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Coupled fixed point results for (φ,G)-contractions of type (b) in b-metric spaces endowed with a graph

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Abstract

The purpose of this paper is to present some existence results for coupled fixed points of generalized contraction type operators in b-metric spaces endowed with a directed graph. Our results generalize the results obtained by Gnana Bhaskar and Lakshmikantham in [T. Gnana Bhaskar, V. Lakshmikantham, Nonlinear Anal., **65** (2006), 1379–1393]. Data dependence, well-posednes and Ulam-Hyres stability of the fixed point problem are also studied.

Keywords: Fixed point, coupled fixed point, b-metric space, connected graph.
2010 MSC: 47H10, 54H25.

1. Preliminaries

In fixed point theory, the importance of study of coupled fixed points is due to their applications to a wide variety of problems. Gnana Bhaskar and Lakshmikantham [8] gave some existence results for coupled fixed point for a mixed monotone type mapping in a metric space endowed with partial order, using a contraction type assumption on the mapping.

The purpose of this paper is to generalize these results using the context of b-metric spaces endowed with a graph. This new research direction in the theory of fixed points was initiated by Jachymski [11], and Gwóźdź-Lukawska and Jachymski [9]. Other results for single-valued and multi-valued operators in such metric spaces were given by Beg et al. in [1], Vetro and Vetro [19], and Chifu and Petrusel in [5].

Our results also generalize and extend some fixed point and coupled fixed point theorems in partially ordered complete metric spaces and b-metric spaces given by Harjani and Sadarangani [10], Nieto and Rodríguez-López [14, 16], Nieto et al. [15], Jleli et al. [13], O'Regan and Petruşel [17], Ran and Reurings [18], Gnana Bhaskar and Lakshmikantham [8], and Chifu and Petrusel in [6].

Let us recall now some essential definitions and fundamental results. We begin with the definition of a b-metric space.

Definition 1.1 ([7])**.** Let X be a nonempty set and let $s \geqslant 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied:

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1. $d(x,y) = 0$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,z) \leqslant s[d(x,y) + d(y,z)]$,

for all $x, y, z \in X$. In this case the pair (X, d) is called a b-metric space.

Remark 1.2. The class of b-metric spaces is larger than the class of metric spaces since a b-metric space is a metric space when $s = 1$. For more details and examples on b-metric spaces, see e.g., [2].

The following example will be useful for our results.

Example 1.3. Let (X, d) be a *b-metric space, with constant* $s \geqslant 1$, and let $Z = X \times X$. *The functional* $\widetilde{d} : Z \times Z \to [0, \infty)$, defined by

$$\widetilde{d}\left((x,y),(u,v)\right) = d\left(x,u\right) + d\left(y,v\right),$$

is a b-metric with the same constant $s \geqslant 1$ for all $(x,y),(u,v) \in Z$. Moreover if (X, d) is a complete b-metric space, then $\left(Z, \widetilde{d}\right)$ is a complete b-metric space, too.

Definition 1.4. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$ for any $t \in [0, \infty)$.

We recall the following essential result.

Lemma 1.5 ([4]). *If* $\varphi : [0, \infty) \to [0, \infty)$ *is a comparison function, then:*

(1) *each iterate* φ^k *of* φ *(where* $k \geqslant 1$*) is also a comparison function;*

(2) φ *is continuous at* 0*;*

(3) $\varphi(t) < t$ *for any* $t > 0$.

Berinde [4] introduced the concept of (c)-*comparison function* in the following way.

Definition 1.6 ([4]). A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(1) φ is increasing;
(2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leqslant a\varphi^k(t) + v_k$ for $k \geqslant k_0$ and any $t \in [0, \infty)$.

The notion of a (c)-comparison function was improved as a (b)-comparison function by Berinde [3], in order to extend some fixed point results to the class of b-metric spaces.

Definition 1.7 ([3]). Let $s \geqslant 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled:

(1) φ is monotone increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leqslant as^k\varphi^k(t) + v_k$ for $k \geqslant k_0$ and any $t \in [0, \infty)$.

It is obvious that the concept of (b)-comparison function reduces to that of (c)-comparison function when $s = 1$.

The following lemma has a crucial role in the proof of our main result.

Lemma 1.8 ([2]). *If* $\varphi : [0, \infty) \to [0, \infty)$ *is a* (b)*-comparison function, then we have the following:*

(1) *the series* $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ *converges for any* $t \in [0, \infty)$*;*

(2) *the function* $S_b : [0, \infty) \to [0, \infty)$ *defined by* $S_b(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t)$, $t \in [0, \infty)$, *is increasing and continuous at* 0.

We note that any (b)-comparison function is a comparison function due to the above lemma.

Let (X, d) be a metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Throughout the paper we shall say that G with the above mentioned properties satisfies standard conditions.

Let us denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Let us consider the function $F : X \times X \to X$.

Definition 1.9. An element $(x, y) \in X \times X$ is called *coupled fixed point* of the mapping F, if $F(x, y) = x$ and $F(y, x) = y$.

We shall denote by $CFix(F)$ the set of all coupled fixed points of mapping F, i.e.,

$$CFix(F) = \{(x, y) \in X \times X : F(x, y) = x \text{ and } F(y, x) = y\}.$$

Definition 1.10 ([6])**.** We say that $F : X \times X \to X$ is edge preserving if

$$(x, u) \in E(G) \text{ and } (y, v) \in E\left(G^{-1}\right) \Rightarrow (F(x, y), F(u, v)) \in E(G)$$

and

$$(F(y, x), F(v, u)) \in E\left(G^{-1}\right).$$

Definition 1.11 ([6])**.** The operator $F : X \times X \to X$ is called G-continuous if for all $(x, y) \in X \times X$, $(x^*, y^*) \in X \times X$ and for any sequence $(n_i)_{i \in \mathbb{N}}$ of positive integers, with $F^{n_i}(x, y) \to x^*$, $F^{n_i}(y, x) \to y^*$, as $i \to \infty$, and $\left(F^{n_i}(x, y), F^{n_i+1}(x, y)\right) \in E(G)$, $\left(F^{n_i}(y, x), F^{n_i+1}(y, x)\right) \in E\left(G^{-1}\right)$, we have that

$$\begin{array}{l} F\left(F^{n_i}(x, y), F^{n_i}(y, x)\right) \to F(x^*, y^*) \\ F\left(F^{n_i}(y, x), F^{n_i}(x, y)\right) \to F(y^*, x^*) \end{array}, \text{ as } i \to \infty.$$

Definition 1.12 ([6])**.** Let (X, d) be a b-metric space, with constant $s \geqslant 1$, and G be a directed graph. We say that the triple (X, d, G) has the property (A_1), if for any sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \to x$, as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have that $(x_n, x) \in E(G)$.

Definition 1.13 ([6])**.** Let (X, d) be a b-metric space, with constant $s \geqslant 1$, and G be a directed graph. We say that the triple (X, d, G) has the property (A_2), if for any sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \to x$, as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G^{-1})$, for $n \in \mathbb{N}$, we have that $(x_n, x) \in E(G^{-1})$.

2. Existence and data dependence results for coupled fixed point problems

Let (X, d) be a *b*-metric space, with constant $s \geqslant 1$, endowed with a directed graph G satisfying the standard conditions. We consider the set denoted by $(X \times X)^F$ and defined as:

$$(X \times X)^F = \left\{(x, y) \in X \times X : (x, F(x, y)) \in E(G) \text{ and } (y, F(y, x)) \in E\left(G^{-1}\right)\right\}.$$

Proposition 2.1 ([6])**.** *If* $F : X \times X \to X$ *is edge preserving, then:*

(i) $(x, u) \in E(G)$ *and* $(y, v) \in E\left(G^{-1}\right)$ *implies* $(F^n(x, y), F^n(u, v)) \in E(G)$ *and* $(F^n(y, x), F^n(v, u)) \in E\left(G^{-1}\right)$ *for all* $n \in \mathbb{N}$;

(ii) $(x,y) \in (X\times X)^F$ *implies* $\left(F^n(x,y), F^{n+1}(x,y)\right) \in E(G)$ *and* $\left(F^n(y,x), F^{n+1}(y,x)\right) \in E\left(G^{-1}\right)$ *for all* $n \in \mathbb{N}$;

(iii) $(x,y) \in (X\times X)^F$ *implies* $(F^n(x,y), F^n(y,x)) \in (X\times X)^F$ *for all* $n \in \mathbb{N}$.

Definition 2.2. The mapping $F : X \times X \to X$ is called (φ, G)-contraction of type (b) if:

(i) F is edge preserving;

(ii) there exists $\varphi : [0,\infty) \to [0,\infty)$ a (b)-comparison function such that

$$d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leqslant \varphi(d(x,u) + d(y,v)),$$
$$\text{for all } (x,u) \in E(G), (y,v) \in E(G^{-1}).$$

Lemma 2.3. *Let* (X,d) *be a b-metric space, with constant* $s \geqslant 1$*, endowed with a directed graph* G *and let* $F : X\times X \to X$ *be a* (φ, G)*-contraction of type* (b)*. Then,*

$$d(F^n(x,y), F^n(u,v)) + d(F^n(y,x), F^n(v,u)) \leqslant \varphi^n(d(x,u) + d(y,v)),$$

for all $(x,u) \in E(G), (y,v) \in E(G^{-1}), n \in \mathbb{N}$.

Proof. Let $(x,u) \in E(G), (y,v) \in E(G^{-1})$. Because F is edge preserving we have

$$(F(x,y), F(u,v)) \in E(G) \text{ and } (F(y,x), F(v,u)) \in E\left(G^{-1}\right).$$

From Proposition 2.1 (i) it follows that

$$(F^n(x,y), F^n(u,v)) \in E(G) \text{ and } (F^n(y,x), F^n(v,u)) \in E\left(G^{-1}\right).$$

Since F is a (φ, G)-contraction of type (b), we obtain

$$\begin{aligned} d\left(F^2(x,y), F^2(u,v)\right) + d\left(F^2(y,x), F^2(v,u)\right) &= d(F(F(x,y), F(y,x)), F(F(u,v), F(v,u))) \\ &\quad + d(F(F(y,x), F(x,y)), F(F(v,u), F(u,v))) \\ &\leqslant \varphi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \\ &\leqslant \varphi(\varphi(d(x,u) + d(y,v))) \\ &\leqslant \varphi^2(d(x,u) + d(y,v)). \end{aligned}$$

Hence, by induction, we reach the conclusion. □

Lemma 2.4. *Let* (X,d) *be a b-metric space, with constant* $s \geqslant 1$*, endowed with a directed graph* G *and let* $F : X\times X \to X$ *be a* (φ, G)*-contraction of type* (b)*. Then, given* $(x,y) \in (X\times X)^F$*, there exists* $r(x,y) \geqslant 0$ *such that*

$$d\left(F^n(x,y), F^{n+1}(x,y)\right) + d\left(F^n(y,x), F^{n+1}(y,x)\right) \leqslant \varphi^n(r(x,y)) \text{ for all } n \in \mathbb{N}.$$

Proof. Let $(x,y) \in (X\times X)^F$. It follows that $(x, F(x,y)) \in E(G)$ and $(y, F(y,x)) \in E\left(G^{-1}\right)$.

If in Lemma 2.3 we consider $u = F(x,y)$ and $v = F(y,x)$ we shall obtain

$$\begin{aligned} &d(F^n(x,y), F^n(F(x,y), F(y,x))) + d(F^n(y,x), F^n(F(y,x), F(x,y))) \\ &\quad \leqslant \varphi^n(d(x, F(x,y)) + d(y, F(y,x))) \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

which is

$$d\left(F^n(x,y), F^{n+1}(x,y)\right) + d\left(F^n(y,x), F^{n+1}(y,x)\right) \leqslant \varphi^n(d(x, F(x,y)) + d(y, F(y,x))) \text{ for all } n \in \mathbb{N}.$$

If we consider $r(x,y) := d(x, F(x,y)) + d(y, F(y,x))$, then

$$d\left(F^n(x,y), F^{n+1}(x,y)\right) + d\left(F^n(y,x), F^{n+1}(y,x)\right) \leqslant \varphi^n(r(x,y)) \quad \text{for all } n \in \mathbb{N}.$$

□

Lemma 2.5. *Let* (X, d) *be a complete b-metric space with constant* $s \geqslant 1$*, endowed with a directed graph* G *and let* $F : X \times X \to X$ *be a* (φ, G)*-contraction of type* (b)*. Then for each* $(x,y) \in (X \times X)^F$*, there exist* $x^*(x) \in X$ *and* $y^*(y) \in X$ *such that* $(F^n(x,y))_{n\in\mathbb{N}}$ *converges to* $x^*(x)$ *and* $(F^n(y,x))_{n\in\mathbb{N}}$ *converges to* $y^*(y)$*, as* $n \to \infty$*.*

Proof. Let $(x,y) \in (X \times X)^F$. It follows that $(x, F(x,y)) \in E(G)$ and $(y, F(y,x)) \in E\left(G^{-1}\right)$. Let $Z = X \times X$ and consider the b-metric given by Example 1.3, $\widetilde{d} : Z \times Z \to [0, \infty)$, defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v) \text{ for all } (x,y),(u,v) \in Z.$$

Consider also, the operator $T : Z \to Z$, defined by

$$T(x,y) = (F(x,y), F(y,x)) \text{ for all } (x,y) \in Z.$$

For (x,y) and $(u,v) \in (X \times X)^F$, we have

$$\widetilde{d}(T(x,y), T(u,v)) = d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)).$$

If $u = F(x,y)$ and $v = F(y,x)$, then $(u,v) \in (X \times X)^F$ and $T(u,v) = T^2(x,y)$. Hence

$$\widetilde{d}\left(T(x,y), T^2(x,y)\right) = d\left(F(x,y), F^2(x,y)\right) + d\left(F(y,x), F^2(y,x)\right).$$

By induction we shall obtain

$$\widetilde{d}\left(T^n(x,y), T^{n+1}(x,y)\right) = d\left(F^n(x,y), F^{n+1}(x,y)\right) + d\left(F^n(y,x), F^{n+1}(y,x)\right).$$

From Lemma 2.4, we have

$$\widetilde{d}\left(T^n(x,y), T^{n+1}(x,y)\right) \leqslant \varphi^n(r(x,y)) \text{ for all } n \in \mathbb{N}.$$

Now we shall prove that $(T^n(x,y))_{n\in\mathbb{N}}$ is a Cauchy sequence. We have

$$\begin{aligned}
\widetilde{d}\left(T^n(x,y), T^{n+p}(x,y)\right) &\leqslant s\widetilde{d}\left(T^n(x,y), T^{n+1}(x,y)\right) + s^2\widetilde{d}\left(T^{n+1}(x,y), T^{n+2}(x,y)\right) \\
&\quad + \cdots + s^{p-1}\widetilde{d}\left(T^{n+p-2}(x,y), T^{n+p-1}(x,y)\right) \\
&\quad + s^p\widetilde{d}\left(T^{n+p-1}(x,y), T^{n+p}(x,y)\right) \\
&\leqslant s\varphi^n(r(x,y)) + s^2\varphi^{n+1}(r(x,y)) + \ldots + s^{p-1}\varphi^{n+p-2}(r(x,y)) \\
&\quad + s^p\varphi^{n+p-1}(r(x,y)) \\
&= \frac{1}{s^{n-1}}\sum_{k=n}^{n+p-1} s^k\varphi^k(r(x,y)).
\end{aligned}$$

Let $S_n = \sum_{k=0}^{n} s^k\varphi^k(r(x,y))$. Hence we have

$$\widetilde{d}\left(T^n(x,y), T^{n+p}(x,y)\right) \leqslant \frac{1}{s^{n-1}}\left(S_{n+p-1} - S_{n-1}\right) \leqslant \frac{1}{s^{n-1}}\sum_{k=0}^{\infty} s^k\varphi^k(r(x,y)).$$

From Lemma 1.8 we have that the series is convergent. In this way, we shall obtain

$$\widetilde{d}\left(T^n(x,y), T^{n+p}(x,y)\right) \leqslant \frac{1}{s^{n-1}}\sum_{k=0}^{\infty} s^k\varphi^k(r(x,y)) \to 0, \text{ as } n \to \infty.$$

In conclusion the sequence $(T^n(x,y))_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since (X, d) is a complete b-metric space, from Example 1.3, we have that $\left(Z, \widetilde{d}\right)$ is a complete b-metric space, and hence there exists $(x^*(x), y^*(y)) \in X \times X$ such that $T^n(x,y) \to (x^*(x), y^*(y))$, as $n \to \infty$. This is equivalent to $(F^n(x,y), F^n(y,x)) \to (x^*(x), y^*(y))$, as $n \to \infty$.

Hence, there exist $x^*(x) \in X$ and $y^*(y) \in X$ such that $(F^n(x,y))_{n\in\mathbb{N}}$ and $(F^n(y,x))_{n\in\mathbb{N}}$ converge to $x^*(x)$ and $y^*(y)$, respectively, as $n \to \infty$. □

Now we shall prove the main results of this section.

Theorem 2.6. *Let* (X, d) *be a complete b-metric space with constant* $s \geqslant 1$*, endowed with a directed graph* G *and let* $F : X \times X \to X$ *be a* (φ, G)*-contraction of type* (b)*. Suppose that:*

(i) *F is G-continuous; or*
(ii) *the triple (X,d,G) has the properties* (A_1) *and* (A_2)*.*

In these conditions $CFix(F) \neq \varnothing$ *if and only if* $(X \times X)^F \neq \varnothing$*.*

Proof. Suppose that $CFix(F) \neq \varnothing$. Let $(x^*, y^*) \in CFix(F)$. We have $(x^*, F(x^*, y^*)) = (x^*, x^*) \in \Delta \subset E(G)$ and $(y^*, F(y^*, x^*)) = (y^*, y^*) \in \Delta \subset E(G^{-1})$.

Hence $(x^*, F(x^*, y^*)) \in E(G)$ and $(y^*, F(y^*, x^*)) \in E(G^{-1})$ which means that $(x^*, y^*) \in (X \times X)^F$ and thus $(X \times X)^F \neq \varnothing$.

Suppose now that $(X \times X)^F \neq \varnothing$. Let $(x, y) \in (X \times X)^F$. It follows that $(x, F(x, y)) \in E(G)$ and $(y, F(y, x)) \in E(G^{-1})$.

Let $(n_i)_{i \in \mathbb{N}}$ be a sequence of positive integers. From Proposition 2.1 (ii), we know that

$$\begin{aligned} &\left(F^{n_i}(x,y), F^{n_i+1}(x,y)\right) \in E(G) \\ &\left(F^{n_i}(y,x), F^{n_i+1}(y,x)\right) \in E\left(G^{-1}\right). \end{aligned} \tag{2.1}$$

Moreover from Lemma 2.5, there exist $x^*(x) \in X$ and $y^*(y) \in X$ such that

$$\begin{aligned} cF^{n_i}(x,y) &\to x^*(x), \\ F^{n_i}(y,x) &\to y^*(y), \end{aligned} \quad \text{as } i \to \infty. \tag{2.2}$$

We shall prove that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$. Suppose that (i) takes place. Since F is G-continuous we shall obtain that

$$\begin{aligned} cF\left(F^{n_i}(x,y), F^{n_i}(y,x)\right) &\to F(x^*, y^*), \\ F\left(F^{n_i}(y,x), F^{n_i}(x,y)\right) &\to F(y^*, x^*), \end{aligned} \quad \text{as } i \to \infty.$$

Now

$$\begin{aligned} d(F(x^*,y^*),x^*) + d(F(y^*,x^*),y^*) \leqslant\; & s\left[d\left(F(x^*,y^*), F^{n_i+1}(x,y)\right) + d\left(F^{n_i+1}(x,y), x^*\right)\right] \\ & + s\left[d\left(F(y^*,x^*), F^{n_i+1}(y,x)\right) + d\left(F^{n_i+1}(y,x), y^*\right)\right]. \end{aligned}$$

Using the G-continuity of F and the convergence of $(F^n(x,y))_{n \in \mathbb{N}}$, we obtain that $d(F(x^*,y^*),x^*) + d(F(y^*,x^*),y^*) = 0$, i.e., $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$.

Thus (x^*, y^*) is a coupled fixed point of the mapping F, so $CFix(F) \neq \varnothing$.

Suppose now that (ii) takes place. From (2.1) and (2.2), using properties (A_1) and (A_2) of the triple (X, d, G), we shall obtain that

$$\left(F^n(x,y), x^*\right) \in E(G), \quad \left(F^n(y,x), y^*\right) \in E\left(G^{-1}\right).$$

We have

$$\begin{aligned} d(F(x^*,y^*),x^*) + d(F(y^*,x^*),y^*) &\leqslant s\left[d\left(F^{n+1}(x,y), F(x^*,y^*)\right) + d\left(F^{n+1}(x,y), x^*\right)\right] \\ &\quad + s\left[d\left(F^{n+1}(y,x), F(y^*,x^*)\right) + d\left(F^{n+1}(y,x), y^*\right)\right] \\ &= s\left[d(F(F^n(x,y), F^n(y,x)), F(x^*,y^*)) + d\left(F^{n+1}(x,y), x^*\right)\right] \\ &\quad + s\left[d(F(F^n(y,x), F^n(x,y)), F(y^*,x^*)) + d\left(F^{n+1}(y,x), y^*\right)\right] \\ &\leqslant s\varphi\left(d\left(F^n(x,y), x^*\right) + d\left(F^n(y,x), y^*\right)\right) + sd\left(F^{n+1}(x,y), x^*\right) \\ &\quad + sd\left(F^{n+1}(y,x), y^*\right) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence $d(F(x^*,y^*),x^*) + d(F(y^*,x^*),y^*) = 0$, which means that $F(x^*,y^*) = x^*$ and $F(y^*,x^*) = y^*$. Thus, $(x^*,y^*) \in CFix(F)$. □

Let us suppose now that for every $(x,y),(u,v) \in X \times X$, there exists $(z,w) \in X \times X$ such that

$$(x,z) \in E(G),(y,w) \in E(G^{-1}), \quad (u,z) \in E(G),(v,w) \in E(G^{-1}). \tag{2.3}$$

Theorem 2.7. *Adding condition (2.3) to the hypotheses of Theorem 2.6 we obtain the uniqueness of the coupled fixed point of F.*

Proof. Let us suppose that there exist $(x^*,y^*),(u^*,v^*) \in X \times X$ two coupled fixed points of F. From (2.3) we have that there exists $(z,w) \in X \times X$ such that

$$(x^*,z) \in E(G),(y^*,w) \in E(G^{-1}), \quad (u^*,z) \in E(G),(v^*,w) \in E(G^{-1}).$$

Using Lemma 2.3, we shall have

$$\begin{aligned} d(x^*,u^*)+d(y^*,v^*) &= d(F^n(x^*,y^*),F^n(u^*,v^*))+d(F^n(y^*,x^*),F^n(v^*,u^*)) \\ &\leqslant s\left[d(F^n(x^*,y^*),F^n(z,w))+d(F^n(z,w),F^n(u^*,v^*))\right] \\ &\quad + s\left[d(F^n(y^*,x^*),F^n(w,z))+d(F^n(w,z),F^n(v^*,u^*))\right] \\ &\leqslant \varphi^n\left(d(x^*,z)+d(y^*,w)\right)+\varphi^n\left(d(u^*,z)+d(v^*,w)\right) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence $d(x^*,u^*)+d(y^*,v^*)=0$ and thus we obtain that $x^*=u^*$ and $y^*=v^*$. □

Remark 2.8. It is obvious that if $(x^*,u^*) \in E(G)$ and $(y^*,v^*) \in E(G^{-1})$, then $x^*=u^*$ and $y^*=v^*$.

Theorem 2.9. *In the conditions of Theorem 2.6, if $(x^*,y^*) \in CFix(F)$ with $(x^*,y^*) \in E(G)$, then $x^*=y^*$.*

Proof. Since $(x^*,y^*) \in E(G)$, then $(y^*,x^*) \in E(G^{-1})$. By the fact that F is a (φ,G)-contraction of type (b), we have

$$\begin{aligned} 2d(x^*,y^*) &= d\left(F(x^*,y^*),F(y^*,x^*)\right)+d\left(F(y^*,x^*),F(x^*,y^*)\right) \\ &\leqslant \varphi\left(d(x^*,y^*)+d(y^*,x^*)\right)=\varphi\left(2d(x^*,y^*)\right). \end{aligned}$$

From the properties of φ, we obtain that $d(x^*,y^*)=0$ and thus $x^*=y^*$. □

Remark 2.10. It is obvious that if we consider a function $f: X \to X, f(x)=F(x,x)$ all these results concerning the coupled fixed point of the mapping F result in the existence and uniqueness results for the fixed point of f.

In what follows we shall give a data-dependence result.

Theorem 2.11 (data dependence)**.** *Let (X,d) be a complete b-metric space with constant $s \geqslant 1$, endowed with a directed graph G and let $F_i : X \times X \to X, i \in \{1,2\}$ be two mappings. Assume that the following conditions are satisfied:*

(i) *F_1 is a (φ,G)-contraction of type (b);*
(ii) *F_1 is G-continuous;*

or

(ii*) *the triple (X,d,G) has the properties (A_1) and (A_2);*
(iii) *for every $(x,y),(u,v) \in X \times X$, there exists $(z,w) \in X \times X$ such (2.3) holds;*
(iv) $CFix(F_2) \neq \varnothing$;
(v) *there exists $\eta > 0$ such that*

$$d\left(F_1(x,y),F_2(x,y)\right) \leqslant \eta, \ \forall (x,y) \in X \times X.$$

In these conditions, if (x^*, y^*) *denotes the unique coupled fixed point of* F_1*, then*

$$d(x^*, \overline{x}) + d(y^*, \overline{y}) \leqslant \sup\{t \in \mathbb{R}_+ \mid t - s\varphi(t) \leqslant 2s\eta\},$$
$$\forall (\overline{x}, \overline{y}) \in CFix(F_2) \text{ and } (x^*, \overline{x}) \in E(G), (y^*, \overline{y}) \in E(G^{-1}).$$

Proof. Let $(x^*, y^*) \in X \times X$ be the unique coupled fixed point of F_1. It follows that

$$\begin{cases} x^* = F_1(x^*, y^*), \\ y^* = F_1(y^*, x^*). \end{cases}$$

Since $CFix(F_2) \neq \emptyset$, let $(\overline{x}, \overline{y}) \in CFix(F_2)$ with $(x^*, \overline{x}) \in E(G), (y^*, \overline{y})\, E(G^{-1})$. Let $Z = X \times X$ and consider the functional $\widetilde{d} : Z \times Z \to [0, \infty)$ defined by

$$\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v) \text{ for all } (x, y), (u, v) \in Z.$$

We have

$$\begin{aligned} \widetilde{d}((x^*, y^*), (\overline{x}, \overline{y})) &= \widetilde{d}((F_1(x^*, y^*), F_1(y^*, x^*)), (F_2(\overline{x}, \overline{y}), F_2(\overline{y}, \overline{x}))) \\ &= d(F_1(x^*, y^*), F_2(\overline{x}, \overline{y})) + d(F_1(y^*, x^*), F_2(\overline{y}, \overline{x})) \\ &\leqslant s[d(F_1(x^*, y^*), F_1(\overline{x}, \overline{y})) + d(F_1(\overline{x}, \overline{y}), F_2(\overline{x}, \overline{y}))] \\ &\quad + s[d(F_1(y^*, x^*), F_1(\overline{y}, \overline{x})) + d(F_1(\overline{y}, \overline{x}), F_2(\overline{y}, \overline{x}))] \\ &\leqslant s\varphi(d(x^*, \overline{x}) + d(y^*, \overline{y})) + 2s\eta. \end{aligned}$$

Hence $d(x^*, \overline{x}) + d(y^*, \overline{y}) \leqslant \sup\{t \in \mathbb{R}_+ \mid t - s\varphi(t) \leqslant 2s\eta\}, \forall (x^*, y^*) \in CFix(F_1)$ and $(\overline{x}, \overline{y}) \in CFix(F_2)$. □

Remark 2.12. In the light of the recent approach in [12], it is an open question to give similar results in the context of K-metric spaces.

3. Well-posedness and Ulam-Hyers stability

Let $F : X \times X \to X$. Consider now the following coupled fixed point problem

$$\begin{cases} x = F(x, y), \\ y = F(y, x), \end{cases} \tag{P1}$$

Definition 3.1. Let (X, d) be a complete b-metric space with constant $s \geqslant 1$. By definition, the coupled fixed point problem (P1) is said to be well-posed if:

(i) $CFix(F) = \{(x^*, y^*)\}$;
(ii) for any sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $X \times X$ for which $d(x_n, F(x_n, y_n)) \to 0$ and respectively $d(y_n, F(y_n, x_n)) \to 0$ as $n \to \infty$, we have that $x_n \to x^*$ and $y_n \to y^*$, as $n \to \infty$.

Theorem 3.2. *Suppose that the operator* $F : X \times X \to X$ *verifies all hypotheses of Theorem 2.7 and for any sequence* $(x_n, y_n)_{n\in\mathbb{N}}$ *in* $X \times X$ *having property that* $d(x_n, F(x_n, y_n)) \to 0$ *and respectively* $d(y_n, F(y_n, x_n)) \to 0$ *as* $n \to \infty$*, we have* $(x_n, x^*) \in E(G)$ *and* $(y_n, y^*) \in E(G^{-1})$*. If the mapping* $\psi : [0, \infty) \to \mathbb{R}$, $\psi(t) = t - s\varphi(t)$*, is such that* $\psi(t) \geqslant 0, \forall t \in \mathbb{R}_+$ *and* $\psi(0) = 0$ *implies that* $t = 0$*, then the coupled fixed point problem (P1) is well-posed.*

Proof. By Theorem 2.7, it follows that the coupled fixed point problem (P1) has a unique solution (x^*, y^*), i.e., $CFix(F) = \{(x^*, y^*)\}$.

Let $(x_n, y_n)_{n\in\mathbb{N}} \subset X \times X$ be a sequence which verifies the following properties:

(a) $d(x_n, F(x_n, y_n)) \to 0$ and respectively $d(y_n, F(y_n, x_n)) \to 0$ as $n \to \infty$;
(b) $(x_n, x^*) \in E(G)$ and $(y_n, y^*) \in E(G^{-1})$.

Let $Z = X \times X$ and consider the functional $\widetilde{d} : Z \times Z \to [0, \infty)$ defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v) \text{ for all } (x,y),(u,v) \in Z.$$

We have

$$\begin{aligned}\widetilde{d}((x_n,y_n),(x^*,y^*)) &= \widetilde{d}((x_n,y_n),(F(x^*,y^*),F(y^*,x^*)))\\ &\leqslant s\widetilde{d}((x_n,y_n),(F(x_n,y_n),F(y_n,x_n)))\\ &\quad + s\widetilde{d}((F(x_n,y_n),F(y_n,x_n)),(F(x^*,y^*),F(y^*,x^*)))\\ &\leqslant s\widetilde{d}((x_n,y_n),(F(x_n,y_n),F(y_n,x_n))) + s\varphi\left(\widetilde{d}((x_n,y_n),(x^*,y^*))\right).\end{aligned}$$

Hence

$$\widetilde{d}((x_n,y_n),(x^*,y^*)) - s\varphi\left(\widetilde{d}((x_n,y_n),(x^*,y^*))\right) \leqslant s\widetilde{d}((x_n,y_n),(F(x_n,y_n),F(y_n,x_n))).$$

Since the mapping $\psi : [0,\infty) \to \mathbb{R}$, $\psi(t) = t - s\varphi(t)$, is such that $\psi(t) \geqslant 0, \forall t \in \mathbb{R}_+$ and $\psi(0) = 0$ implies that $t = 0$, then, letting $n \to \infty$, we get that $(x_n, y_n) \to (x^*, y^*)$. □

In what follows we shall give an Ulam-Hyers stability result for the coupled fixed point problem (P1).

Definition 3.3. Let (X, d) be a complete b-metric space with constant $s \geqslant 1$, and let $\widetilde{d}$ be any b-metric on $Z = X \times X$ generated by d. By definition, the coupled fixed point problem (P1) is said to be Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, increasing, continuous in 0 with $\psi(0) = 0$, such that for each $\varepsilon \in \mathbb{R}_+^*$ and for each solution $(u^*, v^*) \in X \times X$ of the inequality $\widetilde{d}((x,y),(F(x,y),F(y,x))) \leqslant \varepsilon$, there exists a solution $(x^*, y^*) \in X \times X$ of the coupled fixed point problem (P1) such that

$$\widetilde{d}((x^*,y^*),(u^*,v^*)) \leqslant \psi(\varepsilon).$$

Theorem 3.4. *Assume that all the hypotheses of Theorem 2.7 take place. If the mapping $\gamma : [0,\infty) \to \mathbb{R}$, $\gamma(t) = t - s\varphi(t)$ is such that $\gamma(t) \geqslant 0, \forall t \in \mathbb{R}_+$ and $\gamma(0) = 0$ implies that $t = 0$, then the coupled fixed point problem (P1) is Ulam-Hyers stable.*

Proof. By Theorem 2.7 we get that $CFix(F) = \{(x^*, y^*)\}$. Let $\varepsilon > 0$ and let $(u^*, v^*) \in X \times X$ such that $\widetilde{d}((u^*,v^*),(F(u^*,v^*),F(v^*,u^*))) \leqslant \varepsilon$ and $(x^*, u^*) \in E(G)$, $(y^*, v^*) \in E(G^{-1})$.

Let $Z = X \times X$ and consider the functional $\widetilde{d} : Z \times Z \to [0, \infty)$ defined by

$$\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v) \text{ for all } (x,y),(u,v) \in Z.$$

We have

$$\begin{aligned}\widetilde{d}((u^*,v^*),(x^*,y^*)) &= \widetilde{d}((u^*,v^*),(F(x^*,y^*),F(y^*,x^*)))\\ &\leqslant s\widetilde{d}((u^*,v^*),(F(u^*,v^*),F(v^*,u^*)))\, s\widetilde{d}((F(u^*,v^*),F(v^*,u^*)),(F(x^*,y^*),F(y^*,x^*)))\\ &\leqslant s\varepsilon + s\varphi\left(\widetilde{d}((u^*,v^*),(x^*,y^*))\right).\end{aligned}$$

Hence

$$\widetilde{d}((u^*,v^*),(x^*,y^*)) - s\varphi\left(\widetilde{d}((u^*,v^*),(x^*,y^*))\right) \leqslant s\varepsilon.$$

Thus we obtain that

$$\widetilde{d}((u^*,v^*),(x^*,y^*)) \leqslant \psi(\varepsilon),$$

where

$$\psi(\varepsilon) := \sup\{t \in \mathbb{R}_+ | t - s\varphi(t) \leqslant s\varepsilon\}.$$

Since the mapping $\gamma : [0,\infty) \to \mathbb{R}$, $\gamma(t) = t - s\varphi(t)$ is such that $\gamma(t) \geqslant 0, \forall t \in \mathbb{R}_+$ and $\gamma(0) = 0$ implies that $t = 0$, then the coupled fixed point problem (P1) is Ulam-Hyers stable. □

4. Applications

In what follows we shall give an application for Theorem 2.6. Let us consider the following problem:

$$\begin{cases} x''(t) = f(t, x(t), y(t)), \\ y''(t) = f(t, y(t), x(t)), \\ x(0) = x'(1) = y(0) = y'(1), \end{cases} \quad t \in [0,1]. \tag{4.1}$$

Notice now that the problem (4.1) is equivalent with the following integral system

$$\begin{cases} x(t) = \int_0^1 K(t,s) f(s, x(s), y(s)) ds, \\ y(t) = \int_0^1 K(s,t) f(s, y(s), x(s)) ds, \end{cases} \quad t \in [0,1], \tag{4.2}$$

where

$$K(t,s) = \begin{cases} t, & t \leqslant s, \\ s, & t > s. \end{cases}$$

The purpose of this section is to give existence and uniqueness results for the solution of the system (4.2) using Theorem 2.6.

Let us consider $X := C([0,1], \mathbb{R}^n)$ endowed with the following b-metric with $s = 2$

$$d(x,y) = \max_{t \in [0,1]} (x(t) - y(t))^2.$$

Consider also the graph G defined by the partial order relation, i.e.,

$$x, y \in X, x \leqslant y \Leftrightarrow x(t) \leqslant y(t) \text{ for any } t \in [0,1].$$

Since $(X, \leqslant)$ is a lattice, we get that (X, G) has the property (2.3). Hence (X, d) is a complete *b*-metric space endowed with a directed graph G.

If we consider $E(G) = \{(x,y) \in X \times X : x \leqslant y\}$, then the diagonal Δ of $X \times X$ is included in $E(G)$. On the other hand $E(G^{-1}) = \{(x,y) \in X \times X : y \leqslant x\}$. Moreover $(X, \|\cdot\|, G)$ has the properties (A_1) and (A_2). In this case $(X \times X)^F = \{(x,y) \in X \times X : x \leqslant F(x,y) \text{ and } F(y,x) \leqslant y\}$.

Theorem 4.1. *Consider the system (4.1). Suppose:*

(i) $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous;*

(ii) *for all* $x, y, u, v \in \mathbb{R}^n$ *with* $x \leqslant u, v \leqslant y$ *we have* $f(t,x,y) \leqslant f(t,u,v)$ *for all* $t \in [0,1]$;

(iii) *there exists* $\widetilde{\varphi} : [0,\infty) \to [0,\infty)$ *a* (b)*-comparison function and* $\alpha, \beta \in (0,\infty)$, *with* $\max\{\alpha,\beta\} < 1$, *such that*

$$(f(t,x,y) - f(t,u,v))^2 \leqslant \widetilde{\varphi}\left(\alpha (x-u)^2 + \beta (y-v)^2\right) \text{ for each } t \in [0,1], x, y, u, v \in \mathbb{R}^n, x \leqslant u, v \leqslant y;$$

(iv) *there exists* $(x_0, y_0) \in X \times X$ *such that*

$$\begin{cases} x_0(t) \leqslant \int_0^1 K(t,s) f(s, x_0(s), y_0(s)) ds, \\ y_0(t) \geqslant \int_0^1 K(t,s) f(s, y_0(s), x_0(s)) ds, \end{cases} \quad t \in [0,1].$$

Then, there exists a unique solution of the integral system (4.2)*.*

Proof. Let $F : X \times X \to X, (x, y) \longmapsto F(x, y)$, where

$$F(x,y)(t) = \int_0^1 K(t,s)\, f(s,x(s),y(s))ds, t \in [0,1]. \tag{4.3}$$

In this way, the system (4.2) can be written as

$$\begin{cases} x = F(x,y), \\ y = F(y,x). \end{cases} \tag{4.4}$$

It can be seen from (4.4), that a solution of this system is a coupled fixed point of the mapping F.

We shall verify if the conditions of Theorem 2.6 are fulfilled.

Let $x, y, u, v \in X$ such that $x \leqslant u$ and $v \leqslant y$. Using (ii), we have

$$F(x,y)(t) = \int_0^1 K(t,s)\, f(s,x(s),y(s))ds \leqslant \int_0^1 K(t,s)\, f(s,u(s),v(s))ds = F(u,v)(t) \text{ for each } t \in [0,1],$$

$$F(v,u)(t) = \int_0^1 K(t,s)\, f(s,v(s),u(s))ds \leqslant \int_0^1 K(t,s)\, f(s,y(s),x(s))ds = F(y,x)(t) \text{ for each } t \in [0,1].$$

Hence, if $x \leqslant u$ and $v \leqslant y$, then $F(x,y) \leqslant F(u,v)$ and $F(v,u) \leqslant F(y,x)$, which according to the definition of $E(G)$, it shows that F is edge preserving. On the other hand, by Cauchy-Buniakovski-Schwarz inequality, we have

$$\begin{aligned}(F(x,y)(t) - F(u,v)(t))^2 &\leqslant \left[\int_0^1 K(t,s)\,(f(s,x(s),y(s)) - f(s,u(s),v(s)))ds\right]^2 \\ &\leqslant \int_0^1 K^2(t,s)\, ds \int_0^1 (f(s,x(s),y(s)) - f(s,u(s),v(s)))^2\, ds \text{ for each } t \in [0,1].\end{aligned}$$

We have

$$\int_0^1 K^2(t,s)\, ds = \int_0^t s^2 ds + \int_t^1 t^2 ds = t^2\left(1 - \frac{2}{3}t\right) \leqslant \frac{1}{3} \text{ for each } t \in [0,1].$$

Hence

$$\begin{aligned}(F(x,y)(t) - F(u,v)(t))^2 &\leqslant \frac{1}{3}\int_0^1 (f(s,x(s),y(s)) - f(s,u(s),v(s)))^2\, ds \\ &\leqslant \frac{1}{3}\int_0^1 \widetilde{\varphi}(\alpha\,(x(s) - u(s))^2 + \beta\,(y(s) - v(s))^2)ds \\ &\leqslant \frac{1}{3}\widetilde{\varphi}\,(\alpha d\,(x,u) + \beta d\,(y,v)) \\ &\leqslant \frac{1}{3}\widetilde{\varphi}\,(\max\{\alpha,\beta\}\,(d\,(x,u) + d\,(y,v))).\end{aligned}$$

Hence

$$d\left(F(x,y),F(u,v)\right) \leqslant \frac{1}{3}\widetilde{\varphi}\left(\max\{\alpha,\beta\}\left(d\left(x,u\right)+d\left(y,v\right)\right)\right), x \leqslant u, v \leqslant y. \tag{4.5}$$

In a similar way, we obtain

$$d\left(F(y,x),F(v,u)\right) \leqslant \frac{1}{3}\widetilde{\varphi}\left(\max\{\alpha,\beta\}\left(d\left(x,u\right)+d\left(y,v\right)\right)\right), x \leqslant u, v \leqslant y. \tag{4.6}$$

By (4.5) and (4.6) we have

$$d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right) \leqslant \frac{2}{3}\widetilde{\varphi}\left(\max\{\alpha,\beta\}\left(d\left(x,u\right)+d\left(y,v\right)\right)\right), x \leqslant u, v \leqslant y.$$

Let us consider the function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi\left(t\right) = \frac{2}{3}\widetilde{\varphi}\left(kt\right), 0 \leqslant k < 1$, which is a (b)-comparison function. Then, we have

$$d\left(F(x,y),F(u,v)\right)+d\left(F(y,x),F(v,u)\right) \leqslant \varphi\left(d\left(x,u\right)+d\left(y,v\right)\right), x \leqslant u, v \leqslant y.$$

Thus we have that F is a (φ, G)-contraction of type (b). Condition (iv) from Theorem 4.1 shows that there exists $(x_0, y_0) \in X \times X$ such that $x_0 \leqslant F(x_0, y_0)$ and $F(y_0, x_0) \leqslant y_0$ which implies that $(X \times X)^F \neq \varnothing$.

On the other hand, because of (i) and of the fact that $(X, \|\cdot\|, G)$ has the properties (A_1) and (A_2) we have that either (i) or (ii) from Theorem 2.6 is fulfilled.

In this way, we have that $F : X \times X \to X$, defined by (4.3), verifies the conditions of Theorems 2.6 and 2.7. Thus, there exists $(x^*, y^*) \in X \times X$ which is a coupled fixed point of the mapping F and, as a consequence, a solution of the problem (4.1). □

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