Some identities of degenerate Daehee numbers arising from certain differential equations

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Abstract

In this paper, we introduce the degenerate Daehee numbers and study a family of differential equations associated with the generating function of these numbers. From those differential equations, we will be able to obtain some new and interesting combinatorial identities involving the degenerate Daehee numbers and generalized harmonic numbers. ©2017 All rights reserved.

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1. Introduction

The Daehee polynomials $D_n^{(r)}(x)$ of order r are given by the generating function

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^r = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}.$$

For $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers of order r. In particular, if $r = 1$, then $D_n(x) = D_n^{(1)}(x)$ and $D_n = D_n^{(1)}$ are respectively called Daehee polynomials and Daehee numbers (see [1, 6, 8, 10, 12, 18–22, 24]).

As a degenerate version of Daehee numbers D_n, we introduce what we call the degenerate Daehee numbers $D_{n,\lambda}$ defined by

$$\frac{\lambda \log\left(1 + \frac{1}{\lambda} \log(1+\lambda t)\right)}{\log(1+\lambda t)} = \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!}. \tag{1.1}$$

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We observe here that $D_{n,\lambda} \to D_n$ as $\lambda \to 0$. Also it is easy to see that

$$D_{n,\lambda} = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}D_l = \sum_{l=0}^{n} \frac{(-1)^l l!}{l+1}S_1(n,l)\lambda^{n-l}.$$

Here $S_1(n,l)$ is the Stirling number of the first kind.

Many mathematicians have studied the arithmetic and combinatorial properties of degenerate versions of special numbers and polynomials, some of which are the degenerate Bernoulli polynomials (also called Korobov polynomials of the second kind), the degenerate Bernouili polynomials of the second kind (also called Korobov polynomials of the first kind), the degenerate Euler polynomials, the degenerate poly-Bernoulli polynomials, the degenerate poly-Bernoulli polynomials of the second, the degenerate falling factorial polynomials, and the degenerate Changhee polynomials (see [2–5, 9, 12, 14, 16, 17, 23]).

On the other hand, in [7, 11], Kim et al. developed some new methods for obtaining identities related to Bernoulli numbers of the second kind and Frobenius-Euler polynomials of higher order arising from certain non-linear differential equations. This idea of obtaining some interesting combinatorial identities by using differential equations satisfied by the generating function of special numbers or special polynomials turned out to be very fruitful (see [7, 11, 13, 15]).

The generalized harmonic numbers are defined as follows:

$$H_{N,0} = 1, \quad \text{for all } N, \tag{1.2}$$

$$H_{N,1} = H_N = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \tag{1.3}$$

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j}, \quad (2 \leqslant j \leqslant N). \tag{1.4}$$

These special numbers have appeared previously in the paper [11].

The purpose of this paper is to introduce the degenerate Daehee numbers and study a family of differential equations associated with the generating function of these numbers. From those differential equations, we will be able to obtain some new and interesting combinatorial identities involving the degenerate Daehee numbers and generalized harmonic numbers.

2. Differential equations arising from the generating function of degenerate Daehee numbers

Let

$$F(t) = F = \log\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right). \tag{2.1}$$

Then by taking the derivative with respect to t of (2.1), we get

$$\begin{aligned}
F^{(1)} = \frac{d}{dt}F(t) &= \left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)^{-1}\frac{1}{1+\lambda t} \\
&= \frac{1}{1+\lambda t}e^{-\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)} \\
&= \frac{1}{1+\lambda}e^{-F}.
\end{aligned} \tag{2.2}$$

From (2.2), we note that

$$\begin{aligned}
F^{(2)} = \frac{d}{dt}F^{(1)} &= \frac{(-1)\lambda}{(1+\lambda t)^2}e^{-F} + \frac{1}{1+\lambda t}\left(-F^{(1)}\right)e^{-F} \\
&= \frac{(-1)\lambda}{(1+\lambda t)^2}e^{-F} + \frac{(-1)}{(1+\lambda t)^2}e^{-2F}.
\end{aligned} \tag{2.3}$$

Further, by taking the derivative with respect to t of (2.3), we obtain

$$
\begin{aligned}
F^{(3)} &= \frac{d}{dt} F^{(2)} \\
&= \frac{(-1)^2 2\lambda^2}{(1+\lambda t)^3} e^{-F} + \frac{(-1)\lambda}{(1+\lambda t)^2} \left(-F^{(1)}\right) e^{-F} \\
&\quad + \frac{(-1)^2 2\lambda}{(1+\lambda t)^3} e^{-2F} + \frac{(-1)}{(1+\lambda t)^2} \left(-2F^{(1)}\right) e^{-2F} \\
&= \frac{(-1)^2}{(1+\lambda t)^3} \left(2\lambda^2 e^{-F} + 3\lambda e^{-2F} + 2e^{-3F}\right).
\end{aligned}
$$

Continuing this process, we are led to put

$$
F^{(N)} = \frac{(-1)^{N-1}}{(1+\lambda t)^N} \sum_{k=1}^{N} a_k (N \mid \lambda) e^{-kF}, \tag{2.4}
$$

for $N = 1, 2, 3, \cdots$.

On the one hand, from (2.4) we have

$$
\begin{aligned}
F^{(N+1)} &= \frac{d}{dt} F^{(N)} \\
&= \frac{(-1)^N N\lambda}{(1+\lambda t)^{N+1}} \sum_{k=1}^{N} a_k (N \mid \lambda) e^{-kF} \\
&\quad + \frac{(-1)^{N-1}}{(1+\lambda t)^N} \sum_{k=1}^{N} a_k (N \mid \lambda) \left(-kF^{(1)}\right) e^{-kF} \\
&= \frac{(-1)^N N\lambda}{(1+\lambda t)^{N+1}} \sum_{k=1}^{N} a_k (N \mid \lambda) e^{-kF} \\
&\quad + \frac{(-1)^N}{(1+\lambda t)^{N+1}} \sum_{k=1}^{N} a_k (N \mid \lambda) k e^{-(k+1)F} \\
&= \frac{(-1)^N}{(1+\lambda t)^{N+1}} \sum_{k=1}^{N} \lambda N a_k (N \mid \lambda) e^{-kF} \\
&\quad + \frac{(-1)^N}{(1+\lambda t)^{N+1}} \sum_{k=2}^{N+1} (k-1) a_{k-1} (N \mid \lambda) e^{-kF} \\
&= \frac{(-1)^N}{(1+\lambda t)^{N+1}} \left\{ \lambda N a_1 (N \mid \lambda) e^{-F} + \sum_{k=2}^{N} \left(\lambda N a_k (N \mid \lambda) + (k-1) a_{k-1} (N \mid \lambda)\right) e^{-kF} \right. \\
&\quad \left. + N a_N (N \mid \lambda) e^{-(N+1)F} \right\}.
\end{aligned}
\tag{2.5}
$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$
F^{(N+1)} = \frac{(-1)^N}{(1+\lambda t)^{N+1}} \sum_{k=1}^{N+1} a_k (N+1 \mid \lambda) e^{-kF}. \tag{2.6}
$$

Now, by comparing (2.5) and (2.6), we have

$$
a_1 (N+1 \mid \lambda) = \lambda N a_1 (N \mid \lambda), \tag{2.7}
$$

$$a_{N+1}(N+1\mid\lambda)=Na_N(N\mid\lambda),\tag{2.8}$$
$$a_k(N+1\mid\lambda)=\lambda Na_k(N\mid\lambda)+(k-1)a_{k-1}(N\mid\lambda),\tag{2.9}$$

where $2\leqslant k\leqslant N$.

From (2.2) and (2.4), we note

$$F^{(1)}=\frac{1}{1+\lambda t}e^{-F}=a_1(1\mid\lambda)\frac{1}{1+\lambda t}e^{-F}.\tag{2.10}$$

Thus by (2.10), we obtain

$$a_1(1\mid\lambda)=1.$$

In addition, from (2.3) and (2.4), we observe

$$\begin{aligned}F^{(2)}&=\frac{(-1)\lambda}{(1+\lambda t)^2}e^{-F}+\frac{(-1)}{(1+\lambda t)^2}e^{-2F}\\&=\frac{(-1)}{(1+\lambda t)^2}\left(a_1(2\mid\lambda)e^{-F}+a_2(2\mid\lambda)e^{-2F}\right).\end{aligned}\tag{2.11}$$

Hence, from (2.11) we have

$$a_1(2\mid\lambda)=\lambda,\quad a_2(2\mid\lambda)=1.$$

Now, we are ready to determine $a_k(N+1\mid\lambda)$'s appearing in (2.7), (2.8) and (2.9). From (2.7), we get

$$\begin{aligned}a_1(N+1\mid\lambda)&=\lambda Na_1(N\mid\lambda)\\&=\lambda N\lambda(N-1)a_1(N-1\mid\lambda)\\&\quad\vdots\\&=(\lambda N)\lambda(N-1)\cdots\lambda2a_1(2\mid\lambda)\\&=\lambda^N N!.\end{aligned}\tag{2.12}$$

By (2.8), we have

$$\begin{aligned}a_{N+1}(N+1\mid\lambda)&=Na_N(N\mid\lambda)\\&=N(N-1)a_{N-1}(N-1\mid\lambda)\\&\quad\vdots\\&=N(N-1)\cdots2a_2(2\mid\lambda)\\&=N!.\end{aligned}\tag{2.13}$$

We remark here that the $N\times N$ matrix with the (i,j) entry given by $a_i(j\mid\lambda)$ $(1\leqslant i,j\leqslant n)$ is defined by

$$\begin{bmatrix}1&\lambda&\lambda^2 2!&\cdots&\lambda^{N-1}(N-1)!\\0&2!&&&\\0&0&3!&&\\\vdots&\vdots&\ddots&\ddots&\\0&0&\cdots&0&(N-1)!\end{bmatrix}.$$

We now turn our attention to $a_k(N+1\mid\lambda)$, for $2\leqslant k\leqslant N$. For $k=2$ in (2.9), we have

$$\begin{aligned}a_2(N+1\mid\lambda)&=\lambda Na_2(N\mid\lambda)+a_1(N\mid\lambda)\\&=\lambda Na_2(N\mid\lambda)+\lambda^{N-1}(N-1)!\end{aligned}$$

$$= \lambda N \left(\lambda (N-1) a_2 (N-1 \mid \lambda) + \lambda^{N-2} (N-2)! \right) + \lambda^{N-1} (N-1)!$$

$$= \lambda^2 N (N-1) a_2 (N-1 \mid \lambda) + \lambda^{N-1} N! \left(\frac{1}{N-1} + \frac{1}{N} \right)$$

$$= \lambda^2 N (N-1) \left(\lambda (N-2) a_2 (N-2 \mid \lambda) + \lambda^{N-3} (N-3)! \right) + \lambda^{N-1} N! \left(\frac{1}{N-1} + \frac{1}{N} \right)$$

$$= \lambda^3 N (N-1) (N-2) a_2 (N-2 \mid \lambda) + \lambda^{N-1} N! \left(\frac{1}{N-2} + \frac{1}{N-1} + \frac{1}{N} \right)$$

$$\vdots$$

$$= \lambda^{N-1} N (N-1) \cdots 2 a_2 (2 \mid \lambda) + \lambda^{N-1} N! \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right)$$

$$= \lambda^{N-1} N! H_{N,1}.$$

Here and in below $H_{N,j}$ $(0 \leqslant j \leqslant N)$ are as defined in (1.2), (1.3) and (1.4).

For $k = 3$ in (2.9), we obtain

$$a_3 (N+1 \mid \lambda) = \lambda N a_3 (N \mid \lambda) + 2 a_2 (N \mid \lambda)$$

$$= \lambda N a_3 (N \mid \lambda) + 2! \lambda^{N-2} (N-1)! H_{N-1,1}$$

$$= \lambda N \left(\lambda (N-1) a_3 (N-1 \mid \lambda) + 2! \lambda^{N-3} (N-2)! H_{N-2,1} \right)$$
$$\quad + 2! \lambda^{N-2} (N-1)! H_{N-1,1}$$

$$= \lambda^2 N (N-1) a_3 (N-1 \mid \lambda) + 2! \lambda^{N-2} N! \left(\frac{H_{N-2,1}}{N-1} + \frac{H_{N-1,1}}{N} \right)$$

$$= \lambda^2 N (N-1) \left(\lambda (N-2) a_3 (N-2 \mid \lambda) + 2! \lambda^{N-4} (N-3)! H_{N-3,1} \right)$$
$$\quad + 2! \lambda^{N-2} N! \left(\frac{H_{N-2,1}}{N-1} + \frac{H_{N-1,1}}{N} \right)$$

$$= \lambda^3 N (N-1) (N-2) a_3 (N-2 \mid \lambda) + 2! \lambda^{N-2} N! \left(\frac{H_{N-3,1}}{N-2} + \frac{H_{N-2,1}}{N-1} + \frac{H_{N-1,1}}{N} \right)$$

$$\vdots$$

$$= \lambda^{N-2} N (N-1) \cdots 3 a_3 (3 \mid \lambda) + 2! \lambda^{N-2} N! \left(\frac{H_{2,1}}{3} + \frac{H_{3,1}}{4} + \cdots + \frac{H_{N-1,1}}{N} \right)$$

$$= 2! \lambda^{N-2} N! \left(\frac{H_{1,1}}{2} + \frac{H_{2,1}}{3} + \cdots + \frac{H_{N-1,1}}{N} \right)$$

$$= 2! \lambda^{N-2} N! H_{N,2}.$$

Proceeding similarly to $k = 2$ and $k = 3$ cases, we can show that

$$a_4 (N+1 \mid \lambda) = 3! \lambda^{N-3} N! H_{N,3}.$$

Continuing in this fashion, we can find that

$$a_k (N+1 \mid \lambda) = (k-1)! \lambda^{N-k+1} N! H_{N,k-1}, \quad (2 \leqslant k \leqslant N). \tag{2.14}$$

Here we observe that (2.14) holds also for $k = 1$ and $k = N + 1$ (cf. (2.12), (2.13)).

Thus, from (2.14) we obtain the following theorem.

Theorem 2.1. *For* $N = 1, 2, 3, \cdots$, *let us consider the following family of differential equations:*

$$F^{(N)} = \frac{(-1)^{N-1}}{(1+\lambda t)^N} \sum_{k=1}^{N} (k-1)! \lambda^{N-k} (N-1)! H_{N-1,k-1} e^{-kF}, \tag{2.15}$$

where

$$H_{N,0} = 1, \quad \text{for all } N,$$

$$H_{N,1} = H_N = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1},$$

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j}, \quad (2 \leqslant j \leqslant N).$$

Then the above family of differential equations in (2.15) have a solution

$$F = F(t) = \log\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right).$$

3. Applications of differential equations

Here we will use Theorem 2.1 in order to derive some new and interesting identity involving the degenerate Daehee numbers and generalized harmonic numbers.

From (2.1), we get

$$
\begin{aligned}
F(t) &= \frac{\lambda \log\left(1 + \frac{1}{\lambda}\log(1+\lambda t)\right)}{\log(1+\lambda t)} \frac{1}{\lambda}\log(1+\lambda t) \\
&= \left(\sum_{l=0}^{\infty} D_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\lambda^{m-1}t^m\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n-l-1}}{n-l}\right)t^n.
\end{aligned}
\tag{3.1}
$$

On the one hand, from (3.1) we obtain

$$
\begin{aligned}
F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\
&= \sum_{n=N}^{\infty}(n)_N\left(\sum_{l=0}^{n-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n-l-1}}{n-l}\right)t^{n-N} \\
&= \sum_{n=0}^{\infty}(n+N)_N\left(\sum_{l=0}^{n+N-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n+N-l-1}}{n+N-l}\right)t^n \\
&= \sum_{n=0}^{\infty}(n+N)!\left(\sum_{l=0}^{n+N-1}\frac{D_{l,\lambda}}{l!}\frac{(-\lambda)^{n+N-l-1}}{n+N-l}\right)\frac{t^n}{n!},
\end{aligned}
\tag{3.2}
$$

where $(x)_N = x(x-1)\cdots(x-N+1)$, for $N \geqslant 1$ and $(x)_0 = 1$.

Now, we observe that

$$
\begin{aligned}
e^{-kF} &= \sum_{m_1=0}^{\infty}\frac{(-k)^{m_1}}{m_1!}F^{m_1} \\
&= \sum_{m_1=0}^{\infty}(-k)^{m_1}\frac{1}{m_1!}\left(\log\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^{m_1} \\
&= \sum_{m_1=0}^{\infty}(-k)^{m_1}\sum_{m_2=m_1}^{\infty}S_1(m_2,m_1)\left(\frac{1}{\lambda}\right)^{m_2}\frac{(\log(1+\lambda t))^{m_2}}{m_2!}
\end{aligned}
\tag{3.3}
$$

$$
= \sum_{m_2=0}^{\infty} \left(\sum_{m_1=0}^{m_2} (-k)^{m_1} S_1(m_2, m_1) \lambda^{-m_2} \right)
$$

$$
\times \sum_{m_3=m_2}^{\infty} S_1(m_3, m_2) \lambda^{m_3} \frac{t^{m_3}}{m_3!}
$$

$$
= \sum_{m_3=0}^{\infty} \left(\sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-k)^{m_1} S_1(m_2, m_1) S_1(m_3, m_2) \lambda^{m_3-m_2} \right) \frac{t^{m_3}}{m_3!}.
$$

In turn, (3.3) gives us

$$
\frac{1}{(1+\lambda t)^N} e^{-kF} = \left(\sum_{l=0}^{\infty} \binom{N+l-1}{l} (-1)^l \lambda^l t^l \right)
$$

$$
\times \sum_{m_3=0}^{\infty} \left(\sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-k)^{m_1} S_1(m_2, m_1) S_1(m_3, m_2) \lambda^{m_3-m_2} \right) \frac{t^{m_3}}{m_3!} \tag{3.4}
$$

$$
= \sum_{n=0}^{\infty} \left(\sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-1)^{n+m_1+m_3} \binom{n}{m_3} (N+n-m_3-1)_{n-m_3} \right.
$$

$$
\left. \times k^{m_1} \lambda^{n-m_2} S_1(m_2, m_1) S_1(m_3, m_2) \right) \frac{t^n}{n!}.
$$

On the other hand, from (1.1) and (3.4) we have

$$
F^{(N)} = (-1)^{N-1} (N-1)! \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} \sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} \right.
$$

$$
\times (-1)^{n+m_1+m_3} \binom{n}{m_3} (N+n-m_3-1)_{n-m_3} (k-1)! k^{m_1} \tag{3.5}
$$

$$
\left. \times \lambda^{N+n-k-m_2} S_1(m_2, m_1) S_1(m_3, m_2) H_{n-1,k-1} \right) \frac{t^n}{n!}.
$$

By equating (3.2) and (3.5), we finally get the following theorem.

Theorem 3.1. *For* $N = 1, 2, 3, \cdots,$ *and* $n = 0, 1, 2, \cdots,$ *we have*

$$
(-1)^{N+n+1} \frac{(n+N)!}{(N-1)!} \sum_{l=0}^{N+n-1} \frac{D_{l,\lambda}}{l!} \frac{(-\lambda)^{n+N-l-1}}{n+N-l}
$$

$$
= \sum_{k=1}^{N} \sum_{m_3=0}^{n} \sum_{m_2=0}^{m_3} \sum_{m_1=0}^{m_2} (-1)^{m_1+m_3} \binom{n}{m_3} (N+n-m_3-1)_{n-m_3}
$$

$$
\times (k-1)! k^{m_1} \lambda^{N+n-k-m_2} S_1(m_2, m_1) S_1(m_3, m_2) H_{N-1,k-1},
$$

where $H_{N,j}$*'s are as in* (1.2), (1.3) *and* (1.4).

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