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# Existence result for a class of coupled fractional differential systems with integral boundary value conditions

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### Abstract

Applying coincidence degree theory of Mawhin, this paper is concerned with existence result for a coupled fractional differential systems with Riemann-Stieltjes integral boundary value conditions. An example is also given to illustrate the main result. 

Keywords: Coincidence degree, coupled fractional differential systems, Riemann-Stieltjes integral, boundary value conditions.
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## 1. Introduction

The subject of fractional calculus has gained considerable popularity and importance due to its wide applications in widespread fields of science and engineering. For details, see [4, 5, 9] and the references therein. Fractional models can provide a more precise description over things than integral ones. This is owing to the fact that fractional derivatives enable the description of memory and hereditary properties of various material and processes. As a result, fractional differential equations have attracted much attention, and lots of good results have been obtained. See [2–5, 8–11] for a good overview.

Meanwhile, coupled fractional differential systems have been studied in some recent works [8, 13, 14]. For example, in [8], the authors studied the following coupled system of fractional differential equations with nonlocal integral boundary conditions

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) = f(t,u(t),v(t)), \quad t \in [0,1], \\ {}^{C}D_{0+}^{\beta}v(t) = g(t,u(t),v(t)), \quad t \in [0,1], \\ u(0) = \gamma I^{p}u(\eta) = \gamma \int_0^{\eta} \frac{(\eta-s)^{p-1}}{\Gamma(p)}u(s)ds, \quad 0<\eta<1, \\ v(0) = \delta I^{q}v(\xi) = \delta \int_0^{\xi} \frac{(\xi-s)^{q-1}}{\Gamma(q)}v(s)ds, \quad 0<\xi<1, \end{cases}$$

where ${}^{C}D_{0+}^{\alpha}$ denotes the Caputo fractional derivative, $0<\alpha,\beta\leqslant 1$, $f,g \in C([0,1]\times\mathbb{R}^2,\mathbb{R})$, and $p,q,\gamma,\delta \in \mathbb{R}$. Applying nonlinear alternative of Leray-Schauder and Banach's fixed-point theorem, they investigated the existence and uniqueness of solution for this coupled system.

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In [13], Zhang et al. investigated the following three-point boundary value conditions at resonance for the following coupled system of nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, v(t), D_{0+}^{\beta-1}v(t)), \quad 0 < t < 1, \\ D_{0+}^{\beta}v(t) = g(t, u(t), D_{0+}^{\alpha-1}u(t)), \quad 0 < t < 1, \\ u(0) = v(0) = 0, \quad u(1) = \sigma_1 u(\eta_1), \quad v(1) = \sigma_2 v(\eta_2), \end{cases}$$

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $1 < \alpha, \beta \leqslant 2$, $0 < \eta_1, \eta_2 < 1$, $\sigma_1, \sigma_2 > 0$, $\sigma_1\eta_1^{\alpha-1} = \sigma_2\eta_2^{\beta-1} = 1$, and $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. Based on new Banach spaces and by using the coincidence degree theory of Mawhin, the existence results were studied.

Recently, in [2], Cui studied the following differential system at resonance

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t), x'(t), y'(t)), \quad t \in (0, 1), \\ -y''(t) = f_2(t, x(t), y(t), x'(t), y'(t)), \quad t \in (0, 1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x], \end{cases}$$

where $f_1, f_2 : (0, 1) \times \mathbb{R}^4 \to \mathbb{R}$ are continuous and may be singular at $t = 0, 1$. $\alpha[y], \beta[x]$ are bounded linear functionals on $C[0, 1]$ given by

$$\alpha[y] = \int_0^1 y(t)dA(t), \quad \beta[x] = \int_0^1 x(t)dB(t),$$

involving Stieltjes integrals.

To our best knowledge, there are fewer results for coupled fractional differential systems with Riemann-Stieltjes integral boundary value conditions. Motivated by all the above works, we consider the existence of solutions for the following systems

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, y(t), D_{0+}^{\beta-1}y(t)), \quad 0 < t < 1, \\ D_{0+}^{\beta}y(t) = g(t, x(t), D_{0+}^{\alpha-1}x(t)), \quad 0 < t < 1, \\ x(0) = y(0) = 0, \quad x(1) = \alpha[x], \quad y(1) = \beta[y], \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta < 2$, $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Carathéodory conditions, $\alpha[x] = \int_0^1 x(t)dA(t)$, $\beta[y] = \int_0^1 y(t)dB(t)$, and $A(t)$, $B(t)$ are functions of bounded variation satisfying

$$\int_0^1 t^{\alpha-1}dA(t) = 1, \quad \int_0^1 t^{\beta-1}dB(t) = 1, \quad \int_0^1 t^{\alpha}dA(t) \neq 1, \quad \int_0^1 t^{\beta}dB(t) \neq 1.$$

The main features of this paper are as follows. (i) A class of coupled fractional differential systems with Riemann-Stieltjes integral boundary value conditions is firstly studied, which generalizes the existing coupled fractional differential systems [13] and has wider applications. (ii) The coincidence degree theory of Mawhin is used to investigate the existence of solutions for system (1.1), which enriches the theory of coupled fractional differential systems.

The rest of this paper is organized as follows. Section 2 introduces some basic definitions and lemmas. In Section 3, the key outcome is presented. Finally, an example is given to demonstrate the main result in Section 4.

## 2. Background materials and preliminaries

In order to get the corresponding conclusion, we first recall some basic concepts and theorems. For details, please refer to [6, 7] and references therein.

**Definition 2.1.** Let $Y$, $Z$ be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator. $L$ is said to be the Fredholm operator of index zero provided that:

(i) $\mathrm{Im}\,L$ is a closed subset of $Z$;
(ii) $\dim\ker L = \mathrm{co}\dim \mathrm{Im}\,L < +\infty$.

Let $Y, Z$ be real Banach spaces and $L : \mathrm{dom}\,L \subset Y \to Z$ be a Fredholm operator of index zero. $P : Y \to Y$, $Q : Z \to Z$ are continuous projectors such that

$$\mathrm{Im}\,P = \ker L,\ \ker Q = \mathrm{Im}\,L,\ Y = \ker L \oplus \ker P,\ Z = \mathrm{Im}\,L \oplus \mathrm{Im}\,Q.$$

It follows that $L|_{\mathrm{dom}\,L\cap\ker P} : \mathrm{dom}\,L \cap \ker P \to \mathrm{Im}\,L$ is invertible. We denote the inverse of the operator by $K_P$.

**Definition 2.2.** Let $\Omega$ be an open bounded subset of $Y$ such that $\mathrm{dom}\,L \cap \Omega \neq \emptyset$. Then the operator $N : Y \to Z$ is called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, where $I$ is the identical operator.

**Theorem 2.3.** *Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, y) \in [(\mathrm{dom}\,L\setminus \ker L) \cap \partial\Omega] \times (0, 1)$;
(ii) $Nx \notin \mathrm{Im}\,L$ *for every* $x \in \ker L \cap \partial\Omega$;
(iii) $\deg(QN|_{\ker L}, \ker L \cap \Omega, 0) \neq 0$, *where* $Q : Z \to Z$ *is a projector as above with* $\mathrm{Im}\,L = \ker Q$.

*Then the equation* $Lx = Nx$ *has at least one solution in* $\mathrm{dom}\,L \cap \overline{\Omega}$.

Next, we mainly introduce some definitions and lemmas of the fractional calculus. For details, please refer to [1, 5, 9].

**Definition 2.4.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

**Definition 2.5.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n$ is the smallest integer greater than or equal to $\alpha$, provided that the right side is pointwise defined on $(0, \infty)$.

**Lemma 2.6.** *Let* $n-1 < \alpha \leqslant n$, $u \in C(0, 1) \cap L^1(0, 1)$, $c_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$, *then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}.$$

**Lemma 2.7.**

(1) *Let* $g \in L^1[a, b]$, $p > q > 0$. *Then*

$$I^p_{0+}I^q_{0_+}g(t) = I^{p+q}_{0+}g(t) = I^q_{0+}I^p_{0+}g(t),\ D^q_{0+}I^p_{0+}g(t) = I^{p-q}_{0+}g(t),\ D^p_{0+}I^p_{0+}g(t) = g(t).$$

(2) *Let* $p > q > 0$. *Then*

$$D^q_{0+}t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-q)}t^{p-q},\ D^p_{0+}t^q = 0.$$

(3) *Let* $\alpha > 0$, $m \in \mathbb{N}$ *and* $D = d/dx$. *If the fractional derivatives* $D_{0^+}^{\alpha}u(t)$ *and* $D_{0^+}^{\alpha+m}u(t)$ *exist, then* $D^m D_{0^+}^{\alpha}u(t) = D_{0^+}^{\alpha+m}u(t)$.

**Lemma 2.8.** $D_{0+}^{\alpha}u(t) = 0$ *if and only if* $u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}$ *for some* $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, *where* $n$ *is the smallest integer greater than or equal to* $\alpha$.

**Definition 2.9.** We say that the map $f$: $[0, 1] \times \mathbb{R}^n \to R$ satisfies the carathéodory conditions with respect to $L^1[0, 1]$ if the following conditions are satisfied:

(i) for each $z \in \mathbb{R}^n$, the mapping $t \to f(t, z)$ is Lebesgue measurable;
(ii) for almost every $t \in [0, 1]$, the mapping $z \to f(t, z)$ is continuous on $\mathbb{R}^n$;
(iii) for each $r > 0$, there exists $\varphi_r \in L^1([0, 1], \mathbb{R})$ such that $|f(t, z)| \leqslant \varphi_r(t)$ for a.e. $t \in [0, 1]$ and every $|z| \leqslant r$.

We use the following two classical Banach spaces $C[0, 1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and $Z_1 = L^1[0, 1]$ with the norm $\|x\|_1 = \int_0^1 |x(t)|dt$. Let

$$C^{\mu}[0, 1] = \{x \in C[0, 1] : D_{0^+}^{\mu-i}x \in C[0, 1],\ i = 0, 1, \cdots, N - 1\},$$

where $\mu > 0$, $N = [\mu] + 1$. Obviously, $C^{\mu}[0, 1]$ is a Banach space with the norm

$$\|x\|_{C^\mu} = \|D_{0^+}^{\mu}x\|_\infty + \cdots + \|D_{0^+}^{\mu-(N-1)}x\|_\infty + \|x\|_\infty.$$

**Lemma 2.10** ([12]). $F \subset C^{\mu}[0, 1]$ *is compact if and only if* $F$ *is uniformly bounded and equicontinuous. Here to be uniformly bounded means that there exists* $M > 0$ *such that for every* $u \in F$

$$\|u\|_{C^\mu} = \|D_{0^+}^{\mu}u\|_\infty + \cdots + \|D_{0^+}^{\mu-(N-1)}u\|_\infty + \|u\|_\infty < M$$

*and to be equicontinuous means that for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *and for all* $t_1, t_2 \in [0, 1], |t_1 - t_2| < \delta, u \in F$, *and* $i = 0, 1, \ldots, N - 1$, *the following holds*

$$|u(t_1) - u(t_2)| < \varepsilon, \quad |D_{0^+}^{\mu-i}u(t_1) - D_{0^+}^{\mu-i}u(t_2)| < \varepsilon.$$

Let $Y_1 = C^{\alpha-1}[0, 1]$, $Y_2 = C^{\beta-1}[0, 1]$, $1 < \alpha, \beta < 2$. Thus $Y = Y_1 \times Y_2$ is a Banach space with the norm defined by $\|(x, y)\|_Y = \max\{\|x\|_{Y_1}, \|y\|_{Y_2}\}$, and $Z = Z_1 \times Z_1$ is a Banach space with the norm defined by $\|(x, y)\|_Z = \max\{\|x\|_1, \|y\|_1\}$.

Define $L_1$ to be the linear operator from $\operatorname{dom} L_1 \cap Y_1 \to Z_1$ with

$$L_1x = D_{0^+}^{\alpha}x,\ x \in \operatorname{dom} L_1,$$

where $\operatorname{dom} L_1 = \{x \in Y_1 | D_{0^+}^{\alpha}x \in L^1[0, 1], x(0) = 0, x(1) = \alpha[x]\}$.

Define $L_2$ to be the linear operator from $\operatorname{dom} L_2 \cap Y_2 \to Z_1$ with

$$L_2y = D_{0^+}^{\beta}y,\ y \in \operatorname{dom} L_2,$$

where $\operatorname{dom} L_2 = \{y \in Y_2 | D_{0^+}^{\beta}y \in L^1[0, 1], y(0) = 0, y(1) = \beta[y]\}$.

Define $L$ to be the linear operator from $\operatorname{dom} L \subset Y \to Z$ with

$$L(x, y) = (L_1x, L_2y),\ (x, y) \in \operatorname{dom} L,$$

where $\operatorname{dom} L = \{(x, y) \in Y | x \in \operatorname{dom} L_1, y \in \operatorname{dom} L_2\}$.

Let $N : Y \to Z$ be defined by

$$N(x, y) = (N_1(y), N_2(x)),$$

where $N_1 : Y_2 \to Z_1$ is defined by

$$N_1 y(t) = f(t, y(t), D_{0+}^{\beta-1} y(t)),$$

and $N_2 : Y_1 \to Z_1$ is defined by

$$N_2 x(t) = g(t, x(t), D_{0+}^{\alpha-1} x(t)).$$

Then the coupled system of boundary value problems (1.1) can be written by

$$L(x, y) = N(x, y).$$

**Lemma 2.11.** *The operator* $L$: $\operatorname{dom} L \subset Y \to Z$ *is a Fredholm operator of index zero.*

*Proof.* It is clear that

$$\ker L = \{(k_1 t^{\alpha-1}, k_2 t^{\beta-1}) : \ k_1, k_2 \in \mathbb{R}, \ t \in [0, 1]\}.$$

Now we seek the structure of $\operatorname{Im} L$.

Let $(x, y) \in \operatorname{Im} L$, then there exists $(u, v) \in \operatorname{dom} L$ such that $L(u, v) = (x, y)$, which means

$$u \in Y_1, \ D_{0+}^{\alpha} u = x, \ v \in Y_2, \ D_{0+}^{\beta} v = y.$$

By Lemma 2.6, one has

$$\begin{aligned} I_{0+}^{\alpha} x(t) &= u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \\ I_{0+}^{\beta} y(t) &= v(t) + d_1 t^{\beta-1} + d_2 t^{\beta-2}, \end{aligned} \tag{2.1}$$

where $c_i, d_i \in \mathbb{R}$ $(i = 1, 2)$. Since $u(0) = 0$, one can get $c_2 = 0$. Therefore

$$u(t) = I_{0+}^{\alpha} x(t) - c_1 t^{\alpha-1}.$$

By virtue of $u(1) = \alpha[u]$, we have

$$u(1) = I_{0+}^{\alpha} x(1) - c_1 = \alpha[u] = \int_0^1 u(t) dA(t).$$

Thus

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} x(s) ds - c_1 = \int_0^1 [\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds - c_1 t^{\alpha-1}] dA(t),$$

which means

$$\int_0^1 (1-s)^{\alpha-1} x(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} x(s) ds dA(t) = 0. \tag{2.2}$$

Similarly, by (2.1) we can get

$$\int_0^1 (1-s)^{\beta-1} y(s) ds - \int_0^1 \int_0^t (t-s)^{\beta-1} y(s) ds dB(t) = 0. \tag{2.3}$$

On the other hand, suppose that $x$ satisfies (2.2) and $y$ satisfies (2.3). Choose

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds + k_1 t^{\alpha-1}, \quad v(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds + k_2 t^{\beta-1},$$

where $k_1, \ k_2 \in \mathbb{R}$. It is easy to see $(u, v) \in \operatorname{dom} L$ and $L(u, v) = (x, y)$. Hence

$$\operatorname{Im} L = \{(x, y) \in Z | \ x \text{ satisfies (2.2)}, \ y \text{ satisfies (2.3)}\}.$$

In the following, consider the linear operator $Q : Z \to Z$

$$Q(x, y) = (Q_1x, Q_2y),$$

where the linear operators $Q_1, Q_2 : Z_1 \to Z_1$ are defined by

$$Q_1x(t) = \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1}x(s)ds - \int_0^1 \int_0^t (t-s)^{\alpha-1}x(s)dsdA(t)\right],$$

and

$$Q_2y(t) = \frac{\beta}{1 - \int_0^1 t^\beta dB(t)} \left[\int_0^1 (1-s)^{\beta-1}y(s)ds - \int_0^1 \int_0^t (t-s)^{\beta-1}y(s)dsdB(t)\right].$$

Obviously, $Q$ is a continuous linear projector and $(x, y) \in \operatorname{Im} L$ is equivalent to $Q(x, y) = (0, 0)$. In addition, it is not difficult to prove that $\operatorname{Im} L = \ker Q$ and $Q^2(x, y) = Q(x, y)$.

Take $(x, y) \in Z$ in the form $(x, y) = ((x, y) - Q(x, y)) + Q(x, y)$. Then $(x, y) - Q(x, y) \in \operatorname{Im} L = \ker Q$. Thus, $Z = \operatorname{Im} L + \operatorname{Im} Q$. Let $(x, y) \in \operatorname{Im} L \cap \operatorname{Im} Q$. Then, $Q(x, y) = (x, y)$. By $(x, y) \in \operatorname{Im} L = \ker Q$, we have $Q(x, y) = (0, 0)$. Hence $(x, y) = (0, 0)$. Therefore, we can get $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$.

Notice that $\dim \ker L = \operatorname{co}\dim \operatorname{Im} L = 2 < +\infty$. Then $\operatorname{Ind} L = \dim \ker L - \operatorname{co}\dim \operatorname{Im} L = 0$, which means $L$ is a Fredholm operator of index zero. □

Let operator $P : Y \to Y$ be defined by

$$P(x, y) = (P_1x, P_2y),$$

where $P_1 : Y_1 \to Y_1$ and $P_2 : Y_2 \to Y_2$ are defined by

$$P_1x(t) = \frac{D_{0^+}^{\alpha-1}x(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad P_2y(t) = \frac{D_{0^+}^{\beta-1}y(0)}{\Gamma(\beta)} t^{\beta-1}.$$

In fact $P, P_1$, and $P_2$ are continuous linear projectors and

$$\ker P = \{(x, y) \in Y | D_{0^+}^{\alpha-1}x(0) = 0, D_{0^+}^{\beta-1}y(0) = 0\}.$$

It is clear that $P^2(x, y) = P(x, y)$ and $\operatorname{Im} P = \ker L$. Take $(x, y) \in Y$ in the form $(x, y) = ((x, y) - P(x, y)) + P(x, y)$. Then $(x, y) - P(x, y) \in \ker P$ and $P(x, y) \in \ker L = \operatorname{Im} P$. Thus, $Y = \ker P + \ker L$. For any $(x, y) \in \ker P \cap \ker L$, we have $(x, y) = (k_1t^{\alpha-1}, k_2t^{\beta-1})$. From $(x, y) \in \ker P$, it follows that $D_{0^+}^{\alpha-1}(k_1t^{\alpha-1})|_{t=0} = k_1\Gamma(\alpha) = 0$ and $D_{0^+}^{\beta-1}(k_2t^{\beta-1})|_{t=0} = k_2\Gamma(\beta) = 0$. Thus $k_1 = k_2 = 0$. Hence $Y = \ker P \oplus \ker L$. For every $(x, y) \in Y$,

$$\begin{aligned}
\|P(x, y)\|_Y &= \|(P_1x, P_2y)\|_Y = \max\{\|P_1x\|_{Y_1}, \|P_2Y\|_{Y_2}\} \\
&= \max\{\frac{1}{\Gamma(\alpha)}|D_{0^+}^{\alpha-1}x(0)|\|t^{\alpha-1}\|_{Y_1}, \frac{1}{\Gamma(\beta)}|D_{0^+}^{\beta-1}y(0)|\|t^{\beta-1}\|_{Y_2}\} \\
&= \max\{\frac{1}{\Gamma(\alpha)}|D_{0^+}^{\alpha-1}x(0)|[\|t^{\alpha-1}\|_\infty + \|D_{0^+}^{\alpha-1}(t^{\alpha-1})\|_\infty], \frac{1}{\Gamma(\beta)}|D_{0^+}^{\beta-1}y(0)|[\|t^{\beta-1}\|_\infty \\
&\quad + \|D_{0^+}^{\beta-1}(t^{\beta-1})\|_\infty]\} \\
&= \max\{(1 + \frac{1}{\Gamma(\alpha)})|D_{0^+}^{\alpha-1}x(0)|, (1 + \frac{1}{\Gamma(\beta)})|D_{0^+}^{\beta-1}y(0)|\}.
\end{aligned} \tag{2.4}$$

Define $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by

$$K_P(x, y) = (I_{0^+}^\alpha x, I_{0^+}^\beta y).$$

For $(x,y) \in \operatorname{Im} L$, we have

$$LK_P(x,y) = L(I_{0+}^{\alpha}x, I_{0+}^{\beta}y) = (D_{0+}^{\alpha}I_{0+}^{\alpha}x, D_{0+}^{\beta}I_{0+}^{\beta}y) = (x,y). \tag{2.5}$$

On the other hand, for $(x,y) \in \operatorname{dom} L \cap \ker P$, it follows from Lemma 2.6 that

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2}, \quad I_{0+}^{\beta}D_{0+}^{\beta}y(t) = y(t) + d_1t^{\beta-1} + d_2t^{\beta-2}, \ c_i, d_i \in \mathbb{R}(i=1,2).$$

One has $c_2 = d_2 = 0$, since $(x,y) \in \operatorname{dom} L$. In view of $(x,y) \in \ker P$, we have $D_{0+}^{\alpha-1}x(0) = D_{0+}^{\beta-1}y(0) = 0$. So, $c_1 = d_1 = 0$. Thus, we have

$$K_PL(x,y) = K_P(D_{0+}^{\alpha}x, D_{0+}^{\beta}y) = (I_{0+}^{\alpha}D_{0+}^{\alpha}x, I_{0+}^{\beta}D_{0+}^{\beta}y) = (x,y). \tag{2.6}$$

This together with (2.5) and (2.6) guarantees that $K_P = (L|_{\operatorname{dom} L\cap \ker P})^{-1}$. Moreover

$$\begin{aligned}\|K_P(x,y)\|_Y = \|(I_{0+}^{\alpha}x, I_{0+}^{\beta}y)\|_Y &= \max\{\|I_{0+}^{\alpha}x\|_{Y_1}, \|I_{0+}^{\beta}y\|_{Y_2}\}\\ &= \max\{\|D_{0+}^{\alpha-1}I_{0+}^{\alpha}x\|_\infty + \|I_{0+}^{\alpha}x\|_\infty, \|D_{0+}^{\beta-1}I_{0+}^{\beta}y\|_\infty + \|I_{0+}^{\beta}y\|_\infty\}\\ &\leqslant \max\{(1+\frac{1}{\Gamma(\alpha)})\|x\|_1, (1+\frac{1}{\Gamma(\beta)})\|y\|_1\} \leqslant \triangle\|(x,y)\|_Z,\end{aligned} \tag{2.7}$$

where $\triangle = \max\{1+\frac{1}{\Gamma(\alpha)}, 1+\frac{1}{\Gamma(\beta)}\}$.

By Lemma 2.10 and standard arguments, we can derive the following conclusion.

**Lemma 2.12.** $K_P(I-Q)N : Y \to Y$ *is compact.*

## 3. Main results

First, we list the following notations and assumptions for convenience:

$$\nu_1 = 1 + \frac{1}{\Gamma(\alpha)}, \quad \nu_2 = 1 + \frac{1}{\Gamma(\beta)}, \quad \sigma_1 = \triangle + \nu_1, \quad \sigma_2 = \triangle + \nu_2,$$

where $\triangle$ is described as in (2.7).

($A_1$) There exist functions $a_i, b_i, c_i \in L[0,1](i=1,2)$, such that

$$|f(t,x,y)| \leqslant a_1(t) + b_1(t)|x| + c_1(t)|y|, \ (x,y) \in \mathbb{R}^2 \text{ and } t \in [0,1], \tag{3.1}$$
$$|g(t,x,y)| \leqslant a_2(t) + b_2(t)|x| + c_2(t)|y|, \ (x,y) \in \mathbb{R}^2 \text{ and } t \in [0,1]. \tag{3.2}$$

($A_2$) For $(x,y) \in \operatorname{dom} L$, there exist constants $M_1, M_2 > 0$ such that if either $|D_{0+}^{\alpha-1}x(t)| > M_1$ or $|D_{0+}^{\beta-1}y(t)| > M_2$, then $QN(x,y) \neq (0,0)$ for all $t \in [0,1]$.

($A_3$) There exist constants $D_1, D_2 > 0$ such that for $(k_1,k_2) \in \mathbb{R}^2$, then either

$$k_2Q_1N_1(k_2t^{\beta-1}) > 0, \text{ if } |k_2| > D_2, \tag{3.3}$$
$$k_1Q_2N_2(k_1t^{\alpha-1}) > 0, \text{ if } |k_1| > D_1, \tag{3.4}$$

or

$$k_2Q_1N_1(k_2t^{\beta-1}) < 0, \text{ if } |k_2| > D_2,$$
$$k_1Q_2N_2(k_1t^{\alpha-1}) < 0, \text{ if } |k_1| > D_1.$$

**Theorem 3.1.** *Suppose that* ($A_1$)-($A_3$) *hold. Then* (1.1) *has at least one solution in* $Y$, *provided that*

$$\begin{aligned}\max\{&\sigma_1(\|b_1\|_1 + \|c_1\|_1), \sigma_2(\|b_2\|_1 + \|c_2\|_1), \nu_1(\|b_1\|_1 + \|c_1\|_1) + \triangle(\|b_2\|_1 + \|c_2\|_1),\\ &\nu_2(\|b_2\|_1 + \|c_2\|_1) + \triangle(\|b_1\|_1 + \|c_1\|_1)\} < 1.\end{aligned} \tag{3.5}$$

*Proof.* We divide this proof into four steps.

Step 1: Set

$$\Omega_1 = \{(x,y) \in \operatorname{dom} L \setminus \ker L : \ L(x,y) = \lambda N(x,y), \lambda \in (0,1)\}.$$

Now we prove $\Omega_1$ is bounded.

For any $(x,y) \in \Omega_1$, we have $N(x,y) \in \operatorname{Im} L = \ker Q$. Hence $QN(x,y) = (0,0)$. By $(A_2)$, there exist $t_0, t_1 \in [0,1]$ such that

$$|D_{0^+}^{\alpha-1}x(t_0)| \leqslant M_1, \ \ |D_{0^+}^{\beta-1}y(t_1)| \leqslant M_2.$$

Notice $(I-P)(x,y) \in \operatorname{dom} L \cap \ker P$ and $LP(x,y) = (0,0)$, this together with (2.7) guarantees that

$$\begin{aligned} \|(I-P)(x,y)\|_Y = \|K_P L(I-P)(x,y)\|_Y &\leqslant \triangle \|L(I-P)(x,y)\|_Z \\ &= \triangle \|L(x,y)\|_Z = \triangle\lambda \|N(x,y)\|_Z \leqslant \triangle \max\{\|N_1 y\|_1, \|N_2 x\|_1\}. \end{aligned} \tag{3.6}$$

Since

$$D_{0^+}^{\alpha-1}x(t) = D_{0^+}^{\alpha-1}x(t_0) + \int_{t_0}^{t} D_{0^+}^{\alpha}x(s)ds,$$

we have

$$|D_{0^+}^{\alpha-1}x(0)| \leqslant \|D_{0^+}^{\alpha-1}x\|_\infty \leqslant |D_{0^+}^{\alpha-1}x(t_0)| + \|D_{0^+}^{\alpha}x\|_1 \leqslant M_1 + \|L_1 x\|_1 \leqslant M_1 + \|N_1 y\|_1. \tag{3.7}$$

Similarly, one can get

$$|D_{0^+}^{\beta-1}y(0)| \leqslant M_2 + \|N_2 x\|_1. \tag{3.8}$$

It follows from (2.4), (3.7), and (3.8) that

$$\|P(x,y)\|_Y \leqslant \max\{\nu_1 M_1 + \nu_1\|N_1 y\|_1, \nu_2 M_2 + \nu_2 \|N_2 x\|_1\}. \tag{3.9}$$

Then by (3.6) and (3.9), we can obtain

$$\begin{aligned} \|(x,y)\|_Y &\leqslant \|P(x,y)\|_Y + \|(I-P)(x,y)\|_Y \\ &\leqslant \max\{\nu_1 M_1 + \nu_1\|N_1 y\|_1, \nu_2 M_2 + \nu_2\|N_2 x\|_1\} + \triangle \max\{\|N_1 y\|_1, \|N_2 x\|_1\} \\ &\leqslant \max\{\nu_1 M_1 + \sigma_1\|N_1 y\|_1, \nu_1 M_1 + \nu_1\|N_1 y\|_1 + \triangle\|N_2 x\|_1, \nu_2 M_2 \\ &\qquad + \sigma_2\|N_2 x\|_1, \nu_2 M_2 + \nu_2\|N_2 x\|_1 + \triangle\|N_1 y\|_1\}. \end{aligned} \tag{3.10}$$

Based on (3.10), we need only to discuss the following four cases.

Case 1. $\|(x,y)\|_Y \leqslant \nu_1 M_1 + \sigma_1\|N_1 y\|_1$.

By (3.1), we have

$$\|(x,y)\|_Y \leqslant \nu_1 M_1 + \sigma_1(\|a_1\|_1 + \|b_1\|_1\|y\|_\infty + \|c_1\|_1\|D_{0^+}^{\beta-1}y\|_\infty). \tag{3.11}$$

Since $\|x\|_\infty, \|D_{0^+}^{\alpha-1}x\|_\infty, \|y\|_\infty, \|D_{0^+}^{\beta-1}y\|_\infty \leqslant \|(x,y)\|_Y$ and (3.11), we can get

$$\|y\|_\infty \leqslant \frac{\nu_1 M_1 + \sigma_1(\|a_1\|_1 + \|c_1\|_1\|D_{0^+}^{\beta-1}y\|_\infty)}{1 - \sigma_1\|b_1\|_1}. \tag{3.12}$$

Therefore, by (3.11) and (3.12), we have

$$\|D_{0^+}^{\beta-1}y\|_\infty \leqslant \frac{1}{1 - \sigma_1(\|b_1\|_1 + \|c_1\|_1)}(\nu_1 M_1 + \sigma_1\|a_1\|_1). \tag{3.13}$$

It follows from (3.11), (3.12), and (3.13) that $\Omega_1$ is bounded.

Case 2. $\|(x,y)\|_Y \leqslant \nu_2 M_2 + \sigma_2\|N_2 x\|_1$.

Similar to that of Case 1, we can also obtain that $\Omega_1$ is bounded.

Case 3. $\|(x,y)\|_Y \leqslant \nu_1 M_1 + \nu_1\|N_1 y\|_1 + \triangle\|N_2 x\|_1$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|(x,y)\|_Y \leqslant{}& \nu_1 M_1 + \nu_1(\|a_1\|_1 + \|b_1\|_1\|y\|_\infty + \|c_1\|_1\|D_{0^+}^{\beta-1}y\|_\infty)\\ &+\triangle(\|a_2\|_1 + \|b_2\|_1\|x\|_\infty + \|c_2\|_1\|D_{0^+}^{\alpha-1}x\|_\infty).\end{aligned} \tag{3.14}$$

Then, (3.14) implies that

$$\begin{aligned}\|y\|_\infty \leqslant{}& \frac{1}{1-\nu_1\|b_1\|_1}[\nu_1 M_1 + \nu_1(\|a_1\|_1 + \|c_1\|_1\|D_{0^+}^{\beta-1}y\|_\infty)\\ &+\triangle(\|a_2\|_1 + \|b_2\|_1\|x\|_\infty + \|c_2\|_1\|D_{0^+}^{\alpha-1}x\|_\infty)].\end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we get

$$\begin{aligned}\|x\|_\infty \leqslant{}& \frac{1}{1-\nu_1\|b_1\|_1-\triangle\|b_2\|_1}[\nu_1 M_1 + \nu_1(\|a_1\|_1 + \|c_1\|_1\|D_{0^+}^{\beta-1}y\|_\infty)\\ &+\triangle(\|a_2\|_1 + \|c_2\|_1\|D_{0^+}^{\alpha-1}x\|_\infty)],\\ \|D_{0^+}^{\alpha-1}x\|_\infty \leqslant{}& \frac{1}{1-\nu_1\|b_1\|_1-\triangle(\|b_2\|_1+\|c_2\|_1)}[\nu_1 M_1 + \nu_1(\|a_1\|_1\\ &+\|c_1\|_1\|D_{0^+}^{\beta-1}y\|_\infty) + \triangle\|a_2\|_1],\end{aligned}$$

and

$$\|D_{0^+}^{\beta-1}y\|_\infty \leqslant \frac{1}{1-\nu_1(\|b_1\|_1+\|c_1\|_1)-\triangle(\|b_2\|_1+\|c_2\|_1)}(\nu_1 M_1 + \nu_1\|a_1\|_1 + \triangle\|a_2\|_1),$$

which means that $\Omega_1$ is bounded.

Case 4. $\|(x,y)\|_Y \leqslant \nu_2 M_2 + \nu_2\|N_2 x\|_1 + \triangle\|N_1 y\|_1$.

Similar to that of Case 3, we can also obtain that $\Omega_1$ is bounded.

Step 2: Let

$$\Omega_2 = \{(x,y) \in \ker L | N(x,y) \in \operatorname{Im} L\}.$$

Now we are going to prove that $\Omega_2$ is bounded.

For $(x,y) \in \Omega_2$, we have $(x,y) = (k_1 t^{\alpha-1}, k_2 t^{\beta-1})$. Notice $\operatorname{Im} L = \ker Q$, we get $QN(x,y) = (0,0)$. It follows from $(A_2)$ and Lemma 2.7 that

$$|D_{0^+}^{\alpha-1}x(t)| = |k_1|\Gamma(\alpha) \leqslant M_1, \quad |D_{0^+}^{\beta-1}y(t)| = |k_2|\Gamma(\beta) \leqslant M_2.$$

Hence

$$\|D_{0^+}^{\alpha-1}x\|_\infty \leqslant M_1, \quad \|D_{0^+}^{\beta-1}y\|_\infty \leqslant M_2, \quad \|x\|_\infty \leqslant |k_1| \leqslant \frac{M_1}{\Gamma(\alpha)}, \quad \|y\|_\infty \leqslant |k_2| \leqslant \frac{M_2}{\Gamma(\beta)}.$$

Thus, $\Omega_2$ is bounded.

Step 3: If the first part of $(A_3)$ holds, set

$$\Omega_3 = \{(x,y) \in \ker L | \lambda J(x,y) + (1-\lambda)QN(x,y) = (0,0), \lambda \in [0,1]\},$$

where $J : \ker L \to \operatorname{Im} Q$ is given by

$$J(k_1 t^{\alpha-1}, k_2 t^{\beta-1}) = (k_2, k_1), \quad (k_1,k_2) \in \mathbb{R}^2.$$

Now we are in position to prove that $\Omega_3$ is bounded. For $(x,y) \in \Omega_3$, we know

$$\lambda(k_2,k_1) + (1-\lambda)QN(k_1 t^{\alpha-1}, k_2 t^{\beta-1}) = (0,0). \tag{3.16}$$

There are three cases to be considered.

Case 1. If $\lambda = 0$, then $QN(k_1 t^{\alpha-1}, k_2 t^{\beta-1}) = (0, 0)$. By Step 2, we have

$$|k_1| \leqslant \frac{M_1}{\Gamma(\alpha)}, \quad |k_2| \leqslant \frac{M_2}{\Gamma(\beta)}.$$

Case 2. If $\lambda = 1$, then $k_1 = k_2 = 0$.

Case 3. If $\lambda \in (0, 1)$, we can obtain that $|k_1| \leqslant D_1$, $|k_2| \leqslant D_2$. In fact, if $|k_1| > D_1$ or $|k_2| > D_2$, it follows from (3.3), (3.4), and (3.16) that

$$\lambda k_1^2 = -(1-\lambda) k_1 Q_2 N_2 (k_1 t^{\alpha-1}) < 0, \quad \text{or} \quad \lambda k_2^2 = -(1-\lambda) k_2 Q_1 N_1 (k_2 t^{\beta-1}) < 0,$$

which is a contradiction. Thus $\Omega_3 \subset \{(x, y) \in \ker L \mid (x, y) = (k_1 t^{\alpha-1}, k_2 t^{\beta-1}), |k_1| \leqslant \max\{\frac{M_1}{\Gamma(\alpha)}, D_1\}, |k_2| \leqslant \max\{\frac{M_2}{\Gamma(\beta)}, D_2\}\}$ is bounded.

If the second part of $(A_3)$ holds, then

$$\Omega_3 = \{(x, y) \in \ker L | -\lambda J(x, y) + (1-\lambda) QN(x, y) = (0, 0), \lambda \in [0, 1]\},$$

J is described as in above. By a similar way, we can also obtain that $\Omega_3$ is bounded.

Step 4: In the following, we shall prove that all conditions of Theorem 2.3 are satisfied.

Set $\Omega$ to be a bounded open set of Y such that $\cup_{i=1}^{3} \overline{\Omega_i} \subset \Omega$. By Lemma 2.12, we know $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact. Thus N is L-compact on $\overline{\Omega}$ and the following conditions hold:

(i) $L(x, y) \neq \lambda N(x, y)$ for every $((x, y), \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;

(ii) $N(x, y) \notin \operatorname{Im} L$ for every $(x, y) \in \ker L \cap \partial\Omega$.

Finally, we shall prove that (iii) of Theorem 2.3 is satisfied.

Let $H((x, y), \lambda) = \pm\lambda J(x, y) + (1-\lambda) QN(x, y)$. According to the above argument, we have

$$H((x, y), \lambda) \neq (0, 0) \quad \text{for all} \quad (x, y) \in \ker L \cap \partial\Omega.$$

Thus, by the homotopy property of degree

$$\begin{aligned} \deg(QN|_{\ker L}, \ker L \cap \Omega, 0) &= \deg(H(\cdot, 0), \ker L \cap \Omega, 0) = \deg(H(\cdot, 1), \ker L \cap \Omega, 0) \\ &= \deg(\pm J, \ker L \cap \Omega, 0) \neq 0. \end{aligned}$$

Then by Theorem 2.3, (1.1) has at least one solution in Y. □

## 4. An example

Consider the following problem

$$\begin{cases} D_{0+}^{\frac{3}{2}} x(t) = f(t, y(t), D_{0+}^{\frac{1}{4}} y(t)), \quad t \in (0, 1), \\ D_{0+}^{\frac{5}{4}} y(t) = g(t, x(t), D_{0+}^{\frac{1}{2}} x(t)), \quad t \in (0, 1), \\ x(0) = y(0) = 0, \\ x(1) = \int_0^1 x(t) d(\frac{3}{2} t), \ y(1) = \int_0^1 y(t) d(\frac{5}{4} t), \end{cases} \tag{4.1}$$

where

$$f(t, u, v) = \begin{cases} t^4 + t \sin u, & t \in [0, \frac{1}{2}], \\ t^4 + t^3 v, & t \in (\frac{1}{2}, 1], \end{cases} \qquad g(t, u, v) = \begin{cases} t^2 + t \cos u, & t \in [0, \frac{1}{2}], \\ t^2 + t^4 v, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Equation (4.1) can be regarded as a BVP of the form (1.1), where $\alpha = \frac{3}{2}$, $\beta = \frac{5}{4}$, $A(t) = \frac{3}{2} t$, $B(t) = \frac{5}{4} t$.

Choose $a_1(t) = t^4 + t$, $b_1(t) = 0$, $c_1(t) = t^3$, $a_2(t) = t^2 + t$, $b_2(t) = 0$, $c_2(t) = t^4$, and $\varphi_r(t) = t^4 + t + t^3 r$, $\psi_r(t) = t^2 + t + t^4 r$. It is not difficult to see that $f$ and $g$ satisfy Carathéodory conditions. We can easily get

$$\int_0^1 t^{\alpha-1} dA(t) = 1, \quad \int_0^1 t^{\beta-1} dB(t) = 1, \quad \int_0^1 t^{\alpha} dA(t) = \frac{3}{5}, \quad \int_0^1 t^{\beta} dB(t) = \frac{5}{9},$$

$$\begin{aligned} Q_1N_1y &= \frac{15}{4}\left[\int_0^1 (1-s)^{\frac{1}{2}} f(s, y(s), D_{0^+}^{\frac{1}{4}} y(s)) ds - \frac{3}{2}\int_0^1 \int_0^t (t-s)^{\frac{1}{2}} f(s, y(s), D_{0^+}^{\frac{1}{4}} y(s)) ds dt\right] \\ &= \frac{15}{4}\int_0^1 (1-s)^{\frac{1}{2}} s f(s, y(s), D_{0^+}^{\frac{1}{4}} y(s)) ds, \end{aligned}$$

$$\begin{aligned} Q_2N_2x &= \frac{45}{16}\left[\int_0^1 (1-s)^{\frac{1}{4}} g(s, x(s), D_{0^+}^{\frac{1}{2}} x(s)) ds - \frac{5}{4}\int_0^1 \int_0^t (t-s)^{\frac{1}{4}} g(s, x(s), D_{0^+}^{\frac{1}{2}} x(s)) ds dt\right] \\ &= \frac{45}{16}\int_0^1 (1-s)^{\frac{1}{4}} s g(s, x(s), D_{0^+}^{\frac{1}{2}} x(s)) ds. \end{aligned}$$

Choose $M_1 = 32$, $M_2 = 16$, $D_1 = 10$, $D_2 = 18$. It is clear that $(A_1)$-$(A_3)$ hold. By simple computations, we know (3.5) holds. Therefore, all the assumptions of Theorem 3.1 hold, which means that (4.1) has at least one solution in $Y$.

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