuous. If there exists $u_0 \in P \backslash \{\theta\}$ such that $u - Tu \neq \mu u_0$, $\forall \mu \geq o$, $u \in \partial\Omega \cap P$, then $i(T, \Omega \cap P, P) = 0$.*

Lemma 2.13 ([4]). *Let $\Omega \subset E$ be a bounded open set with $\theta \in \Omega$. Suppose that $T : \bar{\Omega} \cap P \to P$ is completely continuous. If $u \neq \mu Tu$, $\forall \mu \in \partial\Omega \cap P$ and $0 \leq \mu \leq 1$, then $i(T, \Omega \cap P) = 1$.*

The main tool of the paper is the following well-known fixed point index result.

Lemma 2.14 ([4]). *Let $T : P \to P$ be a completely continuous mapping and $Ty \neq y$ for $y \in \partial B_r$. Then we have the following conclusions:*

(i) *: If $||y|| \leq ||Ty||$ for $y \in \partial B_r$, then $i(T, B_r, P) = 0$.*

(ii) *: If $||y|| \geq ||Ty||$ for $y \in \partial B_r$, then $i(T, B_r, P) = 1$.*

3. Main results

In this section, we investigate the existence and multiplicity of solutions for the initial value problem of nonlinear fractional differential equations (1.1)–(1.2).

Theorem 3.1. *Suppose that* (H1)–(H3) hold. Then the problem (1.1)–(1.2) has at least one positive solution.

Proof. From (H2), there exist $\epsilon > 0$ and $r > 0$ such that

$$f(t, u) \geq (m + \epsilon)u, \quad u \in [0, r]. \tag{3.1}$$

Therefore, for all $u \in \bar{B}_r \cap P$, by (3.1) we have

$$(Tu)(t) \geq (m + \epsilon) \int_0^t G(t, s)u(s)ds = (m + \epsilon)(Lu)(t), \quad t \in [0, 1]. \tag{3.2}$$

Similar to Lemma 2.11, we can show that the operator L defined in (2.6) is completely continuous and so by Lemma 2.10, it has a fixed point say, ϕ^*, i.e., $\phi^* = L(\phi^*)$. Now, we show that

$$u - Tu \neq \mu\phi^*, \quad \forall u \in \partial B_r \cap P, \ \ \mu \geq 0.$$

Otherwise, if there exist $u_0 \in \partial B_r \cap P$ and μ_0 such that $u_0 - Tu_0 = \mu_0\phi^*$, then

$$u_0 = Tu_0 + \mu_0\phi^* \geq \mu_0\phi^*.$$

If $\tau^* = \sup\{\tau | u_0 \geq \tau\phi^*\}$, then one has

$$(m + \epsilon)L(u_0) \geq mL(u_0) \geq \tau^* mL(\phi^*) = \tau^* m\phi^*.$$

Therefore,

$$u_0 = T(u_0) + \mu_0\phi^* \geq (m + \epsilon)L(u_0) + \mu_0\phi^* \geq \tau^* m\phi^* + \mu_0\phi^* \geq (\tau^* + \mu_0)\phi^*,$$

which contradicts definition of τ^*. So, for all $u \in \partial B_r \cap P$, $\mu \geq 0$. We have $u - Tu \neq \mu\phi^*$. Therefore, by Lemma 2.12 it follows that

$$i(T, B_r \cap P, P) = 0. \tag{3.3}$$

By (H3), there exists $R > r$ such that

$$f(t, u) \leq (k - \epsilon)u, \quad \forall u \in P, \ \geq R. \tag{3.4}$$

Then for $u \in \partial B_R \cap P$, by virtue of (3.4) and (2.11), we know that

$$\begin{aligned}||Tu(t)|| &\leq k\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (t-s)^{\alpha-\beta-1}||u||ds \\ &\leq k\int_0^1 [s(1-s)]^{\alpha-\beta-1}ds||u|| < R.\end{aligned}$$

So, this yields that

$$i(T, B_R \cap P, P) = 1. \tag{3.5}$$

From (3.3) and (3.5), we get

$$i(T,(B_R\backslash\bar{B}_r)\cap P, P) = i(T, B_R\cap P, P) - i(T, B_r \cap P, P) = 1.$$

Therefore, T has at least one fixed point on $(B_R\backslash\bar{B}_r)$. Consequently, problem (1.1)–(1.2) has at least one positive solution. □

Theorem 3.2. *Suppose that* (H1),(H4) and (H5) are satisfied. Then the problem (1.1)–1.2 has at least one positive solution.

Proof. By (H4), there exist $\epsilon > 0$ and $r_1 < 1$ such that

$$f(t,u) \leq ku, \quad u \in [0,r_1], \ \ t \in [0,1]. \tag{3.6}$$

Then for any $u \in \partial B_{r_1} \cap P$, by virtue of (3.6) and (2.11), we have

$$\begin{aligned}||Tu(t)|| &\leq k\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (t-s)^{\alpha-\beta-1}||u||ds \\ &\leq k\int_0^1 [s(1-s)]^{\alpha-\beta-1}ds||u|| \\ &= kB(\alpha-\beta,\alpha-\beta)||u|| < r_1.\end{aligned}$$

Also, by (H5), there exists $R_1 > 1$ such that

$$f(t,u) \geq (m+\epsilon)u, \quad u \geq R_1, \ \ t \in [0,1]. \tag{3.7}$$

Then for any $u \in \partial B_{R_1} \cap P$, by virtue of (3.7) and (2.10), we have

$$\begin{aligned}||Tu(t)|| &\geq (m+\epsilon)\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (t-s)^{\alpha-\beta-1}||u||ds \\ &\geq m\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 [s(1-s)]^{\alpha-\beta-1}ds||u|| \\ &= m\frac{B(\alpha-\beta,\alpha-\beta)}{\Gamma(\alpha-\beta)} > R_1.\end{aligned}$$

Hence, similar to Theorem 3.1, we know that T has a positive fixed point in $(B_{R_1}\backslash\bar{B}_{r_1})\cap P$. That is to say the problem (1.1)–(1.2) has a positive solution in $(B_{R_1}\backslash\bar{B}_{r_1})\cap P$. □

Theorem 3.3. *Suppose that* (H1),(H2),(H5) and (H6) are satisfied. Then the problem (1.1)–(1.2) has at least two positive solutions.

Proof. By the methods of Theorems 3.1 and 3.2 we can select $R > r_0 > r > 0$ such that

$$i(T, B_r \cap P, P) = 0, \tag{3.8}$$

$$i(T, B_R \cap P, P) = 0. \tag{3.9}$$

Then for any $u \in \partial B_{r_0} \cap P$, by Proposition 2.7 and (H6), we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\leq B(\alpha-\beta,\alpha-\beta)\sup\{f(t,u)|0\leq u\leq r_0,\ 0\leq t\leq 1\} \\ &< r_0, \quad 0\leq t\leq 1. \end{aligned} \tag{3.10}$$

This yields that for any $u \in \partial B_{r_0} \cap P$, we have $Tu < u$. In fact if there exist $u_0 \in \partial B_{r_0} \cap P$ such that $u_0(t) \leq Tu_0(t),\ t\in[0,1]$, then $||u_0|| < r_0$ which contradicts the fact that $u_0 \in \partial B_{r_0} \cap P$. Therefore, by the same method as in the proofs of previous theorems we have the following

$$i(T, B_R\backslash\bar{B}_{r_0} \cap P, P) = -1, \tag{3.11}$$

$$i(T, B_{r_0}\backslash\bar{B}_r \cap P, P) = 1. \tag{3.12}$$

Therefore, T has two positive fixed points in $B_R\backslash\bar{B}_{r_0} \cap P$ and $B_{r_0}\backslash\bar{B}_r \cap P$, respectively. Consequently, problem (1.1)–(1.2) has two positive solutions. □

Theorem 3.4. *Suppose that* (H1),(H3),(H4) and (H7) are satisfied. Then the problem (1.1)–(1.2) has at least two positive solutions.

Proof. By the methods of Theorems 3.1 and 3.2, we can select $\bar{R} > \bar{r}_0 > \bar{r} > 0$ such that

$$i(T, B_{\bar{r}} \cap P, P) = 1, \tag{3.13}$$

$$i(T, B_{\bar{R}} \cap P, P) = 1. \tag{3.14}$$

Then for any $u \in \partial B_{\bar{r}_0} \cap P$, by Proposition 2.8 and (H7), we have

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \tag{3.15}$$

$$\geq \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 [s(1-s)]^{\alpha-\beta-1}ds\{\Gamma(\alpha-\beta)\int_0^1 [s(1-s)]^{\alpha-\beta-1}\}^{-1}\bar{r}_0 \tag{3.16}$$

$$= \bar{r}_0 > u, \quad 0 \leq t \leq 1.$$

Therefore,

$$i(T, B_{\bar{r}_0} \cap P, P) = 0, \tag{3.17}$$

that is similar to the proof of Theorem 3.3, T has two positive fixed points in $B_{\bar{R}}\backslash\bar{B}_{\bar{r}_0} \cap P$ and $B_{\bar{r}_0}\backslash\bar{B}_{\bar{r}} \cap P$, respectively. Consequently, problem (1.1)–(1.2) has two positive solutions. □

Example 3.5. Consider the following fractional initial value problems

$$\begin{aligned} {}^cD^{m+\frac{3}{4}}u(t) &= \frac{1}{4}[\frac{1}{\pi^2}u^2 + 1 + \frac{3}{4}\ {}^cD^{m+\frac{1}{4}}\sin u], \quad 0<t<1, \\ u^k(0) &= 0, \qquad k=0,1,2,...,m, \end{aligned} \tag{3.18}$$

where $\alpha = m + \frac{3}{4}, \beta = m + \frac{1}{4}$, m=0,1,2,..., $f(t,u) = \frac{1}{4}[\frac{1}{\pi^2}u^2 + 1 + \frac{3}{4}\ {}^cD^{n+\frac{1}{4}}\sin u]$. By a simple computation, it is easy to see that (H1),(H2) and (H5) hold. Take $r_0 = \pi$, then $f(t,u) = \frac{1}{4}[\frac{1}{\pi^2}u^2 + 1 + \frac{3}{4}\ {}^cD^{n+\frac{1}{4}}\sin u] \leq \frac{3}{4}$ and $\{\int_0^1 [s(1-s)]^{\alpha-\beta-1}ds\}^{-1}r_0 = \frac{r_0}{B(\alpha-\beta,\alpha-\beta)} = 1$. Thus, (H6) holds. Therefore, it follows from Theorem 3.3 that the initial value problems (3.18) for any $m = 0,1,...$ has at least two positive solutions.

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