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Fixed point property for digital spaces

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Abstract

The paper compares the fixed point property (*FPP* for short) of a compact Euclidean plane with its digital versions associated with Khalimsky and Marcus-Wyse topology. More precisely, by using a Khalimsky and a Marcus-Wyse topological digitization, the paper studies digital versions of the *FPP* for Euclidean topological spaces. Besides, motivated by the *digital homotopy fixed point property* (*DHFP* for brevity) [O. Ege, I. Karaca, C. R. Math. Acad. Sci. Paris, **353** (2015), 1029–1033], the present paper establishes the digital homotopy almost fixed point property (*DHAFP* for short) which is more generalized than the *DHFP*. Moreover, the present paper corrects some errors in [O. Ege, I. Karaca, C. R. Math. Acad. Sci. Paris, **353** (2015), 1029–1033] and improves it.

Keywords: Digital space, digitization, Khalimsky topology, Marcus-Wyse topology, fixed point property, digital homotopy almost fixed point property, almost fixed point property.
2010 MSC: 55N35, 55M20, 68R10, 68U05.

1. Introduction

For a nonempty binary symmetric relation set (X, π), we say that X is π-*connected* [10] if for any two elements x and y of X there is a finite sequence $(x_i)_{i \in [0,l]_Z}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in \pi$ for $j \in [0, l-1]_Z$, where for two distinct integers $a, b \in \mathbf{Z}$ we often use the notation $[a, b]_Z := \{x \in \mathbf{Z} \,|\, a \leqslant x \leqslant b\}$ [14]. By using this terminology, we say that a *digital space* is a nonempty, π-connected, symmetric relation set, denoted by (X, π) [10]. It is well-known that a digital space [10] includes a digital image (X, k) with digital k-connectivity (i.e., Rosenfeld model) [17, 18], a Khalimsky topological space with Khalimsky adjacency [12], a Marcus-Wyse topological space with Marcus-Wyse adjacency [22] and so forth [15]. Besides, a digital space can be established in terms of a digitization of a Euclidean space [11]. Thus the paper deals with both the fixed point property (*FPP* for short) for Euclidean topological spaces in $\mathbf{R}^n$ and the *FPP* for their digitized spaces in $\mathbf{Z}^n$ from the viewpoint of digital topology. We say that a digital image (X, k) is k-connected if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [14]. At this moment we need to recall that the digital image (X, k) on $\mathbf{Z}^n$ is not a topological space but just a digital graph on $\mathbf{Z}^n$ with a k-adjacency. We say that a non-empty and k-connected digital image (X, k) has the *FPP* [18] if every k-continuous map

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$f : (X, k) \to (X, k)$ has a point $x \in X$ such that $f(x) = x$. In fixed point theory from the viewpoint of digital topology in a graph-theoretical approach, we also assume that every digital image (X, k) is k-connected and non-empty.

To compare the *FPP* for Euclidean topological spaces with its digital version associated with Rosenfeld, Khalimsky, and Marcus-Wyse model, let us recall basic notions on fixed point theory and digital topology. Hereafter, we use the simplified words 'K-' and 'M-' instead of Khalimsky, and Marcus-Wyse, respectively if there is no danger of ambiguity. In digital topology, since there are several kinds of digitizations [11], Sections 2 and 4 follow the graph-theoretical approach in [17, 18] and Section 3 mainly deals with digitizations associated with a K- and an M-topological structure. Besides, we recall that under some conditions, Euler characteristics of digital images can be used to study the *FPP* for digital images. We say that a non-empty digital image (X, k) has the *almost fixed point property* (*AFPP*, for brevity) [18] if every k-continuous map $f : (X, k) \to (X, k)$ has a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x [18]. It is obvious that the *AFPP* is broader than the *FPP* [18].

We say that a topological space X has the *homotopy fixed point property* (*HFP* for short) [21] if it has the *FPP* with respect to the unit interval $[0, 1]$. Motivated by the *HFP*, the authors of [3] recently established its digital version named by the *digital homotopy fixed point property* (*DHFP* for short) in terms of a (digitally) k-continuous map [18], adjacency relations of Cartesian products of digital images [4], a digital k-homotopy [2] and so forth. This approach can contribute to topology and applied sciences. Besides, we say that a digital image (X, k) has the *DHFP* [3] if it has the *digital fixed point property* (*DFP* for brevity) with respect to any integral interval $[0, m]_{\mathbf{Z}} \subset \mathbf{Z}$ (see Definition 4.7 of the present paper). It is well known that only a singleton has the *FPP* [18]. Although the authors of [3] studied the *DHFP* in terms of a graph-theoretical approach, the work can be simplified as one statement (see Proposition 1.1 below) because it is well-known that a digital image (X, k) does not have the *FPP* with $|X| \geqslant 2$ (see Theorems 3.3 and 4.1 of [18] and the papers [6, 7]). Namely, only a singleton has the *FPP* and further, the *DHFP* of (X, k) requires its *digital fixed point property* (*DFP* for short) (see Corollary 3.3 of [3]). Hence we clearly obtain the following:

Proposition 1.1. *A digital image* (X, k) *has the DHFP if and only if it is a singleton.*

Since the paper [3] has some errors, the present paper corrects them and makes the paper improved. Furthermore, based on Proposition 1.1, the present paper proposes the '*digital homotopy almost fixed point property* (*DHAFP* for short)' which is weaker than the *DHFP* in [3].

The rest of the paper is organized as follows: Section 2 provides basic notions from digital topology. Section 3 investigates some properties of digitizations in a K- and an M-topological approach and further, studies some relations between the *FPP* for spaces in R^n and its digital versions. Furthermore, it refers to the *FPP* for K- and M-topological spaces. Section 4 studies the *DHFP*. Section 5 investigates a non-homotopy equivalence property of the *DHFP*. Section 6 deals with the almost fixed point property of digital images and studies various properties of the *DHAFP*. Besides, we prove that both a bounded simple k-path in $\mathbf{Z}^n$ and a bounded digital plane with 8-adjacency satisfy the *DHAFP*.

2. Preliminaries

To study the *FPP* for digital spaces and the *AFPP* for digital images from the viewpoint of digital topology, we need to recall some basic notions from digital topology such as k-adjacency relations of nD integer grids, a digital k-neighborhood, digital continuity and so forth [5, 17, 18]. Let $\mathbf{N}$, $\mathbf{Z}^n$, and $\mathbf{R}$ represent the sets of natural numbers, points in the Euclidean nD space with integer coordinates, and real numbers, respectively.

To study nD digital images, $n \in \mathbf{N}$, as a generalization of the k-adjacency relations of $\mathbf{Z}^n, n \in \{1, 2, 3\}$, we will take the following property [5].

For a natural number m, $1 \leqslant m \leqslant n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n,$$

are $k(m,n)$-adjacent if

$$\text{at most } m \text{ of their coordinates differ by } \pm 1, \text{ and all others coincide.} \tag{2.1}$$

In terms of the operator of (2.1), the $k(m,n)$-adjacency relations of $\mathbf{Z}^n, n \in \mathbf{N}$, are obtained [5] as follows:

$$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2.2}$$

Rosenfeld [17] called a set $X(\subset \mathbf{Z}^n)$ with a k-adjacency a digital image, denoted by (X,k). Indeed, to study digital images on $\mathbf{Z}^n$ in the graph-theoretical approach [17, 18], using the k-adjacency relations of $\mathbf{Z}^n$ of (2.2), we say that a digital k-neighborhood of p in $\mathbf{Z}^n$ is the set [17] $N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\} \cup \{p\}$. Furthermore, we often use the notation [13] $N_k^*(p) := N_k(p) \setminus \{p\}$. For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a,b]_{\mathbf{Z}} = \{m \in \mathbf{Z} \,|\, a \leqslant m \leqslant b\}$ with 2-adjacency is called a digital interval. Besides, for a k-adjacency relation of $\mathbf{Z}^n$, a simple k-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be a subset $(x_i)_{i \in [0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\,i-j\,| = 1$. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x,y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [5], is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i-j| = 1 \pmod{l}$.

For a digital image (X,k), for $X \subset \mathbf{Z}^n$ we put [5]

$$N_k(x,1) := N_k(x) \cap X. \tag{2.3}$$

As a generalization of $N_k(x,1)$ of (2.3), for a digital image (X,k) let us recall a digital k-neighborhood [5]. Namely, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X [5]

$$N_k(x_0,\varepsilon) := \{x \in X \,|\, l_k(x_0,x) \leqslant \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0,x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.

Definition 2.1. We say that a k-connected digital image (X,k) on $\mathbf{Z}^n$ is bounded if for some point $x_0 \in X$ there is an $N_k(x_0,\varepsilon)$ such that $X = N_k(x_0,\varepsilon)$, where $\varepsilon \lneq \infty$.

For two digital images (X,k_1) in $\mathbf{Z}^{n_1}$ and (Y,k_2) in $\mathbf{Z}^{n_2}$, we have often used the following adjacency for a digital product $X \times Y = \{(x,y) | x \in X, y \in Y\} \subset \mathbf{Z}^{n_1+n_2}$, as follows.

Definition 2.2 ([5])**.** For two digital images (X,k_1) in $\mathbf{Z}^{n_1}$, (Y,k_2) in $\mathbf{Z}^{n_2}$, consider the digital product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. Then we say that two points $(x,y) \in X \times Y$, $(x',y') \in X \times Y$ are normally k-adjacent to each other if and only if

(1) x is k_1-adjacent to x' and $y = y'$; or
(2) y is k_2-adjacent to y' and $x = x'$; or
(3) x is k_1-adjacent to x' and y is k_2-adjacent to y'.

The paper [18] established the notion of digital continuity of a map $f : (X,k_0) \to (Y,k_1)$ by saying that f maps every k_0-connected subset of (X,k_0) into a k_1-connected subset of (Y,k_1) (see Theorem 2.4 of [18]). Motivated by this approach, the digital continuity of maps between digital images was represented in terms of the neighborhood of (2.3), as follows:

Proposition 2.3 ([5])**.** *Let (X,k_0) and (Y,k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function f : $(X,k_0) \to (Y,k_1)$ is (k_0,k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x,1)) \subset N_{k_1}(f(x),1)$.*

Definition 2.4 ([2] (see also [9]))**.** Consider two digital images (X,k_0) and (Y,k_1) in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Then a map $h : X \to Y$ is called a (k_0,k_1)-isomorphism if h is a (k_0,k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1,k_0)-continuous.

3. Comparison the *FPP* of a compact Euclidean plane with its digital versions

Motivated by an Alexandroff space [1], the Khalimsky nD space was established and the study of its properties includes the papers [11–13]. More precisely, *Khalimsky line topology* κ on $\mathbf{Z}$, denoted by $(\mathbf{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ as a subbase [12] (see also [13]). Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or the *Khalimsky nD space*), denoted by $(\mathbf{Z}^n, \kappa^n)$. For the sake of convenience, we say that a point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd; and it is *pure closed* if each of the coordinates is even [13]. The other points in $\mathbf{Z}^n$ are called *mixed* [13]. In $\mathbf{Z}^2$, these points are showed like Figures 2-4, the symbols ■, a black big circle, • mean a pure closed point, a pure open point, and a mixed point, respectively.

As usual, for a subset $X \subset \mathbf{Z}^n$ we will consider (X, κ_X^n), $n \geqslant 1$ [7] as a subspace of $(\mathbf{Z}^n, \kappa^n)$, and it is called a K-topological space. For two K-topological spaces $(X, \kappa_X^{n_0}) := X$ and $(Y, \kappa_Y^{n_1}) := Y$ we say that a map $f : X \to Y$ is K-continuous if it is K-continuous at every point $x \in X$, i.e, $f(SN_K(x)) \subset SN_K(f(x))$ [7, 8, 11], where $SN_K(x)$ is the smallest open set containing the point x from the viewpoint of Khalimsky topology. By using spaces $(X, \kappa_X^n) := X$ and K-continuous maps, we have a Khalimsky topological category, denoted by *KTC* [11], consisting of the following two sets [7]:

(1) for any set $X \subset \mathbf{Z}^n$, the set of spaces (X, κ_X^n) as objects of *KTC* denoted by *Ob*(*KTC*);

(2) for all pairs of elements in *Ob*(*KTC*) the set of all K-continuous maps between them as morphisms.

The paper [7] studied the *FPP* for K-topological spaces and further, the paper [7] proved that not every K-contractible space does not have the *FPP*.

Let us now recall basic concepts from Marcus-Wyse (M-, for short) topology as another digital space. The M-*topology* on $\mathbf{Z}^2$, denoted by $(\mathbf{Z}^2, \gamma)$, is induced by the set $\{U\}$ in (3.1) below as a subbase [22], where for each point $p = (x, y) \in \mathbf{Z}^2$

$$U := SN_M(p) := N_4(p) \text{ if } x + y \text{ is even.} \tag{3.1}$$

In relation to the further statement of a point in $\mathbf{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [20].

In all subspaces of $(\mathbf{Z}^2, \gamma)$ of Figures 2-4, a black jumbo dot means an *even point* and further, the symbol $\lozenge$ means a *double even point* or a *even point*, and the symbol • means an *odd point*. In view of (3.1), we can obviously obtain the following: under $(\mathbf{Z}^2, \gamma)$ the singleton with either a double even point or an even point is the closure containing the given point. In addition, the singleton with an odd point is clearly the smallest open neighborhood of the given point. For a set $X \subset \mathbf{Z}^2$ we can take the subspace, denoted by (X, γ_X), induced by $(\mathbf{Z}^2, \gamma)$. As usual, for a subset $X \subset \mathbf{Z}^2$ we will consider (X, γ_X^2) [22] as a subspace of $(\mathbf{Z}^2, \gamma^2)$, and it is called an M-topological space. For two M-topological spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a function $f : X \to Y$ is said to be M-continuous at a point $x \in X$ [22], if $f(SN_M(x)) \subset SN_M(f(x))$ [11], where $SN_M(x)$ is the smallest open set containing the point x from the viewpoint of M-topology. Besides, a map $f : X \to Y$ is M-continuous if it is M-continuous at every point $x \in X$. By using spaces $(X, \gamma_X) := X$ and M-continuous maps, we have an M-topological category, denoted by *MTC*, consisting of the following two sets [11]:

(1) for any set $X \subset \mathbf{Z}^2$, the set of spaces (X, γ_X) as objects of *MTC* denoted by *Ob*(*MTC*);

(2) for all pairs of elements in *Ob*(*MTC*) the set of all M-continuous maps between them as morphisms.

To compare the *FPP* for Euclidean spaces with its digital version, let us investigate two digitizations such as a K- and an M-digitization. Given a Euclidean space $X \subset \mathbf{R}^n$, K-topological and M-topological digitizations were introduced in [11].

The following local K-neighborhood of a point $p \in \mathbf{Z}^n$ plays an important role in K-digitizing $X \subset \mathbf{R}^n$.

Definition 3.1 ([8])**.** In $\mathbf{R}^n$, for each point $p := (p_i)_{i\in[1,n]_{\mathbf{Z}}} \in \mathbf{Z}^n$, we define the set associated with $(\mathbf{Z}^n, \kappa^n)$, as follows:

$$N_K(p) := \{(x_i)_{i\in[1,n]_{\mathbf{Z}}}\}, \text{ where } \left\{ \begin{array}{l} \text{if } p_i = 2m, \;\; \text{then } x_i \in [2m - \frac{1}{2}, 2m + \frac{1}{2}], \\ \text{if } p_i = 2m+1, \;\; \text{then } \; x_i \in (2m + \frac{1}{2}, 2m + \frac{3}{2}), \end{array} \right\},$$

which is called a local K-neighborhood associated with $(\mathbf{Z}^n, \kappa^n)$.

For instance, we can observe $N_K(p)$ in the Euclidean 3-dimensional space (see Figures 1-4).
Hereafter, we denote by $(\mathbf{R}^n, E^n)$ the Euclidean n-dimensional space.

Definition 3.2 ([8])**.** For two points $x, y \in (\mathbf{R}^n, E^n)$, x is related to y if $x, y \in N_K(p)$ for some point $p \in \mathbf{Z}^n$, denoted by $x \sim_K y$ which is an equivalence relation.

By using this approach, we may consider $\bigcup_{p\in \mathbf{Z}^n} N_K(p)$ as $\mathbf{R}^n$ which is used to digitize $(\mathbf{R}^n, E^n)$ into $(\mathbf{Z}^n, \kappa^n)$. Besides, it is clear that for two points p, q in $\mathbf{Z}^n$ with $p \neq q$, $N_K(p) \cap N_K(q) = \emptyset$ so that the set $\{N_K(p) \,|\, p \in Z^n\}$ is a partition of $\mathbf{R}^n$ [11].

Definition 3.3 ([11])**.** Let X be a subspace in $(\mathbf{R}^n, E^n)$. A K-digitization of X, denoted by $D_K(X)$, is defined as follows:

$$D_K(X) = \{p \in \mathbf{Z}^n \,|\, N_K(p) \cap X \neq \emptyset\}$$

with K-topology.

Example 3.4. Let us consider the curve X in Figure 2 (a). After K-digitizing X, we obtain the set $D_K(X) := \{(1,0), (1,1), (2,1), (3,1), (4,1), (4,2)\}$. In this case the singletons $\{(1,1)\}$ and $\{(3,1)\}$ are the smallest open sets containing the given points, respectively. Besides, the singleton $\{(4,2)\}$ is the closed set.

Let us now consider a digitization of a Euclidean 2D subspace $X \subset \mathbf{R}^2$ in the Marcus-Wyse (for brevity, M-) topological approach. The following local M-neighborhood of a point $p \in \mathbf{Z}^2$ plays an important role in M-digitizing $X \subset \mathbf{R}^2$, as follows:

Definition 3.5 ([11])**.** In $\mathbf{R}^2$, for each point $p \in \mathbf{Z}^2$, we define the following neighborhood of p (see Figures 1 (b), (5)-(7)): for $i \in \{1,2\}$

$$N_M(p) := \left\{ \begin{array}{l} \{(t_1,t_2) \,|\, t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}]\} \text{ if } p = (p_1,p_2) \text{ is a double even point, and} \\ \{(t_1,t_2) \,|\, t_i \in [p_i - \frac{1}{2}, p_i + \frac{1}{2}]\} \setminus \{(p_1 \pm \frac{1}{2}, p_2 \pm \frac{1}{2})\} \text{ if } p = (p_1,p_2) \text{ is an even point, and} \\ \{(t_1,t_2) \,|\, t_i \in (p_i - \frac{1}{2}, p_i + \frac{1}{2})\} \text{ if } p = (p_1,p_2) \text{ is an odd point} \end{array} \right\},$$

which is called the local M-neighborhood of p associated with $(\mathbf{Z}^2, \gamma)$.

Definition 3.6 ([11])**.** For two points $x, y \in (\mathbf{R}^2, E^2)$, we say that x is related to y if $x, y \in N_M(p)$ for some point $p \in \mathbf{Z}^2$, denoted by $x \sim_M y$ which is an equivalence relation.

It is clear that the set $\{N_M(p) \,|\, p \in \mathbf{Z}^2\}$ is a partition of $\mathbf{R}^2$ associated with M-topology [11].

Definition 3.7 ([11])**.** Let X be a subspace in $(\mathbf{R}^2, E^2)$. An M-digitization $D_M(X)$ of X is defined as follows:

$$D_M(X) = \{p \in \mathbf{Z}^2 \,|\, N_M(p) \cap X \neq \emptyset\}$$

with M-topology.

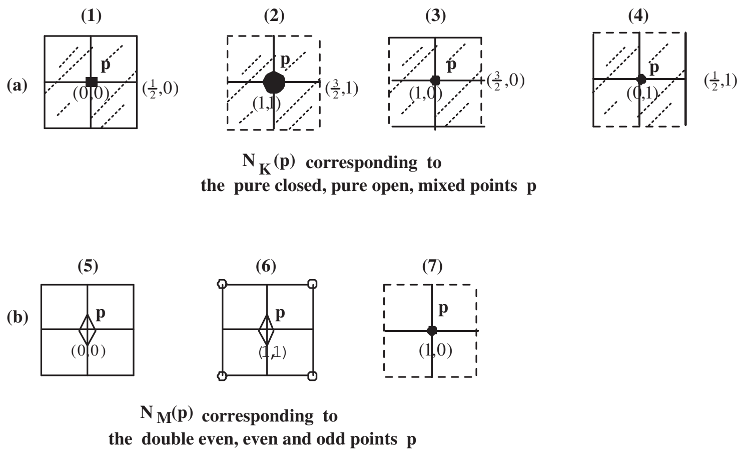

Figure 1: (a) Local K-neighborhoods of given points p in (1)-(4); (b) Local M-neighborhoods of given points p in (5)-(7) [11].

Example 3.8. Let us consider the curve X in Figure 2 (b). After M-digitizing X, we obtain the set $D_M(X) := \{(1,0),(1,1),(2,1),(3,1),(4,1),(4,2)\}$. In this case each of the singletons $\{(1,0)\}$, $\{(2,1)\}$, and $\{(4,1)\}$ is the smallest open set containing the given point. Besides, each of the singletons $\{(1,1)\}$, $\{(3,1)\}$, and $\{(4,2)\}$ is a closed set.

Remark 3.9. In view of Examples 3.4 and 3.8, we see that $D_K(X)$ and $D_M(X)$ have quite different topological structures.

For a closed interval $X \subset \mathbf{R}$ and an open interval $Z \subset \mathbf{R}$, owing to Brouwer fixed point theorem [16], in Euclidean topology it is clear that whereas X has the *FPP*, Y does not have the *FPP*. Thus we have a query of a relation between the *FPP* of X and that of its K- and M-digitizations, as follows:

Remark 3.10. Let $X := \{x \in \mathbf{R} : a \leqslant x \leqslant b, a, b \in \mathbf{Z}\}$ be a closed interval (see Figure 2 (c)), $Y := \{y \in \mathbf{R} : a \leqslant y \lneq b, a, b \in \mathbf{Z}\}$ be a closed-open interval (see Figure 2 (c)), and $Z := \{z \in \mathbf{R} : a \lneq z \lneq b, a, b \in \mathbf{Z}\}$ be an open interval. Then we obtain the followings:

(1) $D_K(X) = D_K(Y) = D_K(Z)$.
(2) Each of $D_K(X), D_K(Y)$, and $D_K(Z)$ has the *FPP* in *KTC*.
(3) $D_M(X) = D_M(Y) = D_M(Z)$.
(4) Each of $D_M(X), D_M(Y)$, and $D_M(Z)$ has the *FPP* in *MTC*.

Indeed, according to Definitions 3.3 and 3.7, since each of $D_K(X), D_K(Y)$, and $D_K(Z)$ is a K-path as a same set, it was proved to have the *FPP* in *KTC* [7, 19]. Besides, each of $D_M(X), D_M(Y)$, and $D_M(Z)$ is, respectively, equivalent to each of $D_K(X), D_K(Y)$, and $D_K(Z)$ up to K- or M-homeomorphism. In addition, each of $D_M(X)$, $D_M(Y)$, and $D_M(Z)$ also has the *FPP* in *MTC* because the *FPP* is a topological invariant property [7].

Theorem 3.11. *A compact* M*-topological plane does not have the FPP.*

Proof. It suffices to prove the assertion with a counterexample. Let us consider the M-topological space (X, γ_X) suggested in Figure 3, i.e., $X := \{x_i, y_i, z_j, w_j \,|\, i \in [0,5]_{\mathbf{Z}}, j \in [0,1]_{\mathbf{Z}}\}$. Assume the self-map f of (X, γ_X) given by

$$f(x_i) = y_i, f(y_i) = x_i, f(z_j) = w_j, f(w_j) = z_j, i \in [0,5]_{\mathbf{Z}}, j \in [0,1]_{\mathbf{Z}},$$

which is a rotation of X by 180°. Then it is clear that whereas f is an M-continuous map, f cannot have any fixed point, which implies that (X, γ_X) does not have the *FPP*. $\square$

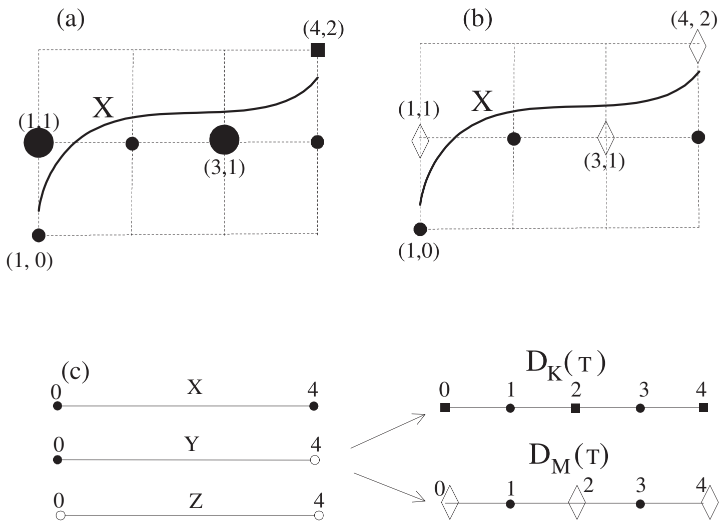

Figure 2: (a) A K-digitization of the given curve $X \subset \mathbf{R}^2$; (b) An M-digitization of the given curve $X \subset \mathbf{R}^2$; (c) A K- and an M-digitization of $T \subset \mathbf{R}$, where $T \in \{X, Y, Z\}$, $X := [0, 4], Y := [0, 4), Z := (0, 4)$.

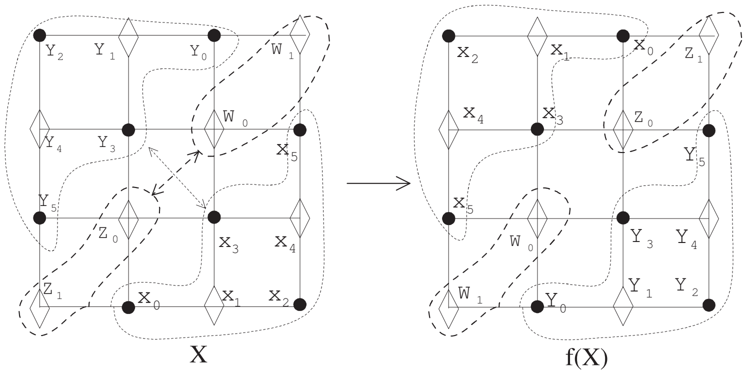

Figure 3: Explanation of the non-*FPP* of a compact M-topological plane.

Based on the above assertions, we will compare the *FPP* among a Euclidean, a K-, and an M-plane as follows:

Proposition 3.12. *For a compact Euclidean topological plane* $[-1, 1] \times [-1, 1] := X$*, whereas its* K*-digitized plane* $(D_K(X), \kappa^2_{D_K(X)})$ *has the FPP, its* M*-digitized plane* $(D_M(X), \gamma_{D_M(X)})$ *does not have the FPP.*

Proof. It is clear that a compact Euclidean plane $X := [-1, 1] \times [-1, 1]$ has the *FPP* [16].

Let us now prove that whereas its K-digitized space $(D_K(X), \kappa^2_{D_K(X)})$ has the *FPP*, its M-digitized space (X, γ_X) does not have the *FPP*.

Case 1: The *FPP* of $(D_K(X), \kappa^2_{D_K(X)})$. Let us consider any K-continuous self-map of $D_K(X)$. For convenience, take $x := (0, 0)$ as a pure closed point.

(1-1) If x is mapped by the map f into x, then the proof is obviously completed.

(1-2) We may assume that x is mapped into one of the mixed points by the map f such as x_1, where x_1 is a mixed point of $D_K(X)$.

For convenience, put $x_1 := (0,1)$ (see Figure 4 (a)). Owing to the K-topological structures of $SN_K(x)$ and $SN_K(x_1)$, it is clear that $f(SN_K(x)) \subset SN_K(x_1) = \{(-1,1), x_1, (1,1)\}$. Then we need to consider the mapping f focused on $SN_K(x_1)$, as follows:

In case $f(x_1) = x_1$, the proof is completed.

In case $f(x_1) = i \in \{(-1,1),(1,1)\}$, we should have $f(i) = i$ because $SN_K(i) = \{i\}$, which completes the proof.

(1-3) Assume that x is mapped by the map f into one of the pure open points of $D_K(X)$ such as x_2, where x_2 is a pure open point in $SN_K(x)$. Then, owing to the K-topological structure of $SN_K(x_2)$, we obviously obtain $f(SN_K(x)) \subset SN_K(x_2) = \{x_2\}$. Hence we obtain the point x_2 as a fixed point of f.

Similarly, the other cases are proved for the *FPP* of $D_K(X)$. Thus a compact K-plane $D_K(X)$ is proved to have the *FPP*.

Case 2: The non-*FPP* of $(D_M(X), \gamma_{D_M(X)})$. Let us now prove the *non-FPP* of $(D_M(X), \gamma_{D_M(X)})$. As a counterexample, consider the self-map g of $(D_M(X), \gamma_{D_M(X)}))$ as follows (see Figure 4 (b)):

$$g(D_M(X) \setminus \{x_1, x_2, x_3\}) = \{x_3\}, g(x_3) = x_0, g(x_1) = x_2, g(x_2) = x_1. \tag{3.2}$$

Then, whereas the map g of (3.2) is obviously an M-continuous map, it does not have any fixed point, which means that $(D_M(X), \gamma_{D_M(X)})$ does not have the *FPP*. □

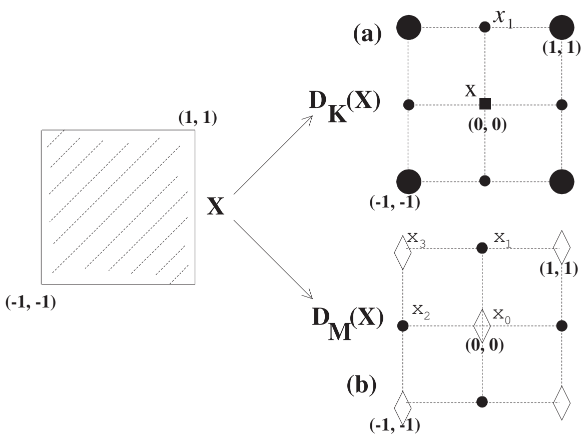

Figure 4: (a) A K-digitization of the plane X, denoted by $D_K(X)$; (b) an M-digitization of the plane P, denoted by$D_M(X)$, where $X := [-1,1]^2 \subset \mathbf{R}^2$.

4. Remarks on the digital homotopy fixed point property

To study the *DHFP* of a digital image (X, k), the paper [3] used the following digital k-homotopy.

Definition 4.1 ([2, 5]). Let (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$ be digital images. Let $f, g : X \to Y$ be (k_0, k_1)-continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $H : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

- for all $x \in X, H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, the induced function $H_x : [0, m]_{\mathbf{Z}} \to Y$ given by $H_x(t) := H(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$-continuous; and
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $H_t : X \to Y$ given by $H_t(x) := H(x, t)$ for all $x \in X$ is (k_0, k_1)-continuous.

Then we say that H is a (k_0, k_1)-homotopy between f and g.

When f and g are (k_0, k_1)-homotopic in Y, we denote by $f \simeq_{(k_0,k_1)} g$ the homotopic relation [2]. In addition, if $n_0 = n_1$ and $k_0 = k_1$, then we say that f and g are k_0-homotopic in Y and use the notation $f \simeq_{k_0} g$. If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map $c_{\{x_0\}}$ in the space (X, x_0), then we say that (X, x_0) is k-*contractible* [2].

In ordinary fixed point theory, Szymik [21] raised the following question: how do the fixed points of a continuous self-map depend on the given self-map? To work this issue out in detail, the author defined the following notions.

Definition 4.2 ([21], Continuous family of self-maps)**.** Let X be a topological space. A continuous family of self-maps of X is a continuous map $f : T \times X \to X$, where T is another topological space. A continuous family $f : T \times X \to X$ of self-maps defines for each $t \in T$ a self-map

$$f_t : X \to X,\ x \to f(t, x), \tag{4.1}$$

of X which is continuous.

Assume every continuous family of self-maps for a given self-map. Continuous family of fixed points need not exist, even if both T and X satisfy the *FPP* (see Example 5.3 of the paper [21]). Hence the above approach can be helpful to study topological spaces from the viewpoint of fixed point theory. To establish the *DHFP*, let us recall the following notions [21] (see Definitions 4.3, 4.4, and 4.5 below).

Definition 4.3 ([21], Continuous family of fixed points)**.** Let $f : T \times X \to X$ be a continuous family of self-maps of X. A continuous family of fixed points of f is a continuous map $p : T \to X$ such that $f(t, p(t)) = p(t)$ for all $t \in T$. The condition (4.1) means that $p(t)$ is a fixed point of f_t for all $t \in T$.

Definition 4.4 ([21], Fixed point property with respect to a topological space)**.** A topological space X has the *FPP* with respect to a topological space T if for all continuous families $f : T \times X \to X$ of self-maps of X, there exists at least one continuous family $p : T \to X$ of fixed points.

Definition 4.5 ([21])**.** A topological space X has the *HFP* if it has the *FPP* with respect to the unit interval $[0, 1]$.

Motivated by these notions, the authors of [3] formulated a digital version of the *HFP* in terms of a normal adjacency relations of digital products (see Definition 2.2). Using the adjacency of Definition 2.2, we propose the following:

Definition 4.6 ([3])**.** A digital image (X, k) has the *DFP* with respect to a digital interval $[0, m]_{\mathbf{Z}}$ if for all (k_*, k)-continuous maps $f : ([0, m]_{\mathbf{Z}} \times X, k_*) \to (X, k)$, where k_* is a normal adjacency for $[0, m]_{\mathbf{Z}} \times X$, there exists at least one $(2, k)$-continuous map $p : [0, m]_{\mathbf{Z}} \to X$ of fixed points.

Definition 4.7 ([3])**.** A digital image (X, k) has the *DHFP* if it has the *DFP* with respect to $[0, m]_{\mathbf{Z}}$ for all integers $m \geqslant 0$.

Definition 4.7 can be stated as follows: a digital image (X, k) has the *DHFP* if for all digital homotopies $f : [0, m]_{\mathbf{Z}} \times X \to X$ in X there is a $(2, k)$-continuous k-path $p : [0, m]_{\mathbf{Z}} \to X$ such that $p(t)$ is a fixed point of f_t for all $t \in [0, m]_{\mathbf{Z}}$. Based on this notion, motivated by the retraction property of *HFP* in [21] (see Propositions 4.1 and 4.2 of [21]), the paper [3] proposed some results (see Propositions 4.8 and 4.9 below) by using both a k-retraction and the *DFP* of a digital image (X, k) with respect to a digital interval $[0, m]_{\mathbf{Z}}$. Since $[0, n]_{\mathbf{Z}}$ is a 2-retract of $[0, m]_{\mathbf{Z}}, n \leqslant m$, we obtain the following:

Proposition 4.8 ([3])**.** *If a digital image (X, k) has the DFP with respect to a digital interval $[0, m]_{\mathbf{Z}}$, then (X, k) has the DFP with respect to $[0, n]_{\mathbf{Z}}$, where $n \lneq m$.*

Proposition 4.9 ([3]). *Assume that a digital image* (X, k) *has the DFP with respect to a digital interval* $[0, m]_Z$*. If* (A, k) *is a* k*-retract of* (X, k)*, then* (A, k) *also has the DFP with respect to* $[0, m]_Z$*.*

Corollary 4.10 ([3]). *The DHFP requires the DFP. It is clear that the converse need not be valid.*

Furthermore, in view of Proposition 4.8, we obtain the following because a digital image (X, k) has the *FPP* with respective to a singleton if and only if it has the *FPP*.

Corollary 4.11. *The DFP requires the FPP. It is clear that the converse need not be true.*

Thus we see that if a given digital image (X, k) does not have the *DFP* (resp. *FPP*), then it does not have the *DHFP* (resp. *DFP*) either. According to Proposition 1.1 and Corollaries 4.10 and 4.11, we obtain the following:

Theorem 4.12. *The digital image* (X, k) *satisfies the DFP if and only if it is a singleton.*

Proof. Since a singleton (X, k) in $\mathbf{Z}^n$, e.g., $X := \{x\}$ has the *FPP*, for every integer interval $[0, m]_Z$ whenever we consider continuous family of fixed points of a self-map f of X is exactly a $(2, k)$-continuous map, denoted by $p_t : X \to X$ whose graph of p_t can be considered as the set $\{(t, x) | \, t \in [0, m]_Z, x \in X\}$. Thus we see that a singleton has the *DFP*.

Conversely, suppose that there is a digital image (X, k) in $\mathbf{Z}^n$ whose cardinality is greater than 1 and having the *DFP*. Then, let us examine if for some $[0, m]_Z$ we obtain a $(2, k)$-continuous map of fixed points

$$F : ([0, m]_Z \times X, k_*) \to (X, k). \tag{4.2}$$

In (4.2) we only consider the case $[0, m]_Z \times X$ has the normal k_*-adjacency. Indeed, by Proposition 1.1, we see that only a singleton has the *FPP*. Thus the digital image (X, k) whose cardinality is greater than 1 cannot have the $(2, k)$-continuous map of fixed points inherited from the homotopy of (4.2) because it does not satisfy the *FPP*. As a result, according to Proposition 1.1, and Corollaries 4.10 and 4.11, we conclude that the given digital image (X, k) should be a singleton. □

To guarantee Corollary 4.11, the authors of [3] proposed the following example.

Example 4.13 ([3, Example 4.1]). Let $X = [a, a + 1]_Z$ be a digital interval where $a \in \mathbf{Z}$. It is easy to see that $(X, 2)$ has both the *DFP* and the *DHFP*.

However, the assertion in Example 4.13 is incorrect because the given digital image $(X, 2)$ does have neither the *FPP* nor the *DFP* (see Proposition 3.12, Corollaries 4.10 and 4.11 and Theorem 4.12). The below Remark 4.14 and Example 4.15 verify the claim.

Remark 4.14. Unlike the assertion of [3] as in Example 4.13 above, the digital image $(X, 2)$ in Example 4.13 satisfies neither the *DFP* nor the *DHFP*. To be specific, assume the self-map

$$f : [a, a + 1]_Z \to [a, a + 1]_Z$$

given by

$$f(a) = a + 1 \text{ and } f(a + 1) = a.$$

Namely, as discussed in Proposition 1.1, it is clear that while the map f is a 2-continuous map, it does not have any fixed point (see also Proposition 1.1). Hence, by Corollaries 4.10, 4.11, and Theorem 4.12, we see that the digital image $(X, 2)$ does not have the *DFP* because the *DHFP* (resp. *DFP*) is stronger than the *DFP* (resp. *FPP* in digital topology).

To be specific, we can guarantee Remark 4.14 more precisely with the following example.

Example 4.15. We see that the given digital image $(X, 2)$ in Example 4.13 does not have the *DHFP* because the digital image $(X, 2)$ does not have the *FPP* from the viewpoint of digital topology (see Remark 4.14).

More precisely, let us assume the two digital images $T := ([0, 3]_{\mathbf{Z}}, 2)$ and $X := ([a, a+1]_{\mathbf{Z}}, 2)$. According to Definition 2.2, we take a normal 8-adjacency for the digital product $T \times X$, i.e., $(T \times X, 8)$. Indeed, in this case only the 8-adjacency is normal for the product $T \times X$. Let us now consider the $(8, 2)$-continuous families of self-maps (see Figure 2) as follows:

$$F : (T \times X, 8) \to (X, 2) \text{ or } f_t : (X, 2) \to (X, 2), t \in [0, 3]_{\mathbf{Z}}$$

given by

$$\left\{\begin{array}{l} f_0 = 1_X, \\ f_1(a) = a+1 \text{ and } f_1(a+1) = a, \\ f_2 = C_{\{a\}} \text{ as a constant map onto the singleton } \{a\}, \\ f_3 = C_{\{a+1\}} \text{ as a constant map onto the singleton } \{a+1\}. \end{array}\right\}$$

Then the fixed point sets of the maps f_t are

$$\{x \in X \,|\, f_t(x) = x\} = \left\{\begin{array}{ll} X, & t = 0, \\ \emptyset, & t = 1, \\ \{a\}, & t = 2, \\ \{a+1\}, & t = 3. \end{array}\right\} \tag{4.3}$$

This property (4.3) makes it clear that there is no 2-continuous family of fixed points. Finally, we prove that Example 4.13 (or Example 1 of the paper [3]) is incorrect.

Theorem 4.16. *A digital image* (X, k) *has the DHFP, the DFP, and the FPP if and only if it is a singleton.*

Proof. In digital topology, by Proposition 1.1 and Corollaries 4.10 and 4.11, since the digital image satisfying the *DHFP* is only a singleton, the proof is completed. □

5. Non-homotopy invariant of the digital homotopy fixed point property

The following notion of a digital homotopy equivalence firstly developed in the paper [4] to classify digital images up to a digital k-homotopy equivalence.

Definition 5.1 ([4]). Let $f : (X, k_0) \to (Y, k_1)$ and $g : (Y, k_1) \to (X, k_0)$ be (k_0, k_1)- and (k_1, k_0)-continuous maps, respectively, such that

$$g \circ f \simeq_{k_0} 1_X \text{ and } f \circ g \simeq_{k_1} 1_Y.$$

Then we say that (X, k_0) and (Y, k_1) have the same (k_0, k_1)-homotopy type (or (X, k_0) is (k_0, k_1)-homotopy equivalent to (Y, k_1)).

Theorem 5.2. *The DHFP is not a digital homotopy equivalence invariant.*

Proof. It suffices to give two examples to guarantee this assertion, e.g., $([0, l]_{\mathbf{Z}}, 2)$ and another digital image (X, k) which is k-homotopy equivalent to a singleton. Consider a singleton of $[0, l]_{\mathbf{Z}}, l \geqslant 1$. It is clear that $([0, l]_{\mathbf{Z}}, 2)$ is 2-homotopy equivalent to any singleton $\{t\}, t \in [0, l]_{\mathbf{Z}}$ such as $\{0\}$. Then, by Remark 4.14 and Example 4.13, we proved that $[0, l]_{\mathbf{Z}}$ does not have the *DHFP* because $([0, l]_{\mathbf{Z}}, 2)$ does not have the *FPP* (see Proposition 1.1). But the singleton $\{t\}, t \in [0, l]_{\mathbf{Z}}$ obviously has the *DHFP*. For convenience, we suffice to prove that the singleton $\{0\}$ has the *DHFP*. To be specific, let us consider the 2-continuous families of self-maps for any $T := [0, m]_{\mathbf{Z}}, m \in \mathbf{N}$ as follows:

$F : (T \times \{0\}, 2) \to (\{0\}, 2)$ (or $f_t : (\{0\}, 2) \to (\{0\}, 2), t \in [0, m]_{\mathbf{Z}}$) given by

$$f_t = 1_{\{0\}}, \quad t \in [0, m]_{\mathbf{Z}}, \tag{5.1}$$

which is the identity map onto the singleton $\{0\}$. Then, according to (5.1), we obtain the fixed point sets of the maps f_t as the set $\{(t,0)\,|t \in [0,l]_{\mathbf{Z}}\}$. Hence there is a $(2,k)$-(or 2-)continuous map of fixed points whose image can be considered to be the set $\{(t,0)\,|\,t \in [0,m]_{\mathbf{Z}}\}$.

In general, consider any k-contractible digital image (X,k) in $\mathbf{Z}^n$ such that $|X| \geqslant 2$. Then it is clear that it is k-homotopy equivalent to any singleton $(\{x\},k) \subset (X,k)$. While the singleton $(\{x\},k)$ has the *DHFP*, the digital image (X,k) does not have the *DHFP* because (X,k) does not have the *FPP* (see Proposition 1.1). □

In addition, we further need to comment on the contents in Section 4 of the paper [3], such a kind of approach as in [3] is trivial with the following reason:

Remark 5.3. Owing to Corollaries 4.10, 4.11, and Proposition 1.1, it is clear that only a singleton digital image satisfies the *DHFP*. Hence the digital images, e.g., $(X,8)$, where $X := [-1,3]^2_{\mathbf{Z}} \setminus \{(0,0)\}$ or $SC_8^{2,4}$, are far from the study of the digital homotopy fixed point theory because none of $(X,8)$ and $SC_8^{2,4}$ does not have the *FPP*.

6. The digital homotopy almost fixed point property

In view of Proposition 1.1, we need to expand *DHFP* into the *DHAFP*. Rosenfeld [18] firstly studied the *AFPP* so that not every k-continuous self-map f of a digital image (X,k) satisfies the *AFPP* [18].

Let us now generalize the *DFP* of Definition 4.6 into the following:

Definition 6.1. A digital image (X,k) has the *DAFP* with respect to a digital interval $[0,m]_{\mathbf{Z}}$ if for all (k_*,k)-continuous maps $f : ([0,m]_{\mathbf{Z}} \times X, k_*) \to (X,k)$ where k_* is a normal adjacency relation for $[0,m]_{\mathbf{Z}} \times X$, there exists at least one $(2,k)$-continuous map $p : [0,m]_{\mathbf{Z}} \to X$ of almost fixed points.

Definition 6.2. A digital image (X,k) has the *DHAFP* (*resp. DHFP*) if it has the *DAFP* (*resp. DFP*) with respect to $[0,m]_{\mathbf{Z}}$ for all integers $m \geqslant 0$.

In view of Proposition 1.1, we need to study the *DHAFP* which is broader than the *DHFP*. Hence, by using the adjacency of Definition 2.2, we propose the following: Definition 6.2 can be represented as follows: a digital image (X,k) has the *DHAFP* if for all digital homotopies $f : [0,m]_{\mathbf{Z}} \times X \to X$ in X there is a $(2,k)$-continuous k-path $p : [0,m]_{\mathbf{Z}} \to X$ such that $p(t)$ is an almost fixed point of f_t for all $t \in [0,m]_{\mathbf{Z}}$.

We obviously obtain the following.

Proposition 6.3. *The DHAFP requires the DAFP. But the converse need not be valid.*

Theorem 6.4. *Let (X,k) be a bounded simple k-path in $\mathbf{Z}^n, n \in \mathbf{N}$. Then it has the DHAFP.*

Proof. It is clear that a bounded simple k-path $(X := \{x_i | i \in [0,t]_{\mathbf{Z}}\}, k)$ in $\mathbf{Z}^n$ is $(k,2)$-isomorphic to $([0,t]_{\mathbf{Z}},2)$. Since $([0,t]_{\mathbf{Z}},2)$ has the *AFPP* [18] and further, the *AFPP* is a digital topological invariant, we obtain that (X,k) has the *AFPP*. Under this situation, for every integral interval $[0,m]_{\mathbf{Z}}$ whenever we consider continuous family of almost fixed points of a self-map f of X is exactly a $(2,k)$-continuous map, denoted by $p_t : X \to X$ whose graph of p_t can be considered as the set $\{(t,x)|\, t \in [0,m]_{\mathbf{Z}}, x \in X\}$. Thus we obtain that (X,k) has the *DHAFP*. □

Example 6.5. The integral interval $(X := [a,a+1]_{\mathbf{Z}},2)$ in Remark 4.14 has the *DHAFP*. To be specific, consider any 2-continuous self-map f of $[a,a+1]_{\mathbf{Z}}$. Then it is clear that the digital image $(X,2)$ has the *AFPP* and further, the *DHAFP* (Proposition 6.3).

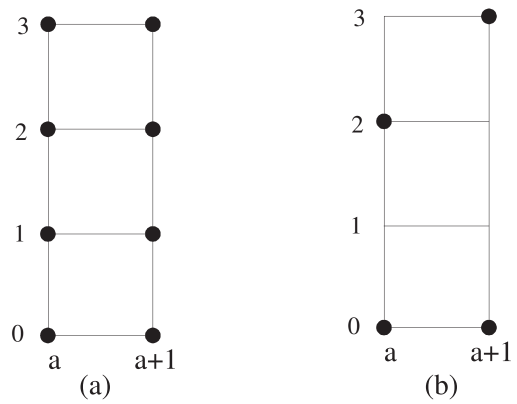

Figure 5: Explanation of the non-*DHFP* of $X := ([a, a+1]_{\mathbf{Z}}, 2)$. (a) $(T \times X, 8)$; (b) Graph of $f_t, t \in [0, 3]_{\mathbf{Z}}$.

Theorem 6.6. *Let* X *be a bounded digital plane in* $\mathbf{Z}^2$ *with* 8*-adjacency. Then it has the DHAFP.*

Proof. It is clear that a bounded digital plane $(X, 8)$ in $\mathbf{Z}^2$ has the *AFPP* [18]. Under this situation, for every integral interval $[0, m]_{\mathbf{Z}}$ whenever we consider continuous family of almost fixed points of a self-map f of X is exactly a $(2, 8)$-continuous map, denoted by $p_t : X \to X$ whose graph of p_t can be considered as the set $\{(t, x) | \, t \in [0, m]_{\mathbf{Z}}, x \in X\}$. Thus we see that $(X, 8)$ has the *DHAFP*. □

Remark 6.7. In Theorem 6.4, if the given adjacency is not an 8-adjacency, the assertion does not hold.

Proof. Let us consider the set $X := [0, 1]_{\mathbf{Z}} \times [0, 1]_{\mathbf{Z}}$ in $\mathbf{Z}^2$. Suppose a 4-adjacency for X. Then we see that $(X, 4)$ is a kind of $SC_4^{2,4} := \{x_0 = (0, 0), x_1 = (1, 0), x_2 = (1, 1), x_3 = (0, 1)\}$. Then it is clear that $(X, 4)$ does not have the *AFPP*. To be specific, consider the self-map of $(X, 4)$ given by $f(x_i) = x_{i+2(\mathrm{mod}\,4)}$. While the map f is a 4-continuous map, it does not have the *AFPP*. By Proposition 6.3, the proof is completed. □

By using the method similar to Theorem 6.4, we obtain the following:

Corollary 6.8. *Let* X *be a bounded digital cube* $X \subset \mathbf{Z}^n$ *with* $(3^n - 1)$*-adjacency. Then* $(X, 3^n - 1)$ *has the DHAFP.*

7. Concluding remarks

We have studied a digital version of the *HFP* for studying the *FPP* of digital images. Besides, we have proved that the *DHFP* is not a digital homotopy equivalence invariant. Finally, according to Proposition 1.1, since only a singleton has the *FPP* from the viewpoint of digital topology followed from Rosenfeld model, it turns out that a digital image satisfies each of the *DHFP*, the *DFP*, and the *FPP* if and only if it is a singleton.

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