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Generalized fractional calculus of the multiindex Bessel function

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Abstract

The present paper is devoted to the study of the fractional calculus operators to obtain a number of key results for the generalized multiindex Bessel function involving Saigo hypergeometric fractional integral and differential operators in terms of generalized Wright function. Various particular cases and consequences of our main fractional-calculus results as classical Riemann-Liouville and Erdélyi-Kober fractional integral and differential formulas are deduced.

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1. Introduction

Recently, the generalized multiindex Bessel function is defined by Choi et al. [1] as follows:

$$J_{(\beta_j)_m,k}^{(\alpha_j)_m,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+1\right)}\frac{(-z)^n}{n!},\quad (m\in\mathbb{N}),\tag{1.1}$$

where $\alpha_j, \beta_j, \gamma \in \mathbb{C}$, $(j=1,\cdots,m)$, $\Re(\gamma)>0$, $\Re(\beta)>-1$, $\sum_{j=1}^{m}\Re(\alpha)_j>\max\{0;\Re(k)-1\}$, $k>0$. The Pohhammer symbol is defined for $\gamma\in\mathbb{C}$ as follows (see [8, p.2 and p.5]):

$$\begin{aligned}(\gamma)_n &= \begin{cases} 1, & n=0,\\ \gamma(\gamma+1)...(\gamma+n+1), & n\in\mathbb{N},\end{cases}\\ &=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},\quad (\gamma\in\mathbb{C}/\mathbb{Z}_0),\end{aligned}$$

and Γ being the Gamma function.

Some important special cases of this function are enumerated below:

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(i) If we put $k = 0$, $m = 1$, $\alpha_1 = 1$, $\beta_1 = \upsilon$ and replace z by $z^2/4$ in (1.1), we obtain

$$J^{1,\gamma}_{\upsilon,0}\left[\frac{z^2}{4}\right] = \left(\frac{2}{z}\right)^{\upsilon} J_{\upsilon}[z],$$

where $J_{\upsilon}[z]$ is a well-known Bessel function of the first kind defined for complex $z \in \mathbb{C}$, $(z \neq 0)$ and $\upsilon \in \mathbb{C}$, $(\Re(\upsilon) > -1,)$ by ([3, 7.2 (2)]) (see also [5])

$$J_{\upsilon}[z] = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\upsilon + k + 1)} \frac{(z/2)^{\upsilon+2k}}{k!};$$

(ii) $J^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}(z)$ has the form:

$$J^{(\alpha_j)_m,\gamma}_{(\beta_j)_m,k}(z) = \frac{1}{\Gamma(\gamma)} {}_1\psi_m \left[\begin{array}{c} (\gamma, 1) \\ (\beta_1 + 1, \alpha_1), ..., (\beta_m + 1, \alpha_m) \end{array} \Big| z \right].$$

A detail account of Bessel function, the reader may be referred to the earlier extensive works by Erdélyi et al. [3] and Watson [9].

The generalized Wright hypergeometric function ${}_p\psi_q(z)$, for $z \in \mathbb{C}$, complex $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$, $(\alpha_i, \beta_j \neq 0;\ i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ is defined as follows:

$${}_p\psi_q(z) = {}_p\psi_q \left[\begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \Big| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k) k!}. \tag{1.2}$$

Wright [10] introduced the generalized Wright function [4] and proved several theorems on the asymptotic expansion of ${}_p\psi_q(z)$ (for instance, see [10–12]) for all values of the argument z, under the condition:

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1.$$

The generalized hypergeometric function for complex $a_i, b_j \in \mathbb{C}$ and $b_j \neq 0, -1, \cdots$ $(i = 1, 2, \cdots, p;\ , j = 1, 2, \cdots, q)$ is given by the power series ([2, Section 4.1 (1)]):

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r z^r}{(b_1)_r \cdots (b_q)_r r!}, \tag{1.3}$$

where for convergence, we have $|z| < 1$ if $p = q + 1$ and for any z if $p \leqslant q$. The function (1.3) is a special case of the generalized Wright function (1.2) for $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} {}_p\psi_q \left[\begin{array}{c} (a_i, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{array} \Big| z \right].$$

The object of this paper is to derive fractional integral and derivative of generalized multiindex Bessel function (1.1) and the left and right sided operators of Saigo fractional calculus [6]. The results derived in this paper are believed to be new.

2. Fractional calculus operators approach

The Riemann-Liouville fractional integral and derivative operators are defined by (see Samko, Kilbas and Marichev [7, Sect. 2]) for $\alpha > 0$:

$$\left(I^{\alpha}_{0+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x - t)^{1-\alpha}} dt, \tag{2.1}$$

$$\left(I_{0-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} dt, \tag{2.2}$$

$$\begin{aligned} \left(D_{0+}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0+}^{1-[\alpha]} f\right)(x), \\ &= \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}} dt, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \left(D_{0-}^{\alpha} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I_{0-}^{1-[\alpha]} f\right)(x), \\ &= \frac{1}{\Gamma(1-\{\alpha\})} \left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}} dt, \end{aligned} \tag{2.4}$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

A useful generalization of the Riemann-Liouville and Erdélyi-Kober fractional integral has been introduced by Saigo [6] in terms of Gauss hypergeometric function.

Let $\alpha, \beta, \gamma \in \mathbb{C}$ and $x \in \mathbb{R}_+$, then the generalized fractional integration and fractional differentiation operators associated with Gauss hypergeometric function are defined as follows:

$$\left(I_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\gamma; \alpha; 1-t/x) f(t) dt, \tag{2.5}$$

$$\left(I_{0-}^{\alpha,\beta,\gamma} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1(\alpha+\beta, -\gamma; \alpha; 1-x/t) f(t) dt, \tag{2.6}$$

$$\left(D_{0+}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{0+}^{-\alpha-\beta,\alpha+\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\gamma-k} f\right)(x), \tag{2.7}$$
$$[\Re(\alpha) > 0;\ k = [\Re(\alpha)] + 1],$$

$$\left(D_{0-}^{\alpha,\beta,\gamma} f\right)(x) = \left(I_{0-}^{-\alpha-\beta,\alpha+\gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^k \left(I_{0-}^{-\alpha+k,-\beta-k,\alpha+\gamma} f\right)(x), \tag{2.8}$$
$$[\Re(\alpha) > 0;\ k = [\Re(\alpha)] + 1].$$

Therefore, if we set $\beta = -\alpha$ in (2.5), (2.6), (2.7), (2.8), reduce to (2.1), (2.2), (2.3), (2.4), of the left and right hand sided Riemann-Liouville fractional integral and derivative operators.

2.1. *Left-sided fractional integration of generalized multiindex Bessel function*

In this section, we establish image formulas for the generalized multiindex Bessel function involving left sided operators of Saigo fractional integral operators (2.1), in term of the generalized Wright function.

Theorem 2.1. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1, \cdots, m)$ are arbitrary parameters and $I_{0+}^{\alpha,\beta,\gamma}(.)$ be the Saigo left-sided fractional integral operator* (2.5)*, then the following result holds:*

$$\left(I_{0+}^{\alpha,\beta,\gamma}\left(t^{\rho-1} J_{(\eta_j)_m,k}^{(\mu_j)_m,\tau}(a\,t^{\upsilon})\right)\right)(x) = \frac{x^{\rho-\beta-1}}{\Gamma(\tau)} {}_3\psi_{m+2}\left[\begin{matrix}(\rho-\beta+\gamma,\upsilon),(\rho,\upsilon),(\tau,k)\\(\rho-\beta,\upsilon),(\rho+\alpha+\gamma,\upsilon),(\eta_j+1,\mu_j)_1^m\end{matrix}\middle| -a\,x^{\upsilon}\right]. \tag{2.9}$$

The conditions for validity of (2.9) *are*

(i) *$\alpha, \beta, \gamma, \Re(\alpha) > 0$ and a are any complex numbers;*

(ii) *ρ and υ are arbitrary, such that $\Re(\rho + \upsilon n) > 0$ and $\Re(\rho + \gamma - \beta + \upsilon n) > 0$.*

Proof. Denote L.H.S. of Theorem 2.1 by I_1, then

$$I_1 \equiv \left(I_{0+}^{\alpha,\beta,\gamma}\left(t^{\rho-1}J_{(\eta_j)_m,k}^{(\mu_j)_m,\tau}(a\,t^{\upsilon})\right)\right)(x),$$

by virtue of (1.1) and (2.5), we have

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\,\alpha;\,1-t/x\right)(t^{\rho-1})\,J_{(\eta_j)_m,k}^{(\mu_j)_m,\tau}(a\,t^{\upsilon})dt,$$

now, interchanging the order of integration and summation is permissible under the conditions stated with the theorem due to convergence of the integrals involved in the process and evaluating the inner integral by beta-function, it gives

$$= \frac{x^{\rho-\beta-1}}{\Gamma(\tau)}\sum_{n=0}^{\infty}\frac{\Gamma(\rho-\beta+\gamma+\upsilon\,n)\,\Gamma(\rho+\upsilon\,n)\,\Gamma(\tau+kn)}{\Gamma(\rho-\beta+\upsilon\,n)\,\Gamma(\rho+\alpha+\gamma+\upsilon\,n)\Gamma\left(\eta_j+1+\mu_j\,n\right)}\frac{(-a\,t^{\upsilon})^n}{n!}$$
$$= \frac{x^{\rho-\beta-1}}{\Gamma(\tau)}{}_3\psi_{m+2}\left[\begin{matrix}(\rho-\beta+\gamma,\upsilon),(\rho,\upsilon),(\tau,k)\\(\rho-\beta,\upsilon),(\rho+\alpha+\gamma,\upsilon),(\eta_j+1,\mu_j)_1^m\end{matrix}\,\middle|-ax^{\upsilon}\right],$$

which completes the proof of Theorem 2.1. □

Corollary 2.2. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1,\cdots,m)$ are arbitrary parameters and $I_{0+}^{\alpha}(.)$ be the Riemann-Lioville left-sided fractional integral operator* (2.1)*, then the following result holds:*

$$\left(I_{0+}^{\alpha}\left(t^{\rho-1}J_{(\eta_j)_m,k}^{(\mu_j)_m,\tau}(a\,t^{\upsilon})\right)\right)(x) = \frac{x^{\rho+\alpha-1}}{\Gamma(\tau)}\,{}_2\psi_{m+1}\left[\begin{matrix}(\rho,\upsilon),(\tau,k)\\(\rho-\beta,\upsilon),(\eta_j+1,\mu_j)_1^m\end{matrix}\,\middle|-a\,x^{\upsilon}\right]. \tag{2.10}$$

The conditions for validity of (2.10) *are*

(i) α, $\Re(\alpha) > 0$ *and a are any complex numbers;*

(ii) ρ *and* υ *are arbitrary, such that* $\Re(\rho+\upsilon n) > 0$.

2.2. Right-sided fractional integration of generalized multiindex Bessel function

In this section, we establish image formulas for the generalized multiindex Bessel function involving right sided operators of Saigo fractional integral operators (2.6), in term of the generalized Wright function.

Theorem 2.3. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1,\cdots,m)$ are arbitrary parameters and $I_{0-}^{\alpha,\beta,\gamma}(.)$ be the Saigo's right-sided fractional integral operator* (2.6)*, then the following result holds:*

$$\left(I_{0-}^{\alpha,\beta,\gamma}\left(t^{\rho}J_{(\eta_j)_m,k}^{(\mu_j)_m,\tau}(a\,t^{-\upsilon})\right)\right)(x) = \frac{x^{\rho-\beta}}{\Gamma(\tau)}{}_3\psi_{m+2}\left[\begin{matrix}(\beta-\rho,\upsilon),(\gamma-\rho,\upsilon),(\tau,k)\\(-\rho,\upsilon),(\alpha+\beta+\gamma-\rho,\upsilon),(\eta_j+1,\mu_j)_1^m\end{matrix}\,\middle|-a\,x^{-\upsilon}\right]. \tag{2.11}$$

The conditions for validity of (2.11) *are*

(i) α,β,γ, $\Re(\alpha) > 0$ *and a are any complex numbers;*

(ii) ρ *and* υ *are arbitrary such that* $\Re(\beta-\rho+\upsilon n) > 0$ *and* $\Re(\gamma-\rho+\upsilon\,n) > 0$.

Proof. Denote L.H.S. of Theorem 2.3 by I_2, then

$$I_2 \equiv \left(I_{0-}^{\alpha,\beta,\gamma}\left(t^{\rho}J_{(\eta_j)_m,k}^{(\mu_j)_m,\tau}(a\,t^{-\upsilon})\right)\right)(x),$$

using the definition of generalized multiindex Bessel function (1.1), fractional integral formula (2.6) and proceeding similarly to the proof of Theorem 2.1, we obtain

$$= \frac{x^{\rho-\beta}}{\Gamma(\tau)} \sum_{n=0}^{\infty} \frac{\Gamma(\beta-\rho+\upsilon n)\,\Gamma(\gamma-\rho+\upsilon n)\;\Gamma(\tau+kn)}{\Gamma(-\rho+\upsilon n)\,\Gamma(\alpha+\beta+\gamma-\rho+\upsilon n)\Gamma\left(\eta_j+1+\mu_j n\right)} \frac{(-a\,t^{-\upsilon})^n}{n!}$$
$$= \frac{x^{\rho-\beta}}{\Gamma(\tau)}\, {}_3\psi_{m+2}\left[{}^{(\beta-\rho,\upsilon),(\gamma-\rho,\upsilon),(\tau,k)}_{(-\rho,\upsilon),(\alpha+\beta+\gamma-\rho,\upsilon),(\eta_j 1,,\mu_j)_1^m} \Big| -a\,x^{-\upsilon} \right],$$

which completes the proof of Theorem 2.3. □

Corollary 2.4. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1, \cdots, m)$ are arbitrary parameters and $I^{\alpha}_{0-}(.)$ be the Riemann Liouville right-sided fractional integral operator* (2.2)*, then the following result holds:*

$$\left(I^{\alpha}_{0-}\left(t^{\rho} J^{(\mu_j)_m,\tau}_{(\eta_j)_m,k}(a\,t^{-\upsilon})\right)\right)(x) = \frac{x^{\rho+\alpha}}{\Gamma(\tau)}\, {}_2\psi_{m+1}\left[{}^{(-\alpha-\rho,\upsilon),(\tau,k)}_{(-\rho,\upsilon),(\eta_j+1,\mu_j)_1^m} \Big| -a\,x^{-\upsilon} \right]. \tag{2.12}$$

The conditions for validity of (2.12) *are*

(i) α, $\Re(\alpha) > 0$ *and a are any complex numbers;*

(ii) ρ *and* υ *are arbitrary such that* $\Re(-\alpha-\rho+\upsilon n) > 0$.

2.3. Left-sided fractional differentiation of generalized multiindex Bessel function

In this section, we establish image formulas for the generalized multiindex Bessel function involving left sided of Saigo fractional differentiation operators (2.7), in term of the generalized Wright function.

Theorem 2.5. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1, \cdots, m)$ are arbitrary parameters and $D^{\alpha,\beta,\gamma}_{0+}(.)$ be the Saigo left-sided fractional differentiation operator* (2.7)*, then the following result holds:*

$$\left(D^{\alpha,\beta,\gamma}_{0+}\left(t^{\rho-1} J^{(\mu_j)_m,\tau}_{(\eta_j)_m,k}(a\,t^{\upsilon})\right)\right)(x) = \frac{x^{\rho+\beta-1}}{\Gamma(\tau)}\, {}_3\psi_{m+2}\left[{}^{(\rho+\alpha+\beta+\gamma,\upsilon),(\rho,\upsilon),(\tau,k)}_{(\rho+\beta,\upsilon),(\rho+\gamma,\upsilon),(\eta_j+1,\mu_j)_1^m} \Big| -ax^{\upsilon} \right]. \tag{2.13}$$

The conditions for validity of (2.13) *are*

(i) α, β, γ, $\Re(\alpha) > 0$ *and a are any complex numbers;*

(ii) ρ *and* υ *are arbitrary such that* $\Re(\rho+\upsilon n) > 0$ *and* $\Re(\rho+\alpha+\beta+\gamma+\upsilon n) > 0$.

Proof. Denote L.H.S. of Theorem 2.5 by I_3, then

$$I_3 \equiv \left(D^{\alpha,\beta,\gamma}_{0+}\left(t^{\rho-1} J^{(\mu_j)_m,\tau}_{(\eta_j)_m,k}(a\,t^{\upsilon})\right)\right)(x),$$

by virtue of (1.1) and (2.7), we have

$$= \left(\frac{d}{dx}\right)^k \left(I^{-\alpha+k,-\beta-k,\alpha+\gamma-k}_{0+}\right)(t^{\rho-1}) J^{(\mu_j)_m,\tau}_{(\eta_j)_m,k}(a\,t^{\upsilon}),$$
$$= \left(\frac{d}{dx}\right)^k \frac{x^{\alpha+\beta}}{\Gamma(-\alpha+k)} \int_0^x (x-t)^{-\alpha+k-1}\, {}_2F_1(-\alpha-\beta, -\gamma-\alpha+k; -\alpha+k; 1-t/x)$$
$$\times\ (t^{\rho-1})\, J^{(\mu_j)_m,\tau}_{(\eta_j)_m,k}(a\,t^{\upsilon})dt,$$

now, interchanging the order of integration and summation is permissible under the conditions

$$= \frac{x^{\rho+\beta-1}}{\Gamma(\tau)} \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\alpha+\beta+\gamma+\upsilon n)\,\Gamma(\rho+\upsilon n)\,\Gamma(\tau+kn)}{\Gamma(\rho+\beta+\upsilon n)\,\Gamma(\rho+\gamma+\upsilon n)\Gamma\left(\eta_j+1+\mu_j n\right)} \frac{(-a\,t^{\upsilon})^n}{n!}$$
$$= \frac{x^{\rho+\beta-1}}{\Gamma(\tau)}\, {}_3\psi_{m+2}\left[{}^{(\rho+\alpha+\beta+\gamma,\upsilon),(\rho,\upsilon),(\tau,k)}_{(\rho+\beta,\upsilon),(\rho+\gamma,\upsilon),(\eta_j+1,\mu_j)_1^m} \Big| -ax^{\upsilon} \right],$$

which completes the proof of Theorem 2.5. □

Corollary 2.6. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1, \cdots, m)$ are arbitrary parameters and $D_{0+}^{\alpha}(.)$ be the Riemann Liouville left-sided fractional differentiation operator (2.3), then the following result holds:*

$$\left(D_{0+}^{\alpha}\left(t^{\rho-1} J_{(\eta_j)_m, k}^{(\mu_j)_m, \tau}(a t^{\upsilon})\right)\right)(x) = \frac{x^{\rho-\alpha-1}}{\Gamma(\tau)}\, {}_2\psi_{m+1}\left[\begin{matrix}(\rho, \upsilon), (\tau, k) \\ (\rho-\alpha, \upsilon), (\eta_j+1, \mu_j)_1^m\end{matrix} \middle| -a x^{\upsilon}\right]. \tag{2.14}$$

The conditions for validity of (2.14) are

(i) α, $\Re(\alpha) > 0$ *and a are any complex numbers;*

(ii) ρ *and* υ *are arbitrary such that* $\Re(\rho + \upsilon n) > 0$.

2.4. Right-sided fractional differentiation of generalized multiindex Bessel function

In this section, we establish image formulas for the generalized multiindex Bessel function involving right sided operators of Saigo fractional differentiation operators (2.8), in term of the generalized Wright function.

Theorem 2.7. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1, \cdots, m)$ are arbitrary parameters and $D_{0-}^{\alpha,\beta,\gamma}(.)$ be the Saigo left-sided fractional differentiation operator (2.8), then the following result holds:*

$$\left(D_{0-}^{\alpha,\beta,\gamma}\left(t^{\rho} J_{(\eta_j)_m, k}^{(\mu_j)_m, \tau}(a t^{-\upsilon})\right)\right)(x) = \frac{x^{\alpha+\beta+\rho}}{\Gamma(\tau)}\, {}_3\psi_{m+2}\left[\begin{matrix}(-\beta-\rho, \upsilon), (\gamma-\rho, \upsilon), (\tau, k) \\ (-\rho, \upsilon), (\gamma-\beta, -\rho\,\upsilon), (\eta_j+1, \mu_j)_1^m\end{matrix} \middle| -a x^{-\upsilon}\right]. \tag{2.15}$$

The conditions for validity of (2.15) are

(i) α, β, γ, $\Re(\alpha) > 0$ *and a are any complex numbers;*

(ii) ρ *and* υ *are arbitrary, such that* $\Re(\gamma - \rho + \upsilon n) > 0$ *and* $\Re(-\beta - \rho + \upsilon n) > 0$.

Proof. Denote L.H.S. of Theorem 2.7 by I_4, then

$$I_4 \equiv \left(D_{0-}^{\alpha,\beta,\gamma}\left(t^{\rho} J_{(\eta_j)_m, k}^{(\mu_j)_m, \tau}(a t^{-\upsilon})\right)\right)(x),$$

by virtue of (1.1) and (2.8), we have

$$\begin{aligned}
&= \left(-\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k, -\beta-k, \alpha+\gamma}\right)(t^{\rho})\, J_{(\eta_j)_m, k}^{(\mu_j)_m, \tau}(a t^{-\upsilon}), \\
&= \left(-\frac{d}{dx}\right)^k \frac{x^{\alpha+\beta}}{\Gamma(-\alpha+k)} \int_x^{\infty} (t-x)^{-\alpha+k-1} t^{\alpha+\beta}\, {}_2F_1(-\alpha-\beta, -\gamma-\alpha; -\alpha+k; 1-x/t) \\
&\quad \times (t^{\rho}) J_{(\eta_j)_m, k}^{(\mu_j)_m, \tau}(a t^{-\upsilon}) dt,
\end{aligned}$$

now, interchanging the order of integration and summation is permissible under the conditions

$$\begin{aligned}
&= \frac{x^{\alpha+\beta+\rho}}{\Gamma(\tau)} \sum_{n=0}^{\infty} \frac{\Gamma(-\beta-\rho+\upsilon n)\,\Gamma(\gamma-\rho+\upsilon n)\,\Gamma(\tau+kn)}{\Gamma(-\rho+\upsilon n)\,\Gamma(\gamma-\beta-\rho+\upsilon n)\Gamma(\eta_j+1+\mu_j n)} \frac{(-a t^{-\upsilon})^n}{n!} \\
&= \frac{x^{\alpha+\beta+\rho}}{\Gamma(\tau)}\, {}_3\psi_{m+2}\left[\begin{matrix}(-\beta-\rho, \upsilon), (\gamma-\rho, \upsilon), (\tau, k) \\ (-\rho, \upsilon), (\gamma-\beta, -\rho\,\upsilon), (\eta_j+1, \mu_j)_1^m\end{matrix} \middle| -a x^{-\upsilon}\right],
\end{aligned}$$

which completes the proof of Theorem 2.7. □

Corollary 2.8. *Let $m \geqslant 1$ be an integer, $\Re(\mu_j) > 0$, η_j $(i = 1, \cdots, m)$ are arbitrary parameters and $D_{0-}^{\alpha}(.)$ be the Riemann Liouville left-sided fractional differentiation operator (2.8), then the following result holds:*

$$\left(D_{0-}^{\alpha}\left(t^{\rho} J_{(\eta_j)_m, k}^{(\mu_j)_m, \tau}(a t^{-\upsilon})\right)\right)(x) = \frac{x^{\rho}}{\Gamma(\tau)}\, {}_2\psi_{m+1}\left[\begin{matrix}(\gamma-\rho, \upsilon), (\tau, k) \\ (\gamma+\alpha, -\rho\,\upsilon), (\eta_j+1, \mu_j)_1^m\end{matrix} \middle| -a x^{-\upsilon}\right]. \tag{2.16}$$

The conditions for validity of (2.16) are

(i) $\alpha, \Re(\alpha) > 0$ *and* a *are any complex numbers;*

(ii) ρ *and* υ *are arbitrary, such that* $\Re(\gamma - \rho + \upsilon n) > 0$.

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