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Fourier series of higher-order ordered Bell functions

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Abstract

In this paper, we consider higher-order ordered Bell functions and derive their Fourier series expansions. Moreover, we express those functions in terms of Bernoulli functions.

Keywords: Fourier series, higher-order ordered Bell functions, higher-order ordered Bell polynomials.
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1. Introduction

For $r \in \mathbb{Z}_{>0}$, the ordered Bell polynomials $b_m^{(r)}(x)$ of order r are defined by the generating function

$$\left(\frac{1}{2-e^t}\right)^r e^{xt} = \sum_{m=0}^{\infty} b_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.1}$$

When $x = 0$, $b_m^{(r)} = b_m^{(r)}(0)$ are called the ordered Bell numbers of order r. In particular, $b_m(x) = b_m^{(1)}(x)$ and $b_m = b_m^{(1)}$ are respectively called the ordered Bell polynomials and the ordered Bell numbers.

The first appearance of the ordered Bell numbers b_m goes back to as early as 1859, when Cayley used them to count certain plane trees with $m+1$ totally ordered leaves. Since then, they have been studied in many counting problems in number theory and enumerative combinatorics (see [2, 3, 5, 8, 12, 14, 15]). The ordered Bell numbers b_m are all positive integers, as we can see from

$$b_m = \sum_{n=0}^{m} n! S_2(m,n) = \sum_{n=0}^{\infty} \frac{n^m}{2^{n+1}}, \quad (m \geqslant 0).$$

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On the other hand, the ordered Bell polynomial $b_m(x)$ has degree m and is a monic polynomial with integral coefficients, as we can see, for example, from

$$b_0(x) = 1, b_m(x) = x^m + \sum_{l=0}^{m-1} \binom{m}{l} b_l(x), \quad (m \geqslant 1).$$

From (1.1), we can derive

$$\frac{d}{dx} b_m^{(r)}(x) = m b_{m-1}^{(r)}(x), \quad (m \geqslant 1), \qquad b_m^{(r)}(x+1) - b_m^{(r)}(x) = b_m^{(r)}(x) - b_m^{(r-1)}(x), \quad (m \geqslant 0).$$

In turn, from these we obtain

$$\begin{aligned} & b_m^{(r)}(1) - b_m^{(r)} = b_m^{(r)} - b_m^{(r-1)}, \quad (m \geqslant 0), \\ & \int_0^1 b_m^{(r)}(x)dx = \frac{1}{m+1}\left(b_{m+1}^{(r)}(1) - b_{m+1}^{(r)}\right) = \frac{1}{m+1}\left(b_{m+1}^{(r)} - b_{m+1}^{(r-1)}\right). \end{aligned} \tag{1.2}$$

As is well-known, the Bernoulli polynomials $B_m(x)$ are given by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

For any real number x, let

$$< x >= x - [x] \in [0, 1)$$

denote the fractional part of x.

The reader may refer to any book (for example, see [1, 13, 16]) for elementary facts about Fourier analysis. Also, we will need the following well-known facts about Bernoulli functions $B_n(< x >)$:

(a) for $m \geqslant 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m};$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n \neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Here we will consider the higher-order ordered Bell functions $b_m^{(r)}(< x >)$, and derive its Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions.

As to the higher-order ordered Bell functions $b_m^{(r)}(< x >)$, we note that the polynomial identity (1.3) follows immediately from Theorems 2.1 and 2.2, which can be derived in turn from the Fourier series expansion of $b_m^{(r)}(< x >)$,

$$b_m^{(r)}(x) = \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} \left(b_{m-j+1}^{(r)} - b_{m-j+1}^{(r-1)}\right) B_j(x). \tag{1.3}$$

Finally, the reader may refer to [4, 6, 7, 9–11] for some recent related works.

2. Fourier series of higher-order ordered Bell functions

From now on, we will assume that $m \geqslant 1$ and $r \geqslant 2$. The case of $r = 1$ has been treated as a special case of the results in [4].

$b_m^{(r)}(<x>)$ is piecewise C^∞. Moreover, in view of (1.2), $b_m^{(r)}(<x>)$ is continuous for those integers (r,m) with $b_m^{(r)} = b_m^{(r-1)}$, and is discontinuous with jump discontinuities at integers for those (r,m) with $b_m^{(r)} \neq b_m^{(r-1)}$.

The Fourier series of $b_m^{(r)}(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(r,m)} e^{2\pi i n x},$$

where

$$A_n^{(r,m)} = \int_0^1 b_m^{(r)}(<x>)e^{-2\pi i n x}dx = \int_0^1 b_m^{(r)}(x)e^{-2\pi i n x}dx.$$

Now, we would like to determine the Fourier coefficients $A_n^{(r,m)}$.

Case 1: $n \neq 0$.

For $r \geqslant 2$, and $m \geqslant 1$, we set

$$\begin{aligned}
\Delta_{r,m} &= b_m^{(r)}(1) - b_m^{(r)} = b_m^{(r)} - b_m^{(r-1)},\\
A_n^{(r,m)} &= \int_0^1 b_m^{(r)}(x)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\left[b_m^{(r)}(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \left(\frac{d}{dx}b_m^{(r)}(x)\right)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\left(b_m^{(r)}(1) - b_m^{(r)}\right) + \frac{m}{2\pi i n}\int_0^1 b_{m-1}^{(r)}(x)e^{-2\pi i n x}dx\\
&= \frac{m}{2\pi i n}A_n^{(r,m-1)} - \frac{1}{2\pi i n}\Delta_{r,m}\\
&= \frac{m}{2\pi i n}\left(\frac{m-1}{2\pi i n}A_n^{(r,m-2)} - \frac{1}{2\pi i n}\Delta_{r,m-1}\right) - \frac{1}{2\pi i n}\Delta_{r,m}\\
&= \frac{(m)_2}{(2\pi i n)^2}A_n^{(r,m-2)} - \sum_{j=1}^{2}\frac{(m)_{j-1}}{(2\pi i n)^j}\Delta_{r,m-j+1}\\
&= \frac{(m)_2}{(2\pi i n)^2}\left(\frac{m-2}{2\pi i n}A_n^{(r,m-3)} - \frac{1}{2\pi i n}\Delta_{r,m-2}\right) - \sum_{j=1}^{2}\frac{(m)_{j-1}}{(2\pi i n)^j}\Delta_{r,m-j+1}\\
&= \frac{(m)_3}{(2\pi i n)^3}A_n^{(r,m-3)} - \sum_{j=1}^{3}\frac{(m)_{j-1}}{(2\pi i n)^j}\Delta_{r,m-j+1}\\
&\ \vdots\\
&= \frac{m!}{(2\pi i n)^m}A_n^{(r,0)} - \sum_{j=1}^{m}\frac{(m)_{j-1}}{(2\pi i n)^j}\Delta_{r,m-j+1}\\
&= -\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi i n)^j}\Delta_{r,m-j+1}.
\end{aligned}$$

Case 2: $n = 0$.

$$A_0^{(r,m)} = \int_0^1 b_m^{(r)}(x)dx = \frac{1}{m+1}\Delta_{r,m+1}.$$

Assume first that $\Delta_{r,m} = 0$. Then $b_m^{(r)}(1) = b_m^{(r)}(0)$. As $b_m^{(r)}(<x>)$ is piecewise C^∞ and continuous, the Fourier series of $b_m^{(r)}(<x>)$ converges uniformly to $b_m^{(r)}(<x>)$, and

$$\begin{aligned}
&b_m^{(r)}(<x>)\\
&=\frac{1}{m+1}\Delta_{r,m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{r,m-j+1}\right)e^{2\pi inx}\\
&=\frac{1}{m+1}\Delta_{r,m+1}+\frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\Delta_{r,m-j+1}\times\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
&=\frac{1}{m+1}\Delta_{r,m+1}+\frac{1}{m+1}\sum_{j=2}^{m}\binom{m+1}{j}\Delta_{r,m-j+1}B_j(<x>)+\Delta_{r,m}\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

We are now ready to state our first theorem.

Theorem 2.1. *For positive integers* r, l *with* $r \geqslant 2$*, we let*

$$\Delta_{r,l}=b_l^{(r)}-b_l^{(r-1)}.$$

Assume that $\Delta_{r,m}=0$ *for positive integers* r, m *with* $r\geqslant 2$*. Then we have the following:*

(a) $b_m^{(r)}(<x>)$ *has the Fourier series expansion*

$$b_m^{(r)}(<x>)=\frac{1}{m+1}\Delta_{r,m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{r,m-j+1}\right)e^{2\pi inx}$$

for all $x\in\mathbb{R}$*, where the convergence is uniform;*

(b)

$$b_m^{(r)}(<x>)=\frac{1}{m+1}\Delta_{r,m+1}+\frac{1}{m+1}\sum_{j=2}^{m}\binom{m+1}{j}\Delta_{r,m-j+1}B_j(<x>)$$

for all $x\in\mathbb{R}$*, where* $B_j(<x>)$ *is the Bernoulli function.*

Assume next that $\Delta_{r,m}\neq 0$ for positive integers r, m with $r\geqslant 2$. Then $b_m^{(r)}(1)\neq b_m^{(r)}(0)$. Thus $b_m^{(r)}(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $b_m^{(r)}(<x>)$ converges pointwise to $b_m^{(r)}(<x>)$ for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(b_m^{(r)}(0)+b_m^{(r)}(1)\right)=b_m^{(r)}+\frac{1}{2}\Delta_{r,m}$$

for $x\in\mathbb{Z}$.

Now, we can state our second theorem.

Theorem 2.2. *For positive integers* r, l *with* $r \geqslant 2$*, we let*

$$\Delta_{r,l}=b_l^{(r)}-b_l^{(r-1)}.$$

Assume that $\Delta_{r,m}\neq 0$ *for positive integers* r, m *with* $r\geqslant 2$*. Then we have the following:*

(a)

$$\frac{1}{m+1}\Delta_{r,m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{r,m-j+1}\right)e^{2\pi inx}=\begin{cases}b_m^{(r)}(<x>), & \textit{for } x\notin\mathbb{Z},\\ b_m^{(r)}+\frac{1}{2}\Delta_{r,m}, & \textit{for } x\in\mathbb{Z};\end{cases}$$

(b)

$$\frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Delta_{r,m-j+1}B_j(<x>) = b_m^{(r)}(<x>),\ \textit{for}\ x \notin \mathbb{Z},$$

$$\frac{1}{m+1}\sum_{\substack{j=0\\ j\neq 1}}^{m}\binom{m+1}{j}\Delta_{r,m-j+1}B_j(<x>) = b_m^{(r)} + \frac{1}{2}\Delta_{r,m},\ \textit{for}\ x \in \mathbb{Z}.$$

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