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Existence of fixed points for γ-FG-contractive condition via cyclic (α, β)-admissible mappings in b-metric spaces

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Abstract

In this paper, we introduce a new concept of cyclic (α, β)-type γ-FG-contractive mapping and we prove some fixed point theorems for such mappings in complete b-metric spaces. Suitable examples are introduced to verify the main results. As an application, we obtain sufficient conditions for the existence of solutions for nonlinear integral equation which are illustrated by an example.

Keywords: b-metric space, cyclic (α, β)-admissible, γ-FG-contractive.
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1. Introduction

Banach contraction principle is the most celebrated result in fixed point theory which illustrates that in a complete metric space, each contractive mapping has a unique fixed point. There have been a great number of generalizations of this principle by using different forms of contractive conditions in various spaces. In recent years, many interesting but different generalizations of the Banach-contraction principle have been given by Samet et al. [10] and Wardowski [11]. In 2012, Wardowski [11] introduced the notion of an F-contraction mapping and investigated the existence of fixed points for such mappings. Afterwards, the concept of F-contraction has been generalized by many other authors, see for example [2, 4]. Wardowski and Van Dung [12], as well as Piri and Kumam [9] generalized the concept of F-contraction and proved certain fixed and common fixed point results. Very recently, Parvaneh et al. [8] used slightly modified family of functions, denoted by $\Delta_{G,\beta}$ and generalized the Wardowski fixed point results in b-metric and ordered b-metric spaces.

On the other hand, Samet et al. [10] introduced the concept of α-admissible maps and gave the concept of α-ψ-contractive mapping, thus generalizing BCP. Afterwards, several other authors used α-admissible

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mappings to obtain various fixed point results. Following this line of work, Alizadeh et al. [3] introduced the notion of cyclic (α, β)-admissible mapping and proved basic fixed point results.

Following this direction, in this paper, we introduce new concepts of cyclic (α, β)-type γ-FG- contractive mapping and we prove some fixed point theorems concerning such contractive mapping and supported by examples. Some consequences are given along with the results for a cyclic mapping. As an application, a solution for nonlinear integral equation is also given and indeed illustrated by an example.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}, \mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers and real numbers, respectively.

Definition 2.1 ([6]). Let X be a nonempty set, and let $s \geqslant 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric space if for all $x, y, z \in X$ the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$.

Then (X, d) is said to be a b-metric space, and the number s is called the coefficient of (X, d).

Then, the concepts of b-convergent, b-Cauchy sequence, b-continuity and completeness in b-metric spaces are naturally given below:

Definition 2.2 ([5]). Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$;
(b) b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;
(c) the b-metric space (X, d) is complete metric space if every b-Cauchy sequence in X is b-convergent.

Each b-convergent sequence in a b-metric space has a unique limit and it is also a b-Cauchy sequence. Moreover, in general, a b-metric is not continuous. We need the following simple lemma about b-convergent sequences in the proof of our main results.

Lemma 2.3 ([1]). *Let (X, d) be a b-metric space with coefficient $s \geqslant 1$ and let $\{x_n\}$, $\{y_n\}$ be b-convergent sequences, which converge to $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leqslant \liminf_{n\to\infty} d(x_n, y_n) \leqslant \limsup_{n\to\infty} d(x_n, y_n) \leqslant s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$$\frac{1}{s} d(x, z) \leqslant \liminf_{n\to\infty} d(x_n, z) \leqslant \limsup_{n\to\infty} d(x_n, z) \leqslant s d(x, z).$$

Definition 2.4 ([5]). Let (X, d) and (X', d') be two b-metric spaces. Then a function $f : X \to X'$ is called b-continuous at a point $x \in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.

Alizadeh et al. [3] introduced the concept of cyclic (α, β)-admissible mapping as follows:

Definition 2.5 ([3]). Let X be a nonempty set, f be a self-mapping on X and let $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that the mapping f is a cyclic (α, β)-admissible mapping if

$$x \in X, \text{ with } \alpha(x) \geqslant 1 \Rightarrow \beta(fx) \geqslant 1,$$

and

$$x \in X, \text{ with } \beta(x) \geqslant 1 \Rightarrow \alpha(fx) \geqslant 1.$$

3. Main results

In this section, we investigate some fixed point results for the new concept of cyclic (α,β)-type γ-FG-contractive mappings and then we prove some fixed point results in b-metric and partially ordered b-metric spaces.

To prove our main result, we will use the following notations cited in Parvaneh et al. [8]. We will consider the following classes of functions.
Δ_F will denote the set of all functions $F:\mathbb{R}_+\to\mathbb{R}$ such that:

(Δ_1) F is continuous and strictly increasing;

(Δ_2) for each sequence $\{t_n\}\subseteq\mathbb{R}_+$, $\lim_{n\to\infty} t_n=0$ if and only if $\lim_{n\to\infty} F(t_n)=-\infty$.

$\Delta_{G,\gamma}$ will denote the set of pairs (G,γ), where $G:\mathbb{R}_+\to\mathbb{R}$ and $\gamma:[0,\infty)\to[0,1)$ such that:

(Δ_3) for each sequence $\{t_n\}\subseteq\mathbb{R}_+$, $\limsup_{n\to\infty} G(t_n)\geqslant 0$ if and only if $\limsup_{n\to\infty} t_n\geqslant 1$;

(Δ_4) for each sequence $\{t_n\}\subseteq[0,\infty)$, $\limsup_{n\to\infty}\gamma(t_n)=1$ implies $\lim_{n\to\infty} t_n=0$;

(Δ_5) for each sequence $\{t_n\}\subseteq\mathbb{R}_+$, $\sum_{n=1}^{\infty} G(\gamma(t_n))=-\infty$.

Example 3.1 ([8])**.** If $F(t)=G(t)=\ln t$ and $\gamma(t)=k\in(0,1)$, then $F\in\Delta_F$ and $(G,\gamma)\in\Delta_{G,\gamma}$. Let $F(t)=-\frac{1}{\sqrt{t}}$, $G(t)=\ln t$ and $\gamma(t)=\frac{1}{k}e^{-t}$ for $t>0$ and $\gamma(t)=0$. Then $F\in\Delta_F$ and $(G,\gamma)\in\Delta_{G,\gamma}$.

Definition 3.2. Let (X,d) be a b-metric space with coefficient $s\geqslant 1$. Suppose that $\alpha,\beta:X\to[0,\infty)$ and $f:X\to X$ is a self-mapping on X. Then f is called cyclic (α,β)-type γ-FG-contractive mapping, if there exist $F\in\Delta_F$, $(G,\gamma)\in\Delta_{G,\gamma}$ such that the following condition holds:

$$\alpha(x)\beta(y)\geqslant 1, d(fx,fy)>0 \Rightarrow \alpha(x)\beta(y)F(s^3 d(fx,fy))\leqslant F(M_s(x,y))+G(\gamma(M_s(x,y))) \tag{3.1}$$

for all $x,y\in X$, where

$$M_s(x,y)=\max\left\{d(x,y),d(y,fy),d(x,fx),\frac{d(x,fy)+d(y,fx)}{2s}\right\}.$$

Theorem 3.3. *Let (X,d) be a complete b-metric space with coefficient $s\geqslant 1$, $\alpha,\beta:X\to[0,\infty)$ and let $f:X\to X$ be a cyclic (α,β)-type γ-FG-contractive mapping satisfying the following conditions:*

(1) *one of the following conditions holds:*
  (a) *there exists $x_0\in X$ such that $\alpha(x_0)\geqslant 1$;*
  (b) *there exists $y_0\in X$ such that $\beta(y_0)\geqslant 1$;*
(2) *f is b-continuous;*
(3) *f is a cyclic (α,β)-admissible mapping.*

Then f has a unique fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n=fx_{n-1}$ for all $n\in\mathbb{N}$ is such that x_0 is an initial point in condition (a) *and the sequence $\{y_n\}$ in X defined by $y_n=fy_{n-1}$ for all $n\in\mathbb{N}$ is such that y_0 is an initial point in condition* (b)*, then $\{x_n\}$ and $\{y_n\}$ converge to a fixed point of f.*

Proof. Case I: Let $x_0\in X$ such that $\alpha(x_0)\geqslant 1$. Define the sequence $\{x_n\}$ by $x_{n+1}=fx_n$. If there exists $n_0\in\mathbb{N}$ such that $x_{n_0}=x_{n_0+1}$, then x_{n_0} is the fixed point of f, and hence the proof is completed. So we assume that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$. It follows that

$$d(x_n,x_{n+1})>0,\ \forall\, n\in\mathbb{N}.$$

Now, we need to prove that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.$$

As f is cyclic (α, β)-admissible mapping, we have

$$\alpha(x_0) \geqslant 1 \Rightarrow \beta(x_1) = \beta(fx_0) \geqslant 1 \Rightarrow \alpha(x_2) = \alpha(fx_1) \geqslant 1. \tag{3.2}$$

By induction, we obtain

$$\alpha(x_{2k}) \geqslant 1, \quad \text{and} \quad \beta(x_{2k+1}) \geqslant 1$$

for all $k \in \mathbb{N}$. Since $\alpha(x_0)\beta(x_1) \geqslant 1$, we get

$$\begin{aligned} F(d(fx_0, fx_1)) &\leqslant \alpha(x_0)\beta(x_1)F(s^3 d(fx_0, fx_1)) \\ &\leqslant F(M_s(x_0, x_1)) + G(\gamma(M_s(x_0, x_1))). \end{aligned}$$

Proceeding in the same manner, we get $\alpha(x_n)\beta(x_{n+1}) \geqslant 1$, for all $n \in \mathbb{N}$.

$$\begin{aligned} F(d(fx_n, fx_{n+1})) &\leqslant \alpha(x_n)\beta(x_{n+1})F(s^3 d(fx_n, fx_{n+1})) \\ &\leqslant F(M_s(x_n, x_{n+1})) + G(\gamma(M_s(x_n, x_{n+1}))). \end{aligned} \tag{3.3}$$

Now, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} M_s(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, fx_{n+1}), d(x_n, fx_n), \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2s}\right\} \\ &\leqslant \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}{2s}\right\} \\ &\leqslant \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right\} \\ &\leqslant \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\right\}. \end{aligned}$$

If $M_s(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$, then inequality (3.3) implies that

$$\begin{aligned} F(d(x_{n+1}, x_{n+2})) &\leqslant \alpha(x_n)\beta(x_{n+1})F(s^3 d(x_{n+1}, x_{n+2})) \\ &< F(d(x_{n+1}, x_{n+2})) + G(\gamma(M_s(x_n, x_{n+1}))). \end{aligned}$$

So, $G(\gamma(M_s(x_n, x_{n+1}))) \geqslant 0$ which implies that $\gamma(M_s(x_n, x_{n+1})) \geqslant 1$ which is a contradiction. Therefore, for all $n \in \mathbb{N}$

$$M_s(x_n, x_{n+1}) = d(x_n, x_{n+1}).$$

From (3.1), we have

$$\begin{aligned} F(d(x_{n+1}, x_{n+2})) &\leqslant \alpha(x_n)\beta(x_{n+1})F(s^3 d(x_{n+1}, x_{n+2})) \\ &\leqslant F(d(x_n, x_{n+1})) + G(\gamma(M_s(x_n, x_{n+1}))) \end{aligned}$$

for all $n \in \mathbb{N}$. Consequently, we deduce that

$$F(d(x_{n+1}, x_{n+2})) \leqslant F(d(x_{n-1}, x_n)) + G(\gamma(M_s(x_{n-1}, x_n))) + G(\gamma(M_s(x_n, x_{n+1}))).$$

Iteratively, we find that

$$F(d(x_n, x_{n+1})) \leqslant F(d(x_0, x_1)) + \sum_{i=1}^{n} G(\gamma(M_s(x_{i-1}, x_i))).$$

Taking $n \to \infty$ we obtain $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$, since $(G, \gamma) \in \Delta_{G,\gamma}$ and since $F \in \Delta_F$ gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.4}$$

Next, we prove that $\{x_n\}$ is a b-Cauchy sequence in X. Suppose not, then there exists $\epsilon_0 > 0$ for which we can find subsequences $\{x_{p(r)}\}$ and $\{x_{q(r)}\}$ of $\{x_n\}$ such that $p(r) > q(r) \geqslant r$ and

$$d(x_{p(r)}, x_{q(r)}) \geqslant \epsilon_0, \tag{3.5}$$

and $q(r)$ is the smallest number such that (3.5) holds:

$$d(x_{p(r)}, x_{q(r)-1}) < \epsilon_0. \tag{3.6}$$

By (b_3), (3.5) and (3.6), we get

$$\begin{aligned}\epsilon_0 \leqslant d(x_{p(r)}, x_{q(r)}) &\leqslant s d(x_{p(r)}, x_{q(r)-1}) + s d(x_{q(r)-1}, x_{q(r)}) \\ &< s\epsilon_0 + s d(x_{q(r)-1}, x_{q(r)}).\end{aligned}$$

Taking the limit supremum as $r \to \infty$ in above inequality which together with (3.4) shows

$$\limsup_{r\to\infty} d(x_{p(r)}, x_{q(r)}) < s\epsilon_0, \tag{3.7}$$

using the triangular inequality, we deduce,

$$d(x_{p(r)}, x_{q(r)}) \leqslant s[d(x_{p(r)}, x_{q(r)+1}) + d(x_{q(r)+1}, x_{q(r)})], \tag{3.8}$$

and

$$d(x_{p(r)}, x_{q(r)+1}) \leqslant s[d(x_{p(r)}, x_{q(r)}) + d(x_{q(r)}, x_{q(r)+1})]. \tag{3.9}$$

Letting $r \to +\infty$ in (3.8) and (3.9), so by (3.4) and (3.7) we obtain

$$\epsilon_0 \leqslant s \limsup_{r\to\infty} d(x_{p(r)}, x_{q(r)+1}),$$

and

$$\limsup_{r\to\infty} d(x_{p(r)}, x_{q(r)+1}) \leqslant s^2\epsilon_0.$$

This implies that

$$\frac{\epsilon_0}{s} \leqslant \limsup_{r\to\infty} d(x_{p(r)}, x_{q(r)+1}) \leqslant s^2\epsilon_0. \tag{3.10}$$

Similarly, we obtain

$$\frac{\epsilon_0}{s} \leqslant \limsup_{r\to\infty} d(x_{q(r)}, x_{p(r)+1}) \leqslant s^2\epsilon_0. \tag{3.11}$$

Finally, we obtain that

$$d(x_{q(r)}, x_{p(r)+1}) \leqslant s[d(x_{q(r)}, x_{q(r)+1}) + d(x_{q(r)+1}, x_{p(r)+1})]. \tag{3.12}$$

Taking the limit supremum as $r \to \infty$ in (3.12), from (3.4) and (3.10), we obtain that

$$\frac{\epsilon_0}{s^2} \leqslant \limsup_{r\to\infty} d(x_{q(r)+1}, x_{p(r)+1}) \leqslant s^3\epsilon_0.$$

Using the cyclic property of α, β we get

$$\alpha(x_{p(r)})\beta(x_{q(r)}) \geqslant 1.$$

Now,

$$\begin{aligned}F(d(fx_{p(r)}, fx_{q(r)})) &\leqslant \alpha(x_{p(r)})\beta(x_{q(r)})F(s^3 d(x_{p(r)+1}, x_{q(r)+1})) \\ &\leqslant F(M_s(x_{p(r)}, x_{q(r)})) + G(\gamma(M_s(x_{p(r)}, x_{q(r)}))),\end{aligned}$$

where

$$\begin{aligned} M_s(x_{p(r)}, x_{q(r)}) &= \max\Big\{ d(x_{p(r)}, x_{q(r)}), d(x_{p(r)}, fx_{p(r)}), \\ &\qquad d(x_{q(r)}, fx_{q(r)}), \frac{d(x_{p(r)}, fx_{q(r)}) + d(x_{q(r)}, fx_{p(r)})}{2s} \Big\} \\ &= \max\Big\{ d(x_{p(r)}, x_{q(r)}), d(x_{p(r)}, x_{p(r)+1}), d(x_{q(r)}, x_{q(r)+1}), \\ &\qquad \frac{d(x_{p(r)}, x_{q(r)+1}) + d(x_{q(r)}, x_{p(r)+1})}{2s} \Big\} \end{aligned} \tag{3.13}$$

for all $r \in \mathbb{N}$. Letting limit supremum as $r \to +\infty$ in (3.13) and using (3.4), (3.7), (3.10), and (3.11), we obtain

$$M_s(x_{p(r)}, x_{q(r)}) = \max\left\{ s\epsilon_0, \frac{s^2\epsilon_0 + s^2\epsilon_0}{2s} \right\} = s\epsilon_0.$$

Now,

$$\begin{aligned} F(s\epsilon_0) &\leqslant F(s^3 \frac{\epsilon_0}{s^2}) \\ &\leqslant F(s^3 \limsup_{r\to\infty} d(x_{q(r)+1}, x_{p(r)+1})) \\ &\leqslant \limsup_{r\to\infty} F(M_s(x_{p(r)}, x_{q(r)})) + \limsup_{r\to\infty} G(\gamma(M_s(x_{p(r)}, x_{q(r)}))) \\ &\leqslant F(s\epsilon_0) + \limsup_{r\to\infty} G(\gamma(M_s(x_{p(r)}, x_{q(r)}))), \end{aligned}$$

which implies that

$$\limsup_{r\to\infty} G(\gamma(M_s(x_{p(r)}, x_{q(r)}))) \geqslant 0.$$

This yields to $\limsup_{r\to\infty} \gamma(M_s(x_{p(r)}, x_{q(r)})) \geqslant 1$, and since $\gamma(t) < 1$ for all $t \geqslant 0$, we have

$$\limsup_{r\to\infty} \gamma(M_s(x_{p(r)}, x_{q(r)})) = 1.$$

Therefore,

$$\limsup_{r\to\infty} M_s(x_{p(r)}, x_{q(r)}) = 0,$$

which is a contradiction because of (3.5) and (3.13). Therefore $\{x_n\}$ is a b-Cauchy sequence in X.

Using the completeness of b-metric space, there exists $x^* \in X$ such that

$$d(x^*, x^*) = \lim_{n\to\infty} d(x_n, x^*) = \lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

By b-continuity of f, we get

$$\lim_{n\to\infty} d(fx_n, fx) = 0.$$

Using (b_3), we have

$$d(x, fx) \leqslant s[d(x, fx_n) + d(fx_n, fx)]$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$d(x, fx) = 0,$$

and then $fx = x$. Let x, y be fixed points of f where $x \neq y$. Now, using (3.2) we have $\alpha(x)\beta(y) \geqslant 1$, and then from

$$\begin{aligned} F(d(fx, fy)) &\leqslant \alpha(x)\beta(y)F(s^3 d(fx, fy)) \\ &\leqslant F(M_s(x, y)) + G(\gamma(M_s(x, y))), \end{aligned}$$

where

$$M_s(x,y) = \left\{ d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(fx,y)}{2s} \right\} = d(x,y),$$

we get

$$F(d(x,y)) \leqslant F(d(x,y)) + G(\gamma(d(x,y))),$$

so $G(\gamma(d(x,y))) \geqslant 0$ which yields that $\gamma(d(x,y)) \geqslant 1$, a contradiction. Hence $x = y$. Therefore, f has a unique fixed point.

Case II: Assume that there exists $y_0 \in X$ such that $\beta(y_0) \geqslant 1$. Proceeding in a similar manner as above, we get the conclusion. Hence the proof is completed. □

Consequently we have the following corollaries:

Taking $\gamma(t) = k, G(t) = \ln t$ where $k \in (0,1)$ and then putting $-\ln k = \tau$ in the above theorem, we obtain a generalization of the results from [11, 12] in the setup of b-metric spaces.

Corollary 3.4. *Let (X, d) be a complete b-metric space with coefficient $s \geqslant 1$, $\alpha, \beta : X \to [0,\infty)$ and let $f : X \to X$ be a mapping satisfying the following conditions:*

(1) *one of the following conditions holds:*
  (a) *there exists $x_0 \in X$ such that $\alpha(x_0) \geqslant 1$;*
  (b) *there exists $y_0 \in X$ such that $\beta(y_0) \geqslant 1$;*

(2)
$$\alpha(x)\beta(y) \geqslant 1, d(fx,fy) > 0 \Rightarrow \tau + \alpha(x)\beta(y)F(s^3 d(fx,fy)) \leqslant F(M_s(x,y))$$
for some $\tau > 0$, for all $x, y \in X$ and M_s is defined as earlier;

(3) *f is b-continuous;*

(4) *f is a cyclic (α, β)-admissible mapping.*

Then f has a unique fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in condition (a) *and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in condition* (b), *then $\{x_n\}$ and $\{y_n\}$ converges to a fixed point of f.*

Taking $F(t) = G(t) = \ln(t)$, and $\alpha(x)\beta(y) = 1$ in the above theorem, we obtain the following result.

Corollary 3.5. *Let (X, d) be a complete b-metric space with coefficient $s \geqslant 1$, $\alpha, \beta : X \to [0,\infty)$, and let $f : X \to X$ be a mapping satisfying the following conditions:*

(1) *one of the following conditions holds:*
  (a) *there exists $x_0 \in X$ such that $\alpha(x_0) \geqslant 1$;*
  (b) *there exists $y_0 \in X$ such that $\beta(y_0) \geqslant 1$;*

(2)
$$s^3 d(fx,fy) \leqslant \gamma(M_s(x,y))M_s(x,y),$$
$d(fx,fy) > 0$ for all $x, y \in X$, and M_s is defined as earlier;

(3) *f is b-continuous;*

(4) *f is a cyclic (α, β)-admissible mapping.*

Then f has a unique fixed point. Moreover, if the sequence $\{x_n\}$ in X defined by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$ is such that x_0 is an initial point in condition (a) *and the sequence $\{y_n\}$ in X defined by $y_n = fy_{n-1}$ for all $n \in \mathbb{N}$ is such that y_0 is an initial point in condition* (b), *then $\{x_n\}$ and $\{y_n\}$ converges to a fixed point of f.*

Taking $F(t) = -\frac{1}{\sqrt{t}}$ and $G(t) = \ln(t)$, and $\alpha(x)\beta(y) = 1$ in the above theorem, we obtain the following result.

Corollary 3.6. *Let (X, d) be a complete b-metric space with coefficient $s \geqslant 1$, $\alpha, \beta : X \to [0,\infty)$, and let $f : X \to X$ be a mapping satisfying the following conditions:*

(1) *one of the following conditions holds:*
   (a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geqslant 1$;
   (b) *there exists* $y_0 \in X$ *such that* $\beta(y_0) \geqslant 1$;

(2)
$$s^3 d(fx, fy) \leqslant \frac{M_s(x,y)}{[1-\sqrt{M_s(x,y)}\ln\gamma(M_s(x,y))]^2}$$
for some $d(fx, fy) > 0$ *for all* $x, y \in X$ *where* $(\ln t, \gamma) \in \Delta_{G,\gamma}$, *and* M_k *is defined as earlier;*

(3) f *is* b*-continuous;*

(4) f *is a cyclic* (α, β)*-admissible mapping.*

Then f *has a unique fixed point. Moreover, if the sequence* $\{x_n\}$ *in* X *defined by* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* x_0 *is an initial point in condition* (a) *and the sequence* $\{y_n\}$ *in* X *defined by* $y_n = fy_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* y_0 *is an initial point in condition* (b), *then* $\{x_n\}$ *and* $\{y_n\}$ *converges to a fixed point of* f.

Taking $\gamma(t) = r$, where $r \in (0, 1)$ and $\alpha(x)\beta(y) = 1$ in the above corollary and denoting $k' = -k$, we obtain the following result.

Corollary 3.7. *Let* (X, d) *be a complete* b*-metric space with coefficient* $s \geqslant 1$, $\alpha, \beta : X \to [0, \infty)$, *and let* $f : X \to X$ *be a mapping satisfying the following conditions:*

(1) *one of the following conditions holds:*
   (a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geqslant 1$;
   (b) *there exists* $y_0 \in X$ *such that* $\beta(y_0) \geqslant 1$;

(2)
$$s^3 d(fx, fy) \leqslant \frac{M_s(x,y)}{[1+k'\sqrt{M_s(x,y)}]^2}$$
for some $d(fx, fy) > 0$ *for all* $x, y \in X$ *where* $k' > 0$, *and* M_s *is defined as earlier;*

(3) f *is* b*-continuous;*

(4) f *is a cyclic* (α, β)*-admissible mapping.*

Then f *has a unique fixed point. Moreover, if the sequence* $\{x_n\}$ *in* X *defined by* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* x_0 *is an initial point in condition* (a) *and the sequence* $\{y_n\}$ *in* X *defined by* $y_n = fy_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* y_0 *is an initial point in condition* (b), *then* $\{x_n\}$ *and* $\{y_n\}$ *converges to a fixed point of* f.

Taking $F(t) = t$, $G(t) = (r-1)t, \gamma(t) = r$ where $r \in [0, \infty)$ and putting $k = 1$, $\alpha(x) = 1$, $\beta(x) = 1$ in the above theorem, we obtain the following result.

Corollary 3.8. *Let* (X, d) *be a complete b-metric space with coefficient* $k \geqslant 1$, $\alpha, \beta : X \to [0, \infty)$, *and let* $f : X \to X$ *be a mapping such that*
$$d(fx, fy) \leqslant rM(x, y)$$
for some $r \in [0, 1)$ *and for all* $x, y \in X$, *and* M *is defined as earlier. Then* f *has a fixed point. Moreover, if the sequence* $\{x_n\}$ *in* X *defined by* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* x_0 *is an initial point then* $\{x_n\}$ *converges to a fixed point of* f.

Taking $s^3 = k$ and $\alpha(x)\beta(y) = 1$ in Theorem 3.3, we obtain the result of Parvaneh et al. [8].

Corollary 3.9. *Let* (X, d) *be a complete* b*-metric space with coefficient* $s > 1$, $\alpha, \beta : X \to [0, \infty)$, *and let* $f : X \to X$ *be a mapping satisfying the following conditions:*

(1) *one of the following conditions holds:*
   (a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geqslant 1$;
   (b) *there exists* $y_0 \in X$ *such that* $\beta(y_0) \geqslant 1$;

(2)
$$\alpha(x)\beta(y) \geqslant 1, d(fx, fy) > 0 \Rightarrow F(kd(fx, fy)) \leqslant F(M(x, y)) + G(\gamma(M(x, y))) \tag{3.14}$$
for all $x, y \in X$, *and*
$$M(x, y) = \max\left\{d(x, y), d(y, fy), d(x, fx), \frac{d(x, fy) + d(y, fx)}{2}\right\};$$

(3) f *is* b*-continuous;*
(4) f *is a cyclic* (α, β)*-admissible mapping.*

Then f *has a unique fixed point. Moreover, if the sequence* $\{x_n\}$ *in* X *defined by* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* x_0 *is an initial point in condition* (a) *and the sequence* $\{y_n\}$ *in* X *defined by* $y_n = fy_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* y_0 *is an initial point in condition* (b)*, then* $\{x_n\}$ *and* $\{y_n\}$ *converges to a fixed point of* f.

Taking $F(t) = t$, $G(t) = (1-k)t, \gamma(t) = k$ where $k \in [0, \infty)$ and putting $\alpha(x) = 1, \beta(x) = 1$ in the above theorem, we obtain the following result.

Corollary 3.10. *Let* (X, d) *be a complete* b*-metric space with coefficient* $s \geqslant 1$, $\alpha, \beta : X \to [0, \infty)$*, and let* $f : X \to X$ *be a mapping such that*
$$s^3 d(fx, fy) \leqslant rM_s(x, y)$$
for some $r \in [0, 1)$ *and for all* $x, y \in X$*, and* M_s *is defined earlier. Then* f *has a fixed point. Moreover, if the sequence* $\{x_n\}$ *in* X *defined by* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *is such that* x_0 *is an initial point then* $\{x_n\}$ *converges to a fixed point of* f.

Now, let us consider the following examples:

Example 3.11. Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. Define the mappings $\alpha, \beta, : X \to [0, \infty)$, $\gamma : [0, \infty) \to [0, 1)$ and $f : X \to X$ as follows:
$$\alpha(x) = \begin{cases} \frac{x+7}{2}, & x \in [0, \frac{1}{2}], \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} \frac{x+6}{2}, & x \in [0, \frac{1}{2}], \\ 1, & \text{otherwise}, \end{cases}$$
and
$$f(x) = \begin{cases} \frac{x^2}{3}, & x \in [0, \frac{1}{2}], \\ x + 0.01, & \text{otherwise}, \end{cases} \quad \text{and} \quad \gamma(t) = \frac{2}{9}.$$
Now, we will prove that f is a cyclic (α, β)-admissible mapping.

For $x \in [0, \frac{1}{2}]$, we have
$$\alpha(x) \geqslant 1 \Rightarrow \beta(fx) = \beta\left(\frac{x^2}{3}\right) = \left(\frac{\frac{x^2}{3} + 6}{2}\right) \geqslant 1,$$
and
$$\beta(x) \geqslant 1 \Rightarrow \alpha(fx) = \alpha\left(\frac{x^2}{3}\right) = \left(\frac{\frac{x^2}{3} + 7}{2}\right) \geqslant 1.$$
Therefore, f is a cyclic (α, β)-admissible mapping.

Next, we will prove that f satisfies the contractive condition (3.14), with the mappings $F, G : \mathbb{R}^+ \to \mathbb{R}$ as $F(t) = G(t) = \ln t$, for all $t \in [0, \infty)$. Assume that $x, y \in X$ are such that $\alpha(x)\beta(y) \geqslant 1$. Then we have $x, y \in [0, \frac{1}{2}]$ and
$$sd(fx, fy) = 2\left|\frac{x^2}{3} - \frac{y^2}{3}\right|^2$$

$$\begin{aligned}
&\leqslant \frac{2}{9}|x^2-y^2|^2\\
&\leqslant \frac{2}{9}(|x-y|^2)\\
&\leqslant \gamma(M(x,y))d(x,y)\\
&\leqslant \gamma(M(x,y))M(x,y),
\end{aligned}$$

hence,

$$F(sd(fx,fy)) \leqslant F(M(x,y)) + G(\gamma(M(x,y))).$$

Then, f satisfies all the conditions of Corollary 3.9, and therefore f has a unique fixed point $x^* = 0$.

Example 3.12. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by $d(x,y) = |x-y|^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with $s = 2$. Define the mappings $\alpha, \beta : X \to [0,\infty)$, $\gamma : [0,\infty) \to [0,1)$ and $f : X \to X$ as follows:

$$\alpha(x) = \begin{cases} \frac{x^2+3}{2}, & x \in [0,1],\\ 0, & \text{otherwise},\end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} \frac{2x^2+5}{4}, & x \in [0,1],\\ 1, & \text{otherwise},\end{cases}$$

and

$$f(x) = \begin{cases} \frac{x}{3\sqrt{3+x^2}}, & x \in [0,1],\\ 2x, & \text{otherwise},\end{cases} \quad \text{and} \quad \gamma(t) = \frac{8}{9}.$$

Now, we will prove that f is a cyclic (α,β)-admissible mapping.

For $x \in [0,1]$, we have

$$\alpha(x) \geqslant 1 \Rightarrow \beta(fx) = \beta\left(\frac{x}{3\sqrt{3+x^2}}\right) = \left(\frac{\left(2\frac{x^2}{9(3+x^2)}\right)+5}{4}\right) \geqslant 1,$$

and

$$\beta(x) \geqslant 1 \Rightarrow \alpha(fx) = \alpha\left(\frac{x}{3\sqrt{3+x^2}}\right) = \left(\frac{\left(\frac{x^2}{9(3+x^2)}\right)+3}{2}\right) \geqslant 1.$$

Therefore, f is a cyclic (α,β)-admissible mapping.

Next, we will prove that f satisfies the contractive condition (3.1), with the mappings $F, G : \mathbb{R}^+ \to \mathbb{R}$ as $F(t) = G(t) = \ln t$, for all $t \in [0,\infty)$. Assume that $x, y \in X$ are such that $\alpha(x)\beta(y) \geqslant 1$. Then we have $x, y \in [0,1]$ and

$$\begin{aligned}
k^3 d(fx,fy) &= 8\left|\frac{x}{3\sqrt{3+x^2}} - \frac{y}{3\sqrt{3+y^2}}\right|^2\\
&\leqslant \frac{8}{9}|x-y|^2\\
&\leqslant \gamma(M(x,y))d(x,y)\\
&\leqslant \gamma(M(x,y))M(x,y),
\end{aligned}$$

hence,

$$F(s^3 d(fx,fy)) \leqslant F(M(x,y)) + G(\gamma(M(x,y))).$$

Then, f satisfies all the conditions of Theorem 3.3, and therefore f has a unique fixed point $x^* = 0$.

In the following, we give some fixed point results involving cyclic mappings which can be regarded as consequences of the previous results.

Definition 3.13 ([7]). Let A and B be nonempty subsets of a set X. A mapping $f : A \cup B \to A \cup B$ is called cyclic if $f(A) \subseteq B$ and $f(B) \subseteq A$.

Definition 3.14. Let (X, d) be a b-metric space with coefficient $s \geqslant 1$. We say that a mapping $f : A \cup B \to A \cup B$ is an (A, B)-γ-FG-contractive mapping if there exist $F \in \Delta_F$, $(G, \gamma) \in \Delta_{G,\gamma}$ such that the following condition holds:

$$A(x)B(y) \geqslant 1, d(fx, fy) > 0 \Rightarrow A(x)B(y)F(s^3 d(fx, fy)) \leqslant F(M_s(x, y)) + G(\gamma(M_s(x, y)))$$

for all $x \in A$ and $y \in B$ where

$$M_s(x, y) = \max\left\{d(x, y), d(y, fy), d(x, fx), \frac{d(x, fy) + d(y, fx)}{2s}\right\}.$$

Theorem 3.15. *Let A and B be two nonempty subsets of the complete b-metric space (X, d) with coefficient $s \geqslant 1$ and $f : A \cup B \to A \cup B$ is an (A, B)-γ-FG-contractive mapping. Then f has a fixed point in $A \cap B$.*

Proof. Define mappings $\alpha, \beta : A \cup B \to [0, \infty)$ by

$$\alpha(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & x \in B, \\ 0, & \text{otherwise}. \end{cases}$$

For $x, y \in A \cup B$ such that $\alpha(x)\beta(y) \geqslant 1$, we get $x \in A$ and $y \in B$. Then we have

$$\alpha(x)\beta(y) \geqslant 1, d(fx, fy) > 0 \Rightarrow \alpha(x)\beta(y)F(s^3 d(fx, fy)) \leqslant F(M_s(x, y)) + G(\gamma(M_s(x, y))),$$

and thus condition (3.1) holds. Therefore, f is an (α, β)-γ-FG-contractive mapping. It is easy to see that f is a cyclic (α, β)-admissible mapping. Since A and B are nonempty subsets, there exists $x_0 \in A$ such that $\alpha(x_0) \geqslant 1$ and there exists $y_0 \in B$ such that $\beta(y_0) \geqslant 1$. Now, all conditions of Theorem 3.3 hold, so f has a fixed point in $A \cup B$, say z. If $z \in A$, then $z = fz \in B$. Similarly, if $z \in B$ then $z \in A$. Hence $z \in A \cup B$. □

Similarly, by replacing $M_s(x, y) = d(x, y)$ we obtain the following corollary.

Corollary 3.16. *Let A and B be two nonempty subsets of the complete b-metric space (X, d) with coefficient $s \geqslant 1$ and $f : A \cup B \to A \cup B$ be a mapping such that*

$$A(x)B(y) \geqslant 1, d(fx, fy) > 0 \Rightarrow A(x)B(y)F(s^3 d(fx, fy)) \leqslant F(d(x, y)) + G(\gamma(d(x, y))).$$

Then f has a fixed point in $A \cap B$.

Taking $F(t) = G(t) = \ln(t)$, and $\alpha(x)\beta(y) = 1$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.17. *Let A and B be two nonempty subsets of the complete b-metric space (X, d) with coefficient $s \geqslant 1$ and $f : A \cup B \to A \cup B$ be a mapping such that*

$$s^3 d(fx, fy) \leqslant M_s(x, y)\gamma(M_s(x, y))$$

for all $x \in A$, $y \in B$ and M_s is defined as earlier. Then f has a fixed point in $A \cap B$.

4. Application to nonlinear integral equations

Consider the nonlinear integral equation

$$x(t) = g(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in I = [0, 1], \quad \lambda \geqslant 0. \tag{4.1}$$

Also, suppose that the following conditions hold:

(a) $g : I \to \mathbb{R}$ is a continuous function;

(b) $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t,x) \geqslant 0$ and there exists a constant $\delta \in [0,1)$ such that for all $x, y \in \mathbb{R}$,
$$|f(s,x(s)) - f(s,y(s))| \leqslant \delta |x(s) - y(s)|;$$

(c) $k \times \mathbb{R} : I \times \to \mathbb{R}$ is continuous at $t \in I$ for every $s \in I$ and measurable at $s \in I$ for all $t \in I$ such that $k(t,x) \geqslant 0$ and
$$\int_0^1 k(t,s)ds \leqslant K;$$

(d) $\lambda^p K^p \delta^p \leqslant \frac{1}{2^{2p-1}}$;

(e) the space $X = C(I)$ of continuous functions defined on $I = [0,1]$, with the standard metric given by
$$\rho(x,y) = \sup_{t \in I} |x(t) - y(t)|, \quad \text{for } x, y \in C(I).$$

Now, for $p \geqslant 1$ we define
$$d(x,y) = (\rho(x,y))^p = \left(\sup_{t\in I} |x(t) - y(t)|\right)^p = \sup_{t\in I} |x(t) - y(t)|^p, \quad \text{for } x, y \in C(I).$$

Also, define
$$M(x,y) = \left\{ d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy) + d(fx,y)}{2} \right\}.$$

Then (X,d) is a complete b-metric space with $s = 2^{p-1}$.

Theorem 4.1. *Under the assumptions* (a)-(e), *the nonlinear integral equation* (4.1) *has a unique solution in* $C(I)$.

Proof. Define the operator $T : X \to X$ by
$$Tx(t) = g(t) + \lambda \int_0^1 k(t,s) f(s,x(s))ds, \quad t \in I = [0,1], \quad \lambda \geqslant 0.$$

Now, for $x, y \in X$ we have
$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| g(t) + \lambda \int_0^1 k(t,s)f(s,x(s))ds - g(t) - \lambda \int_0^1 k(t,s)f(s,y(s))ds \right| \\ &\leqslant \lambda \int_0^1 k(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leqslant \lambda \int_0^1 k(t,s)\delta|x(s) - y(s)|ds. \end{aligned}$$

As $|x(s) - y(s)| \leqslant \sup_{s\in I} |x(s) - y(s)| = \rho(x,y)$,
$$|Tx(t) - Ty(t)| \leqslant \lambda K \delta \rho(x,y).$$

Now,
$$\begin{aligned} d(Tx,Ty) &= \sup_{t\in I} |Tx(t) - Ty(t)|^p \\ &\leqslant (\lambda K \delta \rho(x,y))^p \\ &\leqslant \lambda^p K^p \delta^p d(x,y) \\ &\leqslant \frac{1}{2^{2p-1}} M(x,y). \end{aligned}$$

Therefore, all the assumptions of Corollary 3.8 are satisfied by the operator T and the equation (4.1) has a unique solution in $C(I)$. □

Example 4.2. Consider the following functional integral equation:

$$x(t) = \frac{1}{1+t} + \frac{1}{6}\int_0^1 \frac{s}{2(1+t^2)} \frac{|x(s)|}{3e^t(1+|x(s)|)} ds, \quad t \in I = [0,1].$$

It is observed that the above equation is a special case of (4.1) with

$$g(t) = \frac{1}{1+t}, \quad k(t,s) = \frac{s}{2(1+t^2)}, \quad f(t,x) = \frac{|x|}{3e^t(1+|x|)}.$$

Now, for arbitrary $x, y \in \mathbb{R}$ such that $x \geqslant y$ and for $t \in [0,1]$, we obtain

$$\begin{aligned} |f(t,x) - f(t,x)| &= \left| \frac{|x|}{3e^t(1+|x|)} - \frac{|y|}{3e^t(1+|y|)} \right| \\ &= \frac{1}{3e^t} \left| \frac{|x|}{(1+|x|)} - \frac{|y|}{(1+|y|)} \right| \\ &\leqslant \frac{1}{3}|x-y|. \end{aligned}$$

Thus, f satisfies condition (b) of the integral equation (4.1) with $\delta = \frac{1}{3}$. It can be easily seen that g is a continuous function and k satisfies condition (c) with

$$\int_0^1 k(t,s)ds = \int_0^1 \frac{s}{2(1+t^2)} ds = \frac{1}{4(1+t^2)} \leqslant \frac{1}{4} = K.$$

By substituting $\delta = \frac{1}{6}, K = \frac{1}{4}$ and $\lambda = \frac{1}{6}$ in condition (d), we get

$$\frac{1}{6^p} \times \frac{1}{4^p} \times \frac{1}{3^p} \leqslant \frac{1}{2^{2p-1}}.$$

The above inequality is true for each $p \geqslant 1$. Consequently, all the conditions of Theorem 4.1 are satisfied and hence the integral equation (4.1) has a unique solution in $C(I)$.

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