Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 1791–1800
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Generalized hypergeometric k-functions via (k, s)-fractional calculus

Kottakkaran Sooppy Nisar[a], Gauhar Rahman[b], Junesang Choi[c,*], Shahid Mubeen[d], Muhammad Arshad[b]

[a]Department of Mathematics, College of Arts and Science at Wadi Al-dawaser, Prince Sattam bin Abdulaziz University, Alkharj, Riyadh region 11991, Kingdom of Saudi Arabia.

[b]Department of Mathematics, International Islamic University, Islamabad, Pakistan.

[c]Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea.

[d]Department of Mathematics, University of Sargodha, Sargodha, Pakistan.

Communicated by Y. J. Cho

Abstract

We introduce $(k; s)$-fractional integral operator involving (k, τ)-hypergeometric function and the Riemann-Liouville left-sided and right-sided $(k; s)$-fractional integral and differential operators. Then we present several useful and interesting results involving the introduced operators. Also, the results presented here, being general, are pointed out to reduce to some known results.

Keywords: Generalized hypergeometric function ${}_pF_q$, τ-hypergeometric function, k-hypergeometric function, differential operators, (k, τ)-hypergeometric function, k-Pochhammer symbol, k-gamma function, k-beta function, $(k; s)$-fractional integral.
2010 MSC: 33C20, 33E20, 26A33, 26A99.

1. Introduction and preliminaries

The generalized hypergeometric function ${}_pF_q$ and its various extensions are well-known to have played important roles in applications in a wide range of research fields such as (for example) mathematical physics and engineering (see, e.g., [4, 5, 18]). The generalized hypergeometric series ${}_pF_q$ is defined by (see [12, p. 73]; see also [21, Section 1.5]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p ; \\ \beta_1, \ldots, \beta_q ;\end{matrix} z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \end{aligned} \tag{1.1}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda, \nu \in \mathbb{C}$), in terms of the familiar Gamma

*Corresponding author
Email addresses: ksnisar1@gmail.com (Kottakkaran Sooppy Nisar), gauhar55uom@gmail.com (Gauhar Rahman), junesang@mail.dongguk.ac.kr (Junesang Choi), smjhanda@gmail.com (Shahid Mubeen), marshad_zia@yahoo.com (Muhammad Arshad)

doi:10.22436/jnsa.010.04.40

Received 2017-02-26

function Γ, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ be sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively, and

$$\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}, \quad \mathbb{Z}_0^- := \mathbb{Z}\setminus\mathbb{N}, \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N}\cup\{0\}.$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \dots, \alpha_p$, and the denominator parameters $\beta_1, \dots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is,

$$(\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ j = 1, \dots, q).$$

For the detailed convergence of ${}_pF_q$, one may be referred, for example, to [21, Section 1.5]. The special case of (1.1) when $p = 2$ and $q = 1$ is usually called Gauss's hypergeometric function or series.

Diverse generalizations of the Gauss hypergeometric function ${}_2F_1$ have been presented by many authors (see, e.g., [2, 13, 15–17, 23–25]). Among other things, we choose to recall the following τ-hypergeometric function (see [23])

$${}_2R_1(\alpha,\beta;\gamma;\tau,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n\Gamma(\beta+n\tau)}{\Gamma(\gamma+n\tau)}\frac{z^n}{n!}, \quad \left(\gamma\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ |z|<1;\ \tau\in\mathbb{R}^+\right).$$

If $\Re(\gamma) > \Re(\beta) > 0$, an integral representation of ${}_2R_1(\alpha,\beta;\gamma;\tau,z)$ is given as follows:

$${}_2R_1(\alpha,\beta;\gamma;\tau,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt^\tau)^{-\alpha}dt, \tag{1.2}$$

or, equivalently,

$${}_2R_1(\alpha,\beta;\gamma;\tau,z) = \frac{\Gamma(\gamma)}{\tau\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{\tau}-1}(1-t^{\frac{1}{\tau}})^{\gamma-\beta-1}(1-zt)^{-\alpha}dt. \tag{1.3}$$

The integral representation (1.2) or (1.3) when $\tau = 1$ reduces to the classical Euler's integral representation of ${}_2F_1(\alpha,\beta;\gamma;z)$ (see, e.g., [21, p. 65]).

Rao et al. [15] obtained many properties for the τ-hypergeometric function ${}_2R_1(\alpha,\beta;\gamma;\tau;z)$. Recently, many researchers have investigated fractional integral operators associated with Mittag-Leffler type functions (see, e.g., [10, 20, 22]).

Díaz and Pariguan [1] found that the following expression:

$$(x)_{n,k} := x(x+k)(x+2k)\cdots(x+(n-1)k), \tag{1.4}$$

has appeared repeatedly in a variety of contexts such as combinatorics of creation and annihilation operators and perturbative computation of Feynman integrals. Motivated by this observation, they [1] might use the Gauss's form of the Gamma function (see [21, Eq. (6), p. 2]) to introduce the so-called k-gamma function $\Gamma_k(z)$ defined by

$$\Gamma_k(z) = \lim_{n\to\infty}\frac{n!\,k^n\,(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \quad \left(k\in\mathbb{R}^+;\ z\in\mathbb{C}\setminus k\mathbb{Z}_0^-\right). \tag{1.5}$$

Starting from this definition, they [1] presented a number of properties for the k-Gamma function some of which are recalled:

$$\Gamma_k(z+k) = z\,\Gamma_k(z), \quad \text{and} \quad \Gamma_k(k) = 1, \tag{1.6}$$

the Euler's integral form:

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} \, dt, \quad \left(k \in \mathbb{R}^+; \ \Re(z) > 0\right). \tag{1.7}$$

The k-Pochhammer symbol $(\lambda)_{n,k}$ is defined (for $\lambda, \nu \in \mathbb{C}; \ k \in \mathbb{R}^+$) by

$$\begin{aligned}(\lambda)_{\nu,k} := & \frac{\Gamma_k(\lambda + \nu k)}{\Gamma_k(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \{0\}) \\ = & \begin{cases} 1 & (\nu = 0), \\ \lambda(\lambda + k) \cdots (\lambda + (n-1)k) & (\nu = n \in \mathbb{N}). \end{cases}\end{aligned}$$

From (1.7), it is easy to find the following relationship between the Gamma function Γ and the k-Gamma function Γ_k:

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right). \tag{1.8}$$

For (1.4), (1.5), (1.6), (1.7), (1.8), see also [9]. The k-beta function $B_k(\alpha, \beta)$ is given by (see, e.g., [1])

$$B_k(\alpha, \beta) = \begin{cases} \frac{1}{k} \int_0^1 t^{\frac{\alpha}{k}-1} (1-t)^{\frac{\beta}{k}-1} \, dt, \\ \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}, \end{cases} \quad (\min\{\Re(\alpha), \Re(\beta)\} > 0). \tag{1.9}$$

They [1] used the k-Pochhammer symbol to define the following k-hypergeometric function

$${}_2F_{1,k}(\alpha, \beta; \gamma; z) = \sum_{n=0}^\infty \frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{z^n}{n!}, \quad \left(\gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ |z| < 1; \ k \in \mathbb{R}^+\right). \tag{1.10}$$

Mubeen and Habibullah [6, 7] introduced the so-called k-fractional integral with the aid of the k-gamma function and provided certain integral representations of generalized k-hypergeometric functions and k-confluent hypergeometric functions. In fact, they [6, 7] defined the k-fractional integral by

$$\mathfrak{I}_k^\alpha(f(x)) = \frac{1}{k\Gamma_k(\alpha)} \int_o^x (x-t)^{\frac{\alpha}{k}-1} f(t) \, dt, \quad \left(k \in \mathbb{R}^+; \ \Re(\alpha) > 0\right),$$

whose special case when $k = 1$, obviously, reduces to the familiar Riemann-Liouville fractional integral (see, e.g., [3, p. 69])

$$\mathfrak{I}^\alpha(f(x)) = \frac{1}{\Gamma(\alpha)} \int_o^x (x-t)^{\alpha-1} f(t) \, dt, \quad (\Re(\alpha) > 0).$$

They [6, 7] presented the following k-fractional integral formulas:

$$\mathfrak{I}_k^\alpha\left(x^{\frac{\beta}{k}-1}\right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} x^{\frac{\alpha}{k}+\frac{\beta}{k}-1},$$

and

$$\mathfrak{I}_k^\alpha\left((x-u)^{\frac{\beta}{k}-1}\right) = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} (x-u)^{\frac{\alpha}{k}+\frac{\beta}{k}-1}.$$

Very recently, Sarikaya et al. [19] have introduced the $(k; s)$-Riemann-Liouville fractional integral of order $\mu > 0$ defined by

$${}_k^s\mathfrak{I}_{a+}^\mu f(x) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\mu}{k}-1} t^s f(t) \, dt, \quad \left(x \in [a, b]; \ k \in \mathbb{R}^+; \ s \in \mathbb{R} \setminus \{-1\}\right), \tag{1.11}$$

where $[a,b]$ is a closed bounded interval on the real line $(-\infty,\infty)$. They [19] obtained the following formula

$$ {}^{s}_{k}\mathfrak{I}^{\mu}_{a+}\left[\left(t^{s+1}-a^{s+1}\right)^{\frac{\lambda}{k}-1}\right]=\frac{\Gamma_k(\lambda)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\lambda+\mu)}\left(x^{s+1}-a^{s+1}\right)^{\frac{\lambda+\mu}{k}-1}. \tag{1.12}$$

Rahman et al. [11] defined the following (k,τ)-hypergeometric function ${}_2R_{1,k}(a,b;c;\tau;z)$:

$$ {}_2R_{1,k}(\alpha,\beta;\gamma;\tau;z)=\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}\Gamma_k(\beta+n\tau k)}{\Gamma_k(\gamma+n\tau k)}\frac{z^n}{n!},\quad \left(k,\ \tau\in\mathbb{R}^+;\ |z|<1\right), \tag{1.13}$$

whose special case when $\tau=1$, obviously, reduces to the k-hypergeometric function ${}_2F_{1,k}(\alpha,\beta;\gamma;z)$ in (1.10). They [11] presented an integral representation and recurrence relations for the generalized k-hypergeometric function ${}_2R_{1,k}(a,b;c;\tau;z)$.

It is also noted that Mubeen et al. [8] presented some properties of the generalized k-hypergeometric function and introduced its k-fractional integration and derivation.

Here we give a slight extension of the generalized k-hypergeometric function (1.13) as in the following definition.

Definition 1.1. Let $\alpha,\ \beta,\ \gamma,\ \mu,\ \nu\in\mathbb{C}$ such that the following summation can be defined.

$$ \begin{aligned} {}_3R_{2,k}(\alpha,\beta,\gamma;\mu,\nu;\tau,\rho;z) &:= \frac{\Gamma_k(\mu)\Gamma_k(\nu)}{\Gamma_k(\beta)\Gamma_k(\gamma)}\\ &\times\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}\Gamma_k(\beta+n\tau k)\,\Gamma_k(\gamma+n\rho k)}{\Gamma_k(\mu+n\tau k)\,\Gamma_k(\nu+n\rho k)}\frac{z^n}{n!},\quad \left(k,\ \tau,\ \rho\in\mathbb{R}^+;\ |z|<1\right). \end{aligned} \tag{1.14}$$

2. $(k;s)$-fractional integrals and differentials of the generalized k-hypergeometric function

Here, we introduce $(k;s)$-fractional integrals and $(k;s)$-fractional differentials involving the generalized k-hypergeometric function ${}_2R_{1,k}(a,b;c;\tau;z)$. We also present certain properties for the introduced fractional integrals and fractional differentials.

Definition 2.1. Let $k,\ \tau\in\mathbb{R}^+$, $x\in[a,b]$, and $s\in\mathbb{R}\setminus\{-1\}$. Also, let $\alpha,\ \beta,\ \gamma,\ \rho,\ \omega\in\mathbb{C}$ with

$$\min\{\Re(\alpha),\ \Re(\beta),\ \Re(\gamma),\ \Re(\rho)\}>0.$$

We define the following generalized $(k;s)$-fractional integral by

$$ \left({}^{s}_{k}\mathfrak{R}^{\omega;\alpha,\beta;\rho}_{a+;\tau,\gamma}f\right)(x):=\frac{1}{k}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\gamma}{k}-1}\ {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau;\omega\left(x^{s+1}-t^{s+1}\right)^{\rho}\right)\ t^s\ f(t)\,dt. \tag{2.1}$$

The case $s=0$ of (2.1) reduces to the known fractional integral operator (see [8])

$$ \left({}_{k}\mathfrak{R}^{\omega;\alpha,\beta;\rho}_{a+;\tau,\gamma}f\right)(x)=\int_a^x(x-t)^{\frac{\gamma}{k}-1}\ {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau,\omega(x-t)^{\rho}\right)\ f(t)\,dt,$$

whose further special case when $k=1$ reduces to the known fractional integral operator (see [14])

$$ \left(\mathfrak{R}^{\omega;\alpha,\beta;\rho}_{a+;\tau,\gamma}f\right)(x)=\int_a^x(x-t)^{\gamma-1}\ {}_2R_1(\alpha,\beta;\gamma;\tau,\omega(x-t)^{\rho})\,f(t)\,dt.$$

We also introduce $(k;s)$-fractional integrations and $(k;s)$-fractional differentiations which are denoted by ${}^{s}_{k}\mathfrak{I}^{\mu}_{a+}$, ${}^{s}_{k}\mathfrak{I}^{\mu}_{a-}$ and $\mathfrak{D}^{\mu}_{\rho+,k}$, $\mathfrak{D}^{\mu}_{\rho-,k}$, respectively. We call ${}^{s}_{k}\mathfrak{I}^{\mu}_{a+}$ and ${}^{s}_{k}\mathfrak{I}^{\mu}_{a-}$ the left-sided and right-sided

Riemann-Liouville $(k;s)$-fractional integral operators, respectively, and $\mathfrak{D}^{\mu}_{\rho+,k}$ and $\mathfrak{D}^{\mu}_{\rho-,k}$ the left-sided and right-sided Riemann-Liouville $(k;s)$-fractional differential operators, respectively. To do this, we recall the following class $L(a,b)$ of Lebesgue measurable real or complex-valued integral functions defined on the closed interval $[a,b]$ on the real line $\mathbb{R}$:

$$L(a,b) := \left\{ \phi : [a,b] \to \mathbb{C} \,:\, \int_a^b |\phi(\tau)| d\tau < \infty \right\}.$$

Definition 2.2. Let $f \in L(a,b)$, $s \in \mathbb{R} \setminus \{1\}$, $\mu \in \mathbb{C}$ with $\Re(\mu) > 0$, $k \in \mathbb{R}^+$, and $x > a$. Then we define the Riemann-Liouville left-sided $(k;s)$-fractional integral operator of order μ by

$$\begin{aligned} {}^{s}_{a,k}\mathfrak{D}^{-\mu}_{x} f(x) &= {}^{s}_{a,k}\,\mathfrak{I}^{\mu}_{x} f(x) = {}^{s}_{k}\,\mathfrak{I}^{\mu}_{a+} f(x) \\ &= ({}^{s}_{k}\mathfrak{I}^{\mu}_{a+} f)(x) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_a^x \frac{f(t)}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}\, dt. \end{aligned} \tag{2.2}$$

Similarly, with the same conditions as above except for $x < b$, we define Riemann-Liouville right-sided $(k;s)$-fractional integral operator of order μ by

$$\begin{aligned} {}^{s}_{\rho,k}\mathfrak{D}^{-\mu}_{b} f(x) &= {}^{s}_{\rho,k}\mathfrak{I}^{\mu}_{b} f(x) = {}^{s}_{k}\,\mathfrak{I}^{\mu}_{b-} f(x) = ({}^{s}_{k}\mathfrak{I}^{\mu}_{b-} f)(x) \\ &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_x^b \frac{f(t)}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}\, dt. \end{aligned}$$

Definition 2.3. Let $s \in \mathbb{R} \setminus \{1\}$, $k \in \mathbb{R}^+$, $\mu \in \mathbb{C}$ with $\Re(\mu) > 0$ and $n := [\Re(\mu)] + 1$. Then the Riemann-Liouville left-sided and right-sided $(k;s)$-fractional differential operators are defined, respectively, by

$$({}^{s}_{k}\mathfrak{D}^{\mu}_{a+} f)(x) = \left(\frac{1}{x^s}\frac{d}{dx}\right)^n \left(k^n\, {}^{s}_{k}\mathfrak{I}^{nk-\mu}_{a+} f\right)(x), \tag{2.3}$$

and

$$({}^{s}_{k}\mathfrak{D}^{\mu}_{a-} f)(x) = \left(-\frac{1}{x^s}\frac{d}{dx}\right)^n \left(k^n\, {}^{s}_{k}\mathfrak{I}^{nk-\mu}_{a-} f\right)(x). \tag{2.4}$$

Setting $s = 0$ in (2.3) and (2.4) yields those in [8]. Setting $k = 1$ and $s = 0$ in Definitions 2.2 and 2.3 reduces to the familiar Riemann-Liouville left-sided and right-sided fractional integrals and derivatives (see [14]).

Theorem 2.4. *Let $k, \tau, \rho \in \mathbb{R}^+$, $s \in \mathbb{R} \setminus \{-1\}$, and $m \in \mathbb{N}$. Also, let $x, a \in \mathbb{R}$ with $x > a$ and $x \neq 0$. Further, let $\alpha, \beta, \tau, \omega, c \in \mathbb{C}$ be such that the involved summations can be defined. Then*

$$\begin{aligned} &\left(\frac{1}{x^s}\frac{d}{dx}\right)^m \left\{ (x^{s+1}-a^{s+1})^{\frac{c}{k}-1}\, {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau,\omega(x^{s+1}-a^{s+1})^{\rho}\right) \right\} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(c-mk)}\,\frac{(s+1)^m}{k^m}\,\left(x^{s+1}-a^{s+1}\right)^{\frac{c}{k}-m-1} \\ &\quad \times {}_3R_{2,k}\Big(\alpha,\beta,c;\gamma,c-mk;\tau,\rho;\omega(x^{s+1}-a^{s+1})^{\rho}\Big). \end{aligned} \tag{2.5}$$

Proof. Let $\mathcal{L}$ be the left-hand side of (2.5). Using (1.13) and interchanging the order of summation and differentiation, we have

$$\begin{aligned} \mathcal{L} &= \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}\Gamma_k(\beta+n\tau k)}{\Gamma_k(\gamma+n\tau k)}\,\frac{\omega^n}{n!} \\ &\quad \times \left\{ \left(\frac{1}{x^s}\frac{d}{dx}\right)^m \left(x^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-1} \right\}. \end{aligned} \tag{2.6}$$

We find

$$\begin{aligned}\left(\frac{1}{x^s}\frac{d}{dx}\right)^m&\left(x^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-1}\\&=(s+1)^m\left(\rho n+\frac{c}{k}-1\right)\cdots\left(\rho n+\frac{c}{k}-m\right)\left(x^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-m-1}\\&=(s+1)^m\frac{\Gamma\left(\rho n+\frac{c}{k}\right)}{\Gamma\left(\rho n+\frac{c}{k}-m\right)}\left(x^{s+1}-a^{s+1}\right)^{\rho n+\frac{c}{k}-m-1}.\end{aligned}\tag{2.7}$$

Using (1.8), we get

$$\frac{\Gamma\left(\rho n+\frac{c}{k}\right)}{\Gamma\left(\rho n+\frac{c}{k}-m\right)}=\frac{\Gamma_k\left(c+n\rho k\right)}{k^m\,\Gamma_k\left(c-mk+n\rho k\right)}.\tag{2.8}$$

Combining (2.7) with (2.8) into (2.6), we obtain

$$\begin{aligned}\mathcal{L}=&\frac{(s+1)^m}{k^m}\left(x^{s+1}-a^{s+1}\right)^{\frac{c}{k}-m-1}\\&\times\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}\Gamma_k(\beta+n\tau k)\,\Gamma_k(c+n\rho k)}{\Gamma_k(\gamma+n\tau k)\,\Gamma_k(c-mk+n\rho k)}\frac{\left\{\omega\left(x^{s+1}-a^{s+1}\right)^{\rho}\right\}^n}{n!},\end{aligned}$$

which, upon expressing in terms of (1.14), leads to the right-hand side of (2.5).

□

Theorem 2.5. *Let* $k,\tau\in\mathbb{R}^+$ *and* $s\in\mathbb{R}\setminus\{-1\}$*. Also, let* $\alpha,\beta,\gamma,\mu\in\mathbb{C}$ *with* $\Re(\mu)>0$ *and* $n:=[\Re(\mu)]+1$*. Further, let* $\omega\in\mathbb{C}$*,* $x\in\mathbb{R}$*,* $a\in\mathbb{R}_0^+$ *with* $x>a$ *and*

$$\left|\omega\left(x^{s+1}-a^{s+1}\right)^{\tau}\right|<1.$$

Then

$$\begin{aligned}{}^{s}_{k}\mathfrak{I}^{\mu}_{a+}&\left\{(t^{s+1}-a^{s+1})^{\frac{\gamma}{k}-1}{}_2R_{1,k}(\alpha,\beta;\gamma;\tau;\omega(t^{s+1}-a^{s+1})^{\tau})\right\}(x)\\&=\frac{(x^{s+1}-a^{s+1})^{\frac{\mu+\gamma}{k}-1}\Gamma_k(\gamma)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma+\mu)}\,{}_2R_{1,k}\left(\alpha,\beta;\gamma+\mu;\tau;\omega\left(x^{s+1}-a^{s+1}\right)^{\tau}\right),\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}{}^{s}_{k}\mathfrak{D}^{\mu}_{a+}&\left\{(t^{s+1}-a^{s+1})^{\frac{\gamma}{k}-1}{}_2R_{1,k}(\alpha,\beta;\gamma;\tau;\omega(t^{s+1}-a^{s+1})^{\tau})\right\}(x)\\&=\frac{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma)}{\Gamma_k(\gamma-\mu)}\left(x^{s+1}-a^{s+1}\right)^{\frac{\gamma-\mu}{k}-1}\\&\quad\times{}_2R_{1,k}\left(\alpha,\beta;\gamma-\mu;\tau;\omega\left(x^{s+1}-a^{s+1}\right)^{\tau}\right).\end{aligned}\tag{2.10}$$

Proof. Let $\mathcal{L}_1$ be the left-hand side of (2.9). By applying (2.2), we have

$$\mathcal{L}_1=\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x\frac{\left(t^{s+1}-a^{s+1}\right)^{\frac{\gamma}{k}-1}{}_2R_{1,k}(\alpha,\beta;\gamma;\tau;\omega(t^{s+1}-a^{s+1})^{\tau})}{\left(x^{s+1}-t^{s+1}\right)^{1-\frac{\mu}{k}}}\,dt.$$

Using (1.13) to expand ${}_2R_{1,k}$ and interchanging the order of summation and integral, which is valid under the given conditions here, we obtain

$$\begin{aligned}\mathcal{L}_1=&\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}\Gamma_k(\beta+\tau nk)}{\Gamma_k(\gamma+\tau nk)}\frac{\omega^n}{n!}\\&\times\int_a^x\frac{(t^{s+1}-a^{s+1})^{\frac{\gamma}{k}+\tau n-1}}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}\,dt.\end{aligned}$$

In terms of (2.2), we find

$$\mathcal{L}_1 = \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}\Gamma_k(\beta+\tau n k)}{\Gamma_k(\gamma+\tau n k)} \frac{\omega^n}{n!} \left\{ {}_k^s\mathfrak{I}^{\mu}_{a+} \left((t^{s+1}-a^{s+1})^{\frac{\gamma}{k}+\tau n-1} \right) \right\}.$$

Using (1.12) and making a suitable arrangement, we get

$$\begin{aligned}\mathcal{L}_1 =& \frac{(x^{s+1}-a^{s+1})^{\frac{\mu+\gamma}{k}-1}\Gamma_k(\gamma)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma+\mu)} \\ &\times \left\{ \frac{\Gamma_k(\gamma+\mu)}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}\Gamma_k(\beta+\tau n k)}{\Gamma_k(\gamma+\mu+\tau n k)} \frac{(\omega(x^{s+1}-a^{s+1})^{\tau})^n}{n!} \right\},\end{aligned}$$

which, upon using (1.13), leads to the right-hand side of (2.9).

Let $\mathcal{L}_2$ be the left-hand side of (2.10). Using (2.3), we have

$$\begin{aligned}\mathcal{L}_2 = \left(\frac{1}{x^s}\frac{d}{dx}\right)^n & \left\{ k^n\, {}_k^s\mathfrak{I}^{nk-\mu}_{\alpha+} \left((t^{s+1}-a^{s+1})^{\frac{\gamma}{k}-1} \right.\right. \\ & \left.\left. \times\, {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau;\omega\left(t^{s+1}-a^{s+1}\right)^{\tau}\right) \right) \right\}(x).\end{aligned}$$

Applying (2.9), we obtain

$$\begin{aligned}\mathcal{L}_2 = & \frac{k^n\,\Gamma_k(\gamma)}{(s+1)^{n-\frac{\mu}{k}}\Gamma_k(\gamma+nk-\mu)} \left(\frac{1}{x^s}\frac{d}{dx}\right)^n \left\{ \left(x^{s+1}-a^{s+1}\right)^{\frac{\gamma-\mu}{k}+n-1} \right. \\ & \left. \times\, {}_2R_{1,k}\left(\alpha,\beta;\gamma+nk-\mu;\tau;\omega\left(x^{s+1}-a^{s+1}\right)^{\tau}\right) \right\},\end{aligned}$$

which, upon using (2.5), leads to the right-hand side of (2.10).

□

Setting $s=0$ and $k=1$ in (2.9) and (2.10) yields the known results (see [14]), which are recalled in the following corollary.

Corollary 2.6. *Let* $k, \tau \in \mathbb{R}^+$ *and* $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ *with* $\Re(\mu) > 0$. *Also, let* $\omega \in \mathbb{C}$, $x \in \mathbb{R}$, $a \in \mathbb{R}_0^+$ *with* $x > a$ *and* $|\omega\,(x-a)^{\tau}| < 1$. *Then*

$$\begin{aligned}&\mathfrak{I}^{\mu}_{a+}\left\{(t-a)^{\gamma-1}\quad {}_2R_1(\alpha,\beta;\gamma;\tau;\omega(t-a)^{\tau})\right\}(x) \\ &\qquad = \frac{(x-a)^{\gamma+\mu-1}\,\Gamma(\gamma)}{\Gamma(\gamma+\mu)}\,{}_2R_1\left(\alpha,\beta;\gamma+\mu;\tau;\omega(x-a)^{\tau}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{D}^{\mu}_{a+}\left\{(t-a)^{\gamma-1}\,{}_2R_1(\alpha,\beta;\gamma;\tau;\omega(t-a)^{\tau})\right\}(x) \\ &\qquad = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\mu)}(x-a)^{\gamma-\mu-1}\,{}_2R_1\left(\alpha,\beta;\gamma-\mu;\tau;\omega(x-a)^{\tau}\right).\end{aligned}$$

3. Some properties of the operator (2.1)

Here, we present certain interesting properties for the generalized $(k;s)$-fractional integral operator (2.1).

Theorem 3.1. *Let* $k, \tau \in \mathbb{R}^+$, $x \in [a,b]$, *and* $s \in \mathbb{R}\setminus\{-1\}$. *Also, let* $\alpha, \beta, \gamma, \omega, \mu \in \mathbb{C}$ *with* $\min\{\Re(\gamma), \Re(\mu)\} > 0$. *Then*

$$\begin{aligned}\left({}_k^s\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma}\left(t^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1}\right)(x) = & \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\gamma+\mu}{k}-1}}{s+1} \\ & \times \frac{\Gamma_k(\gamma)\,\Gamma_k(\mu)}{\Gamma_k(\gamma+\mu)}\,{}_2R_{1,k}\left(\alpha,\beta;\gamma+\mu;\tau;\omega(x^{s+1}-a^{s+1})^{\tau}\right).\end{aligned} \tag{3.1}$$

Proof. Let $\mathcal{L}$ be the left-hand side of (3.1). By using (2.1) and (1.13) and exchanging the order of summation and integral, which is valid under the given conditions of this theorem, we have

$$\mathcal{L} = \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}\Gamma_k(\beta+\tau n k)}{\Gamma_k(\gamma+\tau n k)} \frac{\omega^n}{n!} I_a^x(\gamma,\mu;k,\tau;s), \tag{3.2}$$

where

$$I_a^x(\gamma,\mu;k,\tau;s) := \frac{1}{k}\int_a^x \left(t^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1} \left(x^{s+1}-t^{s+1}\right)^{\frac{\gamma}{k}+\tau n-1} t^s \, dt.$$

Making a suitable substitution and using (1.9), we find

$$\begin{aligned} I_a^x(\gamma,\mu;k,\tau;s) &= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\gamma+\mu}{k}+\tau n-1}}{k(s+1)} \int_0^1 v^{\frac{\mu}{k}-1}(1-v)^{\frac{\gamma+n\tau k}{k}-1} dv \\ &= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\gamma+\mu}{k}+\tau n-1}}{s+1} B_k(\mu,\gamma+n\tau k) \\ &= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\gamma+\mu}{k}+\tau n-1}}{s+1} \frac{\Gamma_k(\mu)\,\Gamma_k(\gamma+\tau n k)}{\Gamma_k(\gamma+\mu+\tau n k)}. \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) and expressing in terms of (1.13), we obtain the right-hand side of (3.1). □

The next theorem shows that the two (k,s)-integral operators in (1.11) and (2.1) are commutative.

Theorem 3.2. *Let* $k, \tau \in \mathbb{R}^+$, $x \in [a,b]$, *and* $s \in \mathbb{R}\setminus\{-1\}$. *Also, let* α, β, γ, $\mu\,\omega \in \mathbb{C}$ *with*

$$\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\mu)\} > 0.$$

Then, for any $f \in L(a,b)$,

$$\left\{ {}^s_k\mathfrak{I}^{\mu}_{a+}\left({}^s_k\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma} f\right)\right\}(x) = \left\{ {}^s_k\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma} \left({}^s_k\mathfrak{I}^{\mu}_{a+} f\right)\right\}(x). \tag{3.4}$$

In particular,

$$\left\{ {}^s_k\mathfrak{I}^{\mu}_{a+}\left({}^s_k\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma} f\right)\right\}(x) = \frac{\Gamma_k(\gamma)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma+\mu)} \left({}^s_k\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma+\mu} f\right)(x). \tag{3.5}$$

Proof. Let $\mathcal{L}$ and $\mathcal{R}$ be the left-side and the right-side of (3.4), respectively. We find from (1.11) and (2.1) that

$$\begin{aligned} \mathcal{L} = \frac{(s+1)^{1-\frac{\mu}{k}}}{k^2\,\Gamma_k(\mu)} \int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\mu}{k}-1} &\left\{ \int_a^t (t^{s+1}-u^{s+1})^{\frac{\gamma}{k}-1} \right. \\ &\left. \times\, {}_2R_{1,k}(\alpha,\beta;\gamma;\tau;\omega(t^{s+1}-u^{s+1})^{\tau})\, u^s f(u)\, du \right\} t^s\, dt. \end{aligned}$$

Changing the order of integrals, we have

$$\begin{aligned} \mathcal{L} = \frac{1}{k}\int_a^x \left\{ \frac{(s+1)^{1-\frac{\mu}{k}}}{k\,\Gamma_k(\mu)} \int_u^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\mu}{k}-1} \left(t^{s+1}-u^{s+1}\right)^{\frac{\gamma}{k}-1} \right. \\ \left. \times\, {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau;\omega(t^{s+1}-u^{s+1})^{\tau}\right)\, t^s\, dt \right\} u^s f(u)\, du. \end{aligned}$$

Let $t^{s+1} - u^{s+1} := \lambda^{s+1}$. We get

$$\mathcal{L} = \frac{1}{k}\int_a^x \left\{ \frac{(s+1)^{1-\frac{\mu}{k}}}{k\,\Gamma_k(\mu)} \times \int_0^{\lambda_0} \frac{\left(\lambda^{s+1}\right)^{\frac{\gamma}{k}-1} {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau;\omega(\lambda^{s+1})^{\tau}\right)}{\left(x^{s+1}-u^{s+1}-\lambda^{s+1}\right)^{1-\frac{\mu}{k}}} \lambda^s\, d\lambda \right\} u^s f(u)\, du,$$

where

$$\lambda_0 := \left(x^{s+1}-u^{s+1}\right)^{\frac{1}{s+1}}.$$

Using (2.2), we obtain

$$\mathcal{L} = \frac{1}{k}\int_a^x \left\{ {}^s_k\mathfrak{I}^{\mu}_{0+} \left(\lambda^{s+1}\right)^{\frac{\gamma}{k}-1} {}_2R_{1,k}\left(\alpha,\beta;\gamma;\tau;\omega(\lambda^{s+1})^{\tau}\right) \right\}(\lambda_0)\, u^s f(u)\, du.$$

Applying (2.9), we get

$$\mathcal{L} = \frac{\Gamma_k(\gamma)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma+\mu)} \left\{ \frac{1}{k}\int_a^x \left(x^{s+1}-u^{s+1}\right)^{\frac{\gamma+\mu}{k}-1} \times {}_2R_{1,k}\left(\alpha,\beta;\gamma+\mu;\tau;\omega\left(x^{s+1}-u^{s+1}\right)^{\tau}\right) u^s f(u)\, du \right\}.$$

Expressing the last integral in terms of (2.1), we obtain

$$\mathcal{L} = \frac{\Gamma_k(\gamma)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma+\mu)} \left({}^s_k\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma+\mu} f\right)(x).$$

This proves (3.5).

A similar argument as above gives

$$\mathcal{R} = \frac{\Gamma_k(\gamma)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\gamma+\mu)} \left({}^s_k\mathfrak{R}^{\omega;\alpha,\beta;\tau}_{a+;\tau,\gamma+\mu} f\right)(x).$$

This completes the proof. □

4. Concluding Remarks

Here, we introduced the generalized (k,s)-fractional integral operator involving τ-Gauss hypergeometric k-function (see [11]) and the Riemann-Liouville left-sided and right-sided $(k;s)$-fractional integral and differential operators of order μ. The results presented here when $s = 0$ and $(s,k) = (0,1)$ reduce to some known identities in [8] and [14], respectively. Also, setting $\tau = 1$ in the results here yields some corresponding identities involving the k-hypergeometric function ${}_2F_{1,k}(a,b;c;z)$.

References

[1] R. Díaz, E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat., **15** (2007), 179–192. 1, 1, 1, 1, 1

[2] A. A. Kilbas, M. Saigo, J. J. Trujillo, *On the generalized Wright function*, Dedicated to the 60th anniversary of Prof. Francesco Mainardi, Fract. Calc. Appl. Anal., **5** (2004), 437–460. 1

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, (2006). 1

[4] V. Kiryakova, *Generalized fractional calculus and applications*, Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, (1994). 1
[5] A. M. Mathai, R. K. Saxena, *Generalized hypergeometric functions with applications in statistics and physical sciences*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, (1973). 1
[6] S. Mubeen, G. M. Habibullah, *An integral representation of some* k*-hypergeometric functions*, Int. Math. Forum, **7** (2012), 203–207. 1
[7] S. Mubeen, G. M. Habibullah, k*-fractional integrals and application*, Int. J. Contemp. Math. Sci., **7** (2012), 89–94. 1
[8] S. Mubeen, C. G. Kologiannaki, G. Rahman, M. Arshad, *Properties of generalized hypergeometric* k*-function via* k*-fractional calculus*, Far East J. Appl. Math., (2017), (Accepted). 1, 2, 2, 4
[9] K. S. Nisar, A. F. Eata, M. Al-Dhaifallah, J.-S. Choi, *Fractional calculus of generalized* k*-Mittag-Leffler function and its applications to statistical distribution*, Adv. Difference Equ., **2016** (2016), 17 pages. 1
[10] J. C. Prajapati, R. K. Jana, R. K. Saxena, A. K. Shukla, *Some results on the generalized Mittag-Leffler function operator*, J. Inequal. Appl., **2013** (2013), 6 pages. 1
[11] G. Rahman, M. Arshad, S. Mubeen, *Some results on a generalized hypergeometric* k*-functions*, Bull. Math. Anal. Appl., **8** (2016), 66–77. 1, 1, 4
[12] E. D. Rainville, *Special functions*, Reprint of 1960 first edition. Chelsea Publishing Co., Bronx, N.Y., (1971). 1
[13] S. B. Rao, A. D. Patel, J. C. Prajapati, A. K. Shukla, *Some properties of generalized hypergeometric function*, Commun. Korean Math. Soc., **28** (2013), 303–317. 1
[14] S. B. Rao, J. C. Prajapati, A. D. Patel, A. K. Shukla, *Some properties of Wright-type generalized hypergeometric function via fractional calculus*, Adv. Difference Equ., **2014** (2014), 11 pages. 2, 2, 2, 4
[15] S. B. Rao, J. C. Prajapati, A. K. Shukla, *Wright type hypergeometric function and its properties*, Adv. Pure Math., **3** (2013), 335–342. 1, 1
[16] S. B. Rao, I. A. Salehbhai, A. K. Shukla, *On sequence of functions containing generalized hypergeometric function*, Math. Sci. Res. J., **17** (2013), 98–110.
[17] S. B. Rao, A. K. Shukla, *Note on generalized hypergeometric function*, Integral Transforms Spec. Funct., **24** (2013), 896–904. 1
[18] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, Theory and applications, Edited and with a foreword by S. M. Nikolskiĭ, Translated from the 1987 Russian original, Revised by the authors, Gordon and Breach Science Publishers, Yverdon, (1993). 1
[19] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, F. Ahmad, (k,s)*-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat., **45** (2016), 77–89. 1, 1
[20] A. K. Shukla, J. C. Prajapati, *On a generalized Mittag-Leffler type function and generated integral operator*, Math. Sci. Res. J., **12** (2008), 283–290. 1
[21] H. M. Srivastava, J.-S. Choi, *Zeta and* q*-Zeta functions and associated series and integrals*, Elsevier, Inc., Amsterdam, (2012). 1, 1, 1, 1
[22] H. M. Srivastava, Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput., **211** (2009), 198–210. 1
[23] N. Virchenko, S. L. Kalla, A. Al-Zamel, *Some results on a generalized hypergeometric function*, Integral Transform. Spec. Funct., **12** (2001), 89–100. 1
[24] N. Virchenko, O. Lisetska, S. L. Kalla, *On some fractional integral operators involving generalized Gauss hypergeometric functions*, Appl. Appl. Math., **5** (2010), 1418–1427.
[25] N. Virchenko, O. V. Rumiantseva, *On the generalized associated Legendre functions*, Fract. Calc. Appl. Anal., **11** (2008), 175–185. 1