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Uniform convexity in $\ell_{p(\cdot)}$

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Abstract

In this work, we investigate the variable exponent sequence space $\ell_{p(\cdot)}$. In particular, we prove a geometric property similar to uniform convexity without the assumption $\limsup_{n\to\infty} p(n) < \infty$. This property allows us to prove the analogue to Kirk's fixed point theorem in the modular vector space $\ell_{p(\cdot)}$ under Nakano's formulation.

Keywords: Fixed point, modular vector spaces, nonexpansive mapping, uniformly convex, variable exponent spaces.
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1. Introduction

The origin of function modulars defined in vector spaces goes back to the 1931 early work of Orlicz [15]. In this work, he introduced the following vector space:

$$X = \left\{ (x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} |\lambda\, x_n|^{p(n)} < \infty \text{ for some } \lambda > 0 \right\},$$

where $\{p(n)\} \subset [1, \infty)$. For interested readers about about the topology and the geometry of X, we recommend the references [8, 13, 18, 19]. Note that the vector space X may be seen as a predecessor to the theory of variable exponent spaces [3]. Recently, these spaces have enjoyed a major development. A systematic study of their vector topological properties was initiated in 1991 by Kováčik and Rákosník [9]. But one of the driving forces for the rapid development of the theory of variable exponent spaces has been the model of electrorheological fluids introduced by Rajagopal and Ružička [16, 17]. These fluids are an example of smart materials, whose development is one of the major tools in space engineering.

The general definition of a modular in an abstract vector space was introduced by Nakano [12, 14]. In this work, we focus on establishing a geometric property similar to modular uniform convexity in the vector space X described above. This investigation allows us to discover new unknown properties.

For the readers interested into the metric fixed point theory, we recommend the book by Khamsi and Kirk [4] and the recent book by Khamsi and Kozlowski [5].

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2. Notations and Definitions

First recall the definition of the variable exponent sequence space $\ell_{p(\cdot)}$.

Definition 2.1 ([15]). For a function $p : \mathbb{N} \to [1, \infty)$, define the vector space

$$\ell_{p(\cdot)} = \left\{(x_n) \in \mathbb{R}^{\mathbb{N}};\ \sum_{n=0}^{\infty} \frac{1}{p(n)} |\lambda\, x_n|^{p(n)} < \infty \ \text{ for some } \lambda > 0\right\}.$$

Inspired by the vector space $\ell_{p(\cdot)}$, Nakano [12, 14, 13] came up with the concept of the modular vector structure. The following proposition summarizes Nakano's main ideas.

Proposition 2.2 ([8, 12, 18]). *Consider the function* $\rho : \ell_{p(\cdot)} \to [0, \infty]$ *defined by*

$$\rho(x) = \rho((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)}.$$

Then ρ *satisfies the following properties:*

(1) $\rho(x) = 0$ *if and only if* $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$, *if* $|\alpha| = 1$,
(3) $\rho(\alpha x + (1-\alpha) y) \leqslant \alpha \rho(x) + (1-\alpha)\rho(y)$, *for any* $\alpha \in [0, 1]$,

for any $x, y \in X$. *The function* ρ *is called a convex modular.*

Next, we introduce a kind of modular topology that is similar to the classical metric topology.

Definition 2.3 ([6]).

(a) We say that a sequence $\{x_n\} \subset \ell_{p(\cdot)}$ is ρ-convergent to $x \in \ell_{p(\cdot)}$ if and only if $\rho(x_n - x) \to 0$. Note that the ρ-limit is unique if it exists.
(b) A sequence $\{x_n\} \subset \ell_{p(\cdot)}$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
(c) A set $C \subset \ell_{p(\cdot)}$ is called ρ-closed if for any sequence $\{x_n\} \subset C$ which ρ-converges to x implies that $x \in C$.
(d) A set $C \subset \ell_{p(\cdot)}$ is called ρ-bounded if $\delta_\rho(C) = \sup\{\rho(x - y); x, y \in C\} < \infty$.

Note that ρ satisfies the Fatou property, i.e., $\rho(x - y) \leqslant \liminf_{n\to\infty} \rho(x - y_n)$ holds whenever $\{y_n\}$ ρ-converges to y, for any x, y, y_n in $\ell_{p(\cdot)}$. The Fatou property is very useful. For example, Fatou property holds if and only if the ρ-balls are ρ-closed. Recall that the subset $B_\rho(x, r) = \{y \in \ell_{p(\cdot)};\ \rho(x - y) \leqslant r\}$, with $x \in \ell_{p(\cdot)}$ and $r \geqslant 0$, is known as a ρ-ball.

Recall that ρ is said to satisfy the Δ_2-condition if there exists $K \geqslant 0$ such that

$$\rho(2x) \leqslant K\, \rho(x)$$

for any $x \in \ell_{p(\cdot)}$ [5]. This property is very important in the study of modular functionals. For more on the Δ_2-condition and its variants may be found in [5, 10, 11]. In the case of $\ell_{p(\cdot)}$, it is easy to see that ρ satisfies the Δ_2-condition if and only if $\limsup_{n\to\infty} p(n) < \infty$. Recall that the Minkowski functional associated to the modular unit ball is known as the Luxemburg norm defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0; \rho\left(\frac{1}{\lambda} x\right) \leqslant 1\right\}.$$

Recall that $(\ell_{p(\cdot)}, \|.\|_\rho)$ is a Banach space. Sundaresan [18] proved that $(\ell_{p(\cdot)}, \|.\|_\rho)$ is reflexive if and only if $1 < \liminf_{n\to\infty} p(n) \leqslant \limsup_{n\to\infty} p(n) < \infty$. In this case, $(\ell_{p(\cdot)}, \|.\|_\rho)$ is uniformly convex which implies in fact that $(\ell_{p(\cdot)}, \|.\|_\rho)$ is superreflexive [1]. In the next section, we will introduce a new modular uniform convexity satisfied by $\ell_{p(\cdot)}$ even when $\limsup_{n\to\infty} p(n) < \infty$ is not satisfied.

3. Modular Uniform Convexity

Modular uniform convexity was introduced in general vector spaces by Nakano [14]. Its study in Orlicz function spaces was carried in [3, 11].

Definition 3.1 ([3, 11])**.** We define the following *uniform convexity type* properties of the modular ρ:

(a) [14] Let $r > 0$ and $\varepsilon > 0$. Define

$$D_1(r,\varepsilon) = \left\{(x,y);\ x,y \in \ell_{p(\cdot)}, \rho(x) \leqslant r, \rho(y) \leqslant r, \rho(x-y) \geqslant \varepsilon r\right\}.$$

If $D_1(r,\varepsilon) \neq \emptyset$, let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\left(\frac{x+y}{2}\right);\ (x,y) \in D_1(r,\varepsilon)\right\}.$$

If $D_1(r,\varepsilon) = \emptyset$, we set $\delta_1(r,\varepsilon) = 1$. We say that ρ satisfies the uniform convexity (UC) if for every $r > 0$ and $\varepsilon > 0$, we have $\delta_1(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(b) [5] We say that ρ satisfies (UUC) if for every $s \geqslant 0$ and $\varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$ depending on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0 \text{ for } r > s.$$

(c) [5] Let $r > 0$ and $\varepsilon > 0$. Define

$$D_2(r,\varepsilon) = \left\{(x,y);\ x,y \in \ell_{p(\cdot)}, \rho(x) \leqslant r, \rho(y) \leqslant r, \rho\left(\frac{x-y}{2}\right) \geqslant \varepsilon r\right\}.$$

If $D_2(r,\varepsilon) \neq \emptyset$, let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\left(\frac{x+y}{2}\right);\ (x,y) \in D_2(r,\varepsilon)\right\}.$$

If $D_2(r,\varepsilon) = \emptyset$, we set $\delta_2(r,\varepsilon) = 1$. We say that ρ satisfies $(UC2)$ if for every $r > 0$ and $\varepsilon > 0$, we have $\delta_2(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_2(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(d) [5] We say that ρ satisfies $(UUC2)$ if for every $s \geqslant 0$ and $\varepsilon > 0$, there exists $\eta_2(s,\varepsilon) > 0$ depending on s and ε such that

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \text{ for } r > s.$$

(e) [14] We say that ρ is strictly convex, (SC), if for every $x,y \in \ell_{p(\cdot)}$ such that $\rho(x) = \rho(y)$ and

$$\rho\left(\frac{x+y}{2}\right) = \frac{\rho(x)+\rho(y)}{2},$$

we have $x = y$.

The property (UC) was introduced by Nakano [14]. In all the subsequent research done on $\ell_{p(\cdot)}$, the authors considered (UC). For example, Sundaresan [18] proved that in $\ell_{p(\cdot)}$, ρ satisfies (UC) if and only if $1 < \inf_{n\in\mathbb{N}} p(n) \leqslant \sup_{n\in\mathbb{N}} p(n) < \infty$. Note that (UC) and $(UC2)$ are equivalent if ρ satisfies the Δ_2-condition [5]. In this case, we must have $\sup_{n\in\mathbb{N}} p(n) < \infty$.

The following technical result is very useful.

Lemma 3.2. *The following inequalities are valid:*

(i) [2] *If* $p \geqslant 2$, *then we have*

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leqslant \frac{1}{2}\Big(|a|^p + |b|^p\Big)$$

for any $a, b \in \mathbb{R}$.

(ii) [18] *If* $1 < p \leqslant 2$, *then we have*

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{a-b}{|a|+|b|}\right|^{2-p}\left|\frac{a-b}{2}\right|^p \leqslant \frac{1}{2}\Big(|a|^p + |b|^p\Big)$$

for any $a, b \in \mathbb{R}$ *such that* $|a|+|b| \neq 0$.

Before we state the main result of this work, we will need the following notation:

$$\rho_K(x) = \rho_K((x_n)) = \sum_{n\in K} \frac{1}{p(n)}\, |x_n|^{p(n)}$$

for any $K \subset \mathbb{N}$ and any $x \in \ell_{p(\cdot)}$. If $K = \emptyset$, we set $\rho_K(x) = 0$.

Theorem 3.3. *Consider the vector space* $\ell_{p(\cdot)}$. *If* $\inf_{n\in\mathbb{N}}\ p(n) > 1$, *then the modular* ρ *is* (UUC2).

Proof. Assume $A = \inf_{n\in\mathbb{N}}\ p(n) > 1$. Let $r > 0$ and $\varepsilon > 0$. Let $x, y \in \ell_{p(\cdot)}$ such that

$$\rho(x) \leqslant r,\ \ \rho(y) \leqslant r\ \ \text{and}\ \ \rho\left(\frac{x-y}{2}\right) \geqslant r\ \varepsilon.$$

Since ρ is convex, then we have

$$r\ \varepsilon \leqslant \rho\left(\frac{x-y}{2}\right) \leqslant \frac{\rho(x)+\rho(y)}{2} \leqslant r,$$

which implies $\varepsilon \leqslant 1$. Next, set $I = \{n \in \mathbb{N}; p(n) \geqslant 2\}$ and $J = \{n \in \mathbb{N}; p(n) < 2\} = \mathbb{N} \setminus I$. Note that we have $\rho(z) = \rho_I(z) + \rho_J(z)$, for any $z \in \ell_{p(\cdot)}$. From our assumptions, we have either $\rho_I((x-y)/2) \geqslant r\ \varepsilon/2$ or $\rho_J((x-y)/2) \geqslant r\ \varepsilon/2$.

Assume first $\rho_I((x-y)/2) \geqslant r\ \varepsilon/2$. Using Lemma 3.2, we conclude that

$$\rho_I\left(\frac{x+y}{2}\right) + \rho_I\left(\frac{x-y}{2}\right) \leqslant \frac{\rho_I(x)+\rho_I(y)}{2},$$

which implies

$$\rho_I\left(\frac{x+y}{2}\right) \leqslant \frac{\rho_I(x)+\rho_I(y)}{2} - \frac{r\ \varepsilon}{2}.$$

Since

$$\rho_J\left(\frac{x+y}{2}\right) \leqslant \frac{\rho_J(x)+\rho_J(y)}{2},$$

we get

$$\rho\left(\frac{x+y}{2}\right) \leqslant \frac{\rho(x)+\rho(y)}{2} - \frac{r\ \varepsilon}{2} \leqslant r\left(1 - \frac{\varepsilon}{2}\right).$$

For the second case, assume $\rho_J((x-y)/2) \geqslant r\ \varepsilon/2$. Set $C = \varepsilon/4$,

$$J_1 = \Big\{n \in J;\ |x_n - y_n| \leqslant C(|x_n| + |y_n|)\Big\}\ \ \text{and}\ \ J_2 = J \setminus J_1.$$

We have

$$\rho_{J_1}\left(\frac{x-y}{2}\right) \leqslant \sum_{n\in J_1} \frac{C^{p(n)}}{p(n)}\left|\frac{|x_n|+|y_n|}{2}\right|^{p(n)} \leqslant \frac{C}{2}\sum_{n\in J_1}\frac{|x_n|^{p(n)}+|y_n|^{p(n)}}{p(n)},$$

because $C \leqslant 1$ and the power function is convex. Hence

$$\rho_{J_1}\left(\frac{x-y}{2}\right) \leqslant \frac{C}{2}\Big(\rho_{J_1}(x) + \rho_{J_1}(y)\Big) \leqslant \frac{C}{2}\Big(\rho(x)+\rho(y)\Big) \leqslant C\ r.$$

Since $\rho_J((x-y)/2) \geqslant r\,\varepsilon/2$, we get

$$\rho_{J_2}\left(\frac{x-y}{2}\right) = \rho_J\left(\frac{x-y}{2}\right) - \rho_{J_1}\left(\frac{x-y}{2}\right) \geqslant \frac{r\,\varepsilon}{2} - C\,r.$$

For any $n \in J_2$, we have

$$A-1 \leqslant p(n)(p(n)-1) \;\text{ and }\; C \leqslant C^{2-p(n)} \leqslant \left|\frac{x_n-y_n}{|x_n|+|y_n|}\right|^{2-p(n)},$$

which implies by Lemma 3.2 that

$$\left|\frac{x_n+y_n}{2}\right|^{p(n)} + \frac{(A-1)C}{2}\left|\frac{x_n-y_n}{2}\right|^{p(n)} \leqslant \frac{1}{2}\Big(|x_n|^{p(n)} + |y_n|^{p(n)}\Big).$$

Hence

$$\rho_{J_2}\left(\frac{x+y}{2}\right) + \frac{(A-1)C}{2}\,\rho_{J_2}\left(\frac{x-y}{2}\right) \leqslant \frac{\rho_{J_2}(x)+\rho_{J_2}(y)}{2},$$

which implies

$$\rho_{J_2}\left(\frac{x+y}{2}\right) \leqslant \frac{\rho_{J_2}(x)+\rho_{J_2}(y)}{2} - r\,\frac{(A-1)\varepsilon^2}{8},$$

since $C = \varepsilon/4$. Therefore, we have

$$\rho\left(\frac{x+y}{2}\right) \leqslant r - r\,\frac{(A-1)\varepsilon^2}{8} = r\left(1-\frac{(A-1)\varepsilon^2}{8}\right).$$

Using the definition of $\delta_2(r,\varepsilon)$, we conclude that

$$\delta_2(r,\varepsilon) \geqslant \min\left(\frac{\varepsilon}{2}, (A-1)\frac{\varepsilon^2}{8}\right) > 0.$$

Therefore, ρ is $(UC2)$. Moreover, if we set $\eta_2(r,\varepsilon) = \min\big(\varepsilon/2, (A-1)\varepsilon^2/8\big)$, we conclude that ρ is in fact $(UUC2)$. □

Remark 3.4. Note that in our proof above, we showed that $\eta_2(r,\varepsilon)$ is in fact a function of ε only. We will make use of this fact throughout.

Using this form of uniform convexity, we can prove some interesting modular geometric properties not clear to hold in the absence of the Δ_2-condition. these properties were proved recently in an unpublished work. For the sake of completeness, we include their proofs.

Proposition 3.5. *Consider the space* $\ell_{p(\cdot)}$. *Assume* $\inf_{n\in\mathbb{N}} p(n) > 1$.

(i) *Let* C *be a nonempty* ρ*-closed convex subset of* $\ell_{p(\cdot)}$. *Let* $x \in \ell_{p(\cdot)}$ *be such that*

$$d_\rho(x,C) = \inf\{\rho(x-y);\; y \in C\} < \infty.$$

Then there exists a unique $c \in C$ *such that* $d_\rho(x,C) = \rho(x-c)$.

(ii) $\ell_{p(\cdot)}$ *satisfies the property (R), i.e., for any decreasing sequence* $\{C_n\}_{n\geqslant 1}$ *of* ρ*-closed convex nonempty subsets of* $\ell_{p(\cdot)}$ *such that* $\sup_{n\geqslant 1} d_\rho(x,C_n) < \infty$, *for some* $x \in \ell_{p(\cdot)}$, *then we have* $\bigcap_{n\geqslant 1} C_n$ *is nonempty.*

Proof. In order to prove (i), we may assume that $x \notin C$ since C is ρ-closed. Therefore, we have $d_\rho(x,C) > 0$. Set $R = d_\rho(x,C)$. Hence for any $n \geqslant 1$, there exists $y_n \in C$ such that $\rho(x-y_n) < R(1+1/n)$. We claim that $\{y_n/2\}$ is ρ-Cauchy. Assume otherwise that $\{y_n/2\}$ is not ρ-Cauchy. Then there exists a subsequence

$\{y_{\varphi(n)}\}$ and $\varepsilon_0 > 0$ such that $\rho\Big((y_{\varphi(n)} - y_{\varphi(m)})/2\Big) \geqslant \varepsilon_0$, for any $n > m \geqslant 1$. Moreover, we have $\delta_2(R(1+1/n), 2\varepsilon_0/R) \geqslant \eta_2(\varepsilon_0/2R) > 0$, for any $n \geqslant 1$. Since $\max\Big(\rho(x - y_{\varphi(n)}), \rho(x - y_{\varphi(m)})\Big) \leqslant R(1+1/\varphi(m))$ and

$$\rho\left(\frac{y_{\varphi(n)} - y_{\varphi(m)}}{2}\right) \geqslant \varepsilon_0 \geqslant R\left(1 + \frac{1}{\varphi(m)}\right)\frac{\varepsilon_0}{2R}$$

for any $n > m \geqslant 1$, we conclude that

$$\rho\left(x - \frac{y_{\varphi(n)} + y_{\varphi(m)}}{2}\right) \leqslant R\left(1 + \frac{1}{\varphi(m)}\right)\left(1 - \eta_2\left(\frac{\varepsilon_0}{2R}\right)\right).$$

Hence

$$R = d_\rho(x, C) \leqslant R\left(1 + \frac{1}{\varphi(m)}\right)\left(1 - \eta_2\left(\frac{\varepsilon_0}{2R}\right)\right)$$

for any $m \geqslant 1$. If we let $m \to \infty$, we get

$$R \leqslant R\left(1 - \eta_2\left(\frac{\varepsilon_0}{2R}\right)\right) < R,$$

which is a contradiction since $R > 0$. Therefore, $\{y_n/2\}$ is ρ-Cauchy. Since $\ell_{p(\cdot)}$ is ρ-complete, then $\{y_n/2\}$ ρ-converges to some y. We claim that $2y \in C$. Indeed, for any $m \geqslant 1$, the sequence $\{(y_n + y_m)/2\}$ ρ-converges to $y + y_m/2$. Since C is ρ-closed and convex, we get $y + y_m/2 \in C$. Finally the sequence $\{y + y_m/2\}$ ρ-converges to $2y$, which implies $2y \in C$. Set $c = 2y$. Since ρ satisfies the Fatou property, we have

$$\begin{aligned} d_\rho(x, C) &\leqslant \rho(x - c) \\ &\leqslant \liminf_{m\to\infty} \rho\Big(x - (y + y_m/2)\Big) \\ &\leqslant \liminf_{m\to\infty}\ \liminf_{n\to\infty} \rho\Big(x - (y_n + y_m/2)\Big) \\ &\leqslant \liminf_{m\to\infty}\ \liminf_{n\to\infty} \Big(\rho(x - y_n) + \rho(x - y_m)\Big)/2 \\ &= d_\rho(x, C). \end{aligned}$$

Hence $\rho(x - c) = d_\rho(x, C)$. The uniqueness of the point c follows from the fact that ρ is (SC) since it is $(UUC2)$.

For the proof of (ii), we assume that $x \notin C_{n_0}$ for some $n_0 \geqslant 1$. In fact, the sequence $\{d_\rho(x, C_n)\}$ is increasing and bounded. Set $\lim_{n\to\infty} d_\rho(x, C_n) = R$. We may assume $R > 0$. Otherwise $x \in C_n$, for any $n \geqslant 1$. From (i), there exists a unique $y_n \in C_n$ such that $d_\rho(x, C_n) = \rho(x - y_n)$, for any $n \geqslant 1$. A similar proof will show that $\{y_n/2\}$ ρ-converges to some $y \in \ell_{p(\cdot)}$. Since $\{C_n\}$ are decreasing, convex and ρ-closed, we conclude that $2y \in \bigcap_{n\geqslant 1} C_n$. □

Remark 3.6. It is natural to wonder whether the property (R) extends to any family of decreasing subsets. Indeed, assume $\inf_{n\in\mathbb{N}}\ p(n) > 1$. Let C be a ρ-closed nonempty convex subset of $\ell_{p(\cdot)}$ which is ρ-bounded. Let $\{C_i\}_{i\in I}$ be a family of ρ-closed nonempty convex subsets of C such that $\bigcap_{i\in F} C_i \neq \emptyset$, for any finite subset F of I. Then $\bigcap_{i\in I} C_i \neq \emptyset$. In order to see this, let $x \in C$. Then $\sup_{i\in I}\ d_\rho(x, C_i) \leqslant \delta_\rho(C) < \infty$ holds. For any subset $F \subset I$, set $d_F = d_\rho(x, \bigcap_{i\in F} C_i)$. Note that if $F_1 \subset F_2 \subset I$ are finite subsets, then $d_{F_1} \leqslant d_{F_2}$. Set

$$d_I = \sup\left\{d_\rho\Big(x, \bigcap_{i\in J} C_i\Big),\ J \subset I \text{ such that } \bigcap_{i\in J} C_i \neq \emptyset\right\}.$$

For any $n \geqslant 1$, there exists a subset $F_n \subset I$ such that $d_I - 1/n < d_{F_n} \leqslant d_I$. Set $F_n^* = F_1 \cup \cdots \cup F_n$, for $n \geqslant 1$. Then $\left\{\bigcap_{i\in F_n^*} C_i\right\}$ is a decreasing sequence of nonempty ρ-closed convex subsets of $\ell_{p(\cdot)}$. The property (R) implies $\bigcap_{i\in J} C_i \neq \emptyset$, where $J = \bigcup_{n\geqslant 1} F_n^* = \bigcup_{n\geqslant 1} F_n$. Set $K = \bigcap_{i\in J} C_i$. Note that $d_\rho(x, K) = d_I$ because

$d_I - 1/n < d_{F_n} \leqslant d_\rho(x, K) \leqslant d_I$, for any $n \geqslant 1$. Proposition 3.5 implies the existence of a unique $y \in K$ such that $\rho(x-y) = d_\rho(x, K) = d_I$. Let $i_0 \in I$, then

$$K \cap C_{i_0} = \bigcap_{i \in J \cup \{i_0\}} C_i \neq \emptyset,$$

because of the same argument using the property (R). Hence $d_\rho(x, K) \leqslant d_\rho(x, K \cap C_{i_0}) \leqslant d_I$. Hence $d_\rho(x, K \cap C_{i_0}) = d_\rho(x, K) = d_I$ which implies $y \in K \cap C_{i_0}$. Therefore, we have $y \in \bigcap_{i \in I} C_i$ which proves our claim.

If the property (R) is satisfied by the family of convex and closed (for the Luxemburg norm) subsets, we will deduce that $\ell_{p(\cdot)}$ is reflexive. The work of Sundaresan [18] will imply in this case that $1 < \inf_{n \in \mathbb{N}} p(n) \leqslant \sup_{n \in \mathbb{N}} p(n) < \infty$.

4. Application

In this section, we will show that under the assumption $\inf_{n \in \mathbb{N}} p(n) > 1$, the space $\ell_{p(\cdot)}$ enjoys a nice modular geometric property which will allow us to prove the analogue to Kirk's fixed point theorem [7].

Definition 4.1. $\ell_{p(\cdot)}$ is said to have the ρ-normal structure property if for any nonempty ρ-closed convex ρ-bounded subset C of $\ell_{p(\cdot)}$ not reduced to one point, there exists $x \in C$ such that

$$\sup_{y \in C} \rho(x-y) < \delta_\rho(C).$$

Theorem 4.2. *Assume* $\inf_{n \in \mathbb{N}} p(n) > 1$. *Then* $\ell_{p(\cdot)}$ *has the* ρ*-normal structure property.*

Proof. Since $\inf_{n \in \mathbb{N}} p(n) > 1$, Theorem 3.3 implies that ρ is (UUC2). Let C be a ρ-closed convex ρ-bounded subset of $\ell_{p(\cdot)}$ not reduced to one point. Hence $\delta_\rho(C) > 0$. Set $R = \delta_\rho(C)$. Let $x, y \in C$ such that $x \neq y$. Hence $\rho((x-y)/2) = \varepsilon > 0$. For any $c \in C$, we have $\rho(x-c) \leqslant R$ and $\rho(y-c) \leqslant R$. Hence

$$\rho\left(\frac{x+y}{2} - c\right) = \rho\left(\frac{(x-c)+(y-c)}{2}\right) \leqslant R\, \left(1 - \delta_2\left(R, \frac{\varepsilon}{R}\right)\right)$$

for any $c \in C$. Hence

$$\sup_{c \in C} \rho\left(\frac{x+y}{2} - c\right) \leqslant R\, \left(1 - \delta_2\left(R, \frac{\varepsilon}{R}\right)\right) < R = \delta_\rho(C).$$

This completes the proof of Theorem 4.2 since C is convex. □

Before we state the modular analogue to Kirk's fixed point theorem in $\ell_{p(\cdot)}$, we will need the following definition.

Definition 4.3. Let $C \subset \ell_{p(\cdot)}$ be nonempty. A mapping $T : C \to C$ is called ρ-Lipschitzian if there exists a constant $K \geqslant 0$ such that

$$\rho(T(x) - T(y)) \leqslant K\, \rho(x-y), \quad \text{for any } x, y \in C.$$

If $K = 1$, T is called ρ-nonexpansive. A point $x \in C$ is called a fixed point of T if $T(x) = x$.

Theorem 4.4. *Assume* $\inf_{n \in \mathbb{N}} p(n) > 1$. *Let* C *be a nonempty* ρ*-closed convex* ρ*-bounded subset of* $\ell_{p(\cdot)}$. *Let* $T : C \to C$ *be a* ρ*-nonexpansive mapping. Then* T *has a fixed point.*

Proof. Let C be a nonempty ρ-closed convex ρ-bounded subset of $\ell_{p(\cdot)}$. Let $T : C \to C$ be a ρ-nonexpansive mapping. Without loss of generality, we assume that C is not reduced to one point. Consider the family

$$\mathcal{F} = \{K \subset C;\ K \text{ is nonempty } \rho\text{-closed convex and } T(K) \subset K\}.$$

The family $\mathcal{F}$ is not empty since $C \in \mathcal{F}$. Since $\inf_{n\in\mathbb{N}} p(n) > 1$, ρ is (UUC2). Using Remark 3.6 combined with Zorn's lemma, we conclude that $\mathcal{F}$ has a minimal element K_0. We claim that K_0 is reduced to one point. Assume not, i.e., K_0 has more than one point. Set $co(T(K_0))$ to be the intersection of all ρ-closed convex subset of C containing $T(K_0)$. Hence $co(T(K_0)) \subset K_0$ since $T(K_0) \subset K_0$. So we have $T\Big(co(T(K_0))\Big) \subset T(K_0) \subset co(T(K_0))$. The minimality of K_0 implies $K_0 = co(T(K_0))$. Next, we use Theorem 4.2 to secure the existence of $x_0 \in K_0$ such that

$$r_0 = \sup_{y\in K_0} \rho(x_0 - y) < \delta_\rho(K_0).$$

Define the subset $K = \{x \in K_0;\ \sup_{y\in K_0} \rho(x-y) \leqslant r_0\}$. K is not empty since $x_0 \in K$. Note that we have $K = \bigcap_{y\in K_0} B_\rho(y, r_0) \cap K_0$, where $B_\rho(y, r_0) = \{z \in \ell_{p(\cdot)};\ \rho(y-z) \leqslant r_0\}$. Since ρ satisfies the Fatou property and is convex, $B_\rho(y, r_0)$ is ρ-closed and convex. Hence K is ρ-closed and convex subset of K_0. let us show that $T(K) \subset K$. Let $x \in K$, then $T(x) \in \bigcap_{y\in K_0} B_\rho(T(y), r_0) \cap K_0$ since T is ρ-nonexpansive. Hence $T(K_0) \subset B_\rho(T(x), r_0)$ which implies $K_0 = co(T(K_0)) \subset B_\rho(T(x), r_0)$, i.e., $T(x) \in \bigcap_{y\in K_0} B_\rho(y, r_0) \cap K_0$. Therefore, $T(K) \subset K$ holds. The minimality of K_0 implies $K = K_0$, i.e., for any $x \in K_0$, we have $\sup_{y\in K_0} \rho(x-y) \leqslant r_0$. This clearly will imply $\rho(x-y) \leqslant r_0$, for any $x, y \in K_0$. Hence $\delta_\rho(K_0) \leqslant r_0$. This is our sought contradiction. Therefore, K_0 is reduced to one point. Since $T(K_0) \subset K_0$, we conclude that T has a fixed point in C. □

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References

[1] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, North-Holland, Amsterdam, (1985). 2
[2] J. A. Clarkson, *Uniformly Convex Spaces*, Trans. Amer. Math. Soc., **40** (1936), 396–414. 3.2
[3] L. Diening, P. Harjulehto, P. Hästö, M. Ružiĉka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Berlin, (2011). 1, 3, 3.1
[4] M. A. Khamsi, W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Wiley-Interscience, New York, (2001). 1
[5] M. A. Khamsi, W. M. Kozlowski, *Fixed Point Theory in Modular Function Spaces*, Birkhauser, New York, (2015). 1, 2, 3.1, 3
[6] M. A. Khamsi, W. K. Kozlowski, S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal., **14** (1990), 935–953. 2.3
[7] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006. 4
[8] V. Klee, *Summability in* $\ell(p_{11}, p_{21}, \cdots)$ *Spaces*, Studia Math., **25** (1965), 277–280. 1, 2.2
[9] O. Kováčik, J. Rákosník, *On spaces* $L^{p(x)}$ *and* $W^{1,p(x)}$, Czechoslovak Math. J., **41** (1991), 592–618. 1
[10] W. M. Kozlowski, *Modular Function Spaces*, Marcel Dekker, New York, (1988). 2
[11] J. Musielak, *Orlicz spaces and modular spaces*, Springer-Verlag, Berlin, (1983). 2, 3, 3.1
[12] H. Nakano, *Modulared Semi-ordered Linear Spaces*, Maruzen Co., Tokyo, (1950). 1, 2, 2.2
[13] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad., **27** (1951), 508–512. 1, 2
[14] H. Nakano, *Topology of linear topological spaces*, Maruzen Co. Ltd., Tokyo, 1951. 1, 2, 3, 3.1, 3
[15] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math., **3** (1931), 200–211. 1, 2.1
[16] K. Rajagopal, M. Ružička, *On the modeling of electrorheological materials*, Mech. Research Comm., **23** (1996), 401–407. 1
[17] M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, (2000). 1
[18] K. Sundaresan, *Uniform convexity of Banach spaces* $\ell(\{p_i\})$, Studia Math., **39** (1971), 227–231. 1, 2.2, 2, 3, 3.2, 3
[19] D. Waterman, T. Ito, F. Barber, J. Ratti, *Reflexivity and Summability: The Nakano* $\ell(p_i)$ *spaces*, Studia Math., **33** (1969), 141–146. 1