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# Common fixed points of mappings satisfying implicit relations in partial metric spaces

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**Abstract**

Matthews, [S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728, 1994, pp. 183-197], introduced and studied the concept of partial metric space, as a part of the study of denotational semantics of dataflow networks. He also obtained a Banach type fixed point theorem on complete partial metric spaces. Very recently Berinde and Vetro, [V. Berinde, F. Vetro, Common fixed points of mappings satisfying implicit contractive conditions, Fixed Point Theory and Applications 2012, 2012:105], discussed, in the setting of metric and ordered metric spaces, coincidence point and common fixed point theorems for self-mappings in a general class of contractions defined by an implicit relation. In this work, in the setting of partial metric spaces, we study coincidence point and common fixed point theorems for two self-mappings satisfying generalized contractive conditions, defined by implicit relations. Our results unify, extend and generalize some related common fixed point theorems of the literature.

Keywords: Coincidence point, common fixed point, contraction, implicit relation, partial metric space.
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## 1. Introduction and Preliminaries

In 1992, Matthews [20] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks. Since then, it is widely recognized that partial metric spaces play a fundamental role in developing models in the theory of computation [24, 31, 33, 38]. Here, we recall some definitions and properties [20, 23, 24, 30, 35] of partial metric spaces, see also [4, 5, 14, 15, 18, 25, 37]. Throughout this paper the letters $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all non negative real numbers and the set of all positive integer numbers.

**Definition 1.1.** A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

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(p1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(p2) $p(x,x) \leqslant p(x,y)$;
(p3) $p(x,y) = p(y,x)$;
(p4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair $(X,p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$.

*Remark* 1.2. It is clear that if $p(x,y) = 0$, then from (p1) and (p2), $x = y$, but if $x = y$, then $p(x,y)$ may not be 0.

The pair $(\mathbb{R}_+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}_+$, is a simple example of a partial metric space.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If $p$ is a partial metric on $X$, then the function $p^s : X \times X \to \mathbb{R}_+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

is a metric on $X$.

**Definition 1.3.** Let $(X,p)$ be a partial metric space and $\{x_n\}$ be a sequence in $X$. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$;
(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

**Definition 1.4.** A partial metric space $(X,p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$, such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

It is easy to see that every closed subset of a complete partial metric space is complete.

**Lemma 1.5** ([20, 23])**.** *Let $(X,p)$ be a partial metric space. Then*

*(a) $\{x_n\}$ is a Cauchy sequence in $(X,p)$ if and only if it is a Cauchy sequence in the metric space $(X,p^s)$;*
*(b) $(X,p)$ is complete if and only if the metric space $(X,p^s)$ is complete. Furthermore, $\lim_{n\to+\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Using the above concepts, Matthews [20] obtained the following Banach fixed point theorem on a complete partial metric space.

**Theorem 1.6.** *Let $f$ be a mapping of a complete partial metric space $(X,p)$ into itself such that there is a real number $k$ with $k \in [0,1)$, satisfying for all $x, y \in X$ :*

$$p(fx,fy) \leqslant kp(x,y).$$

*Then $f$ has a unique fixed point.*

It is well know that, starting from the Banach fixed point theorem [7], the study of fixed and common fixed points of mappings satisfying a certain metrical contractive condition attracted many researchers, see for example [32]. In particular, among these results, we refer to the works [8, 9] of Berinde that obtained also a constructive method for finding fixed points by considering self-mappings that satisfy an explicit contractive type condition.

On the other hand, Popa [26, 27], initiated a study of implicit contractive type conditions for proving easily several classical fixed point theorems, see also [2, 3].

In particular, we recall that Berinde [9], to obtain some constructive fixed point theorems for almost contractions satisfying an implicit relation, considered the family $\mathcal{F}$ of all continuous real functions $F : \mathbb{R}_+^6 \to \mathbb{R}_+$ and the following conditions:

($F_{1a}$) $F$ is nonincreasing in the fifth variable and $F(u, v, v, u, u+v, 0) \leqslant 0$ for $u, v \geqslant 0$ implies that there exists $h \in [0, 1)$ such that $u \leqslant hv$;

($F_{1b}$) $F$ is nonincreasing in the fourth variable and $F(u, v, 0, u+v, u, v) \leqslant 0$ for $u, v \geqslant 0$ implies that there exists $h \in [0, 1)$ such that $u \leqslant hv$;

($F_{1c}$) $F$ is nonincreasing in the third variable and $F(u, v, u+v, 0, v, u) \leqslant 0$ for $u, v \geqslant 0$ implies that there exists $h \in [0, 1)$ such that $u \leqslant hv$;

($F_2$) $F(u, u, 0, 0, u, u) > 0$, for all $u > 0$.

In this way Berinde unified and extended various results, see [1], [6], [8]-[11], [17, 19], [26, 28].

**Example 1.7.** The following functions $F \in \mathcal{F}$ satisfy the properties $(F_2)$ and $(F_{1a})$-$(F_{1c})$ (see Examples 1-6, 9 and 11 of [9]).

(i) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2$, where $a \in [0, 1)$;
(ii) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4)$, where $b \in [0, 1/2)$;
(iii) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c(t_5 + t_6)$, where $c \in [0, 1/2)$;
(iv) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$, where $a \in [0, 1)$;
(v) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where $a, b, c \in [0, 1)$ and $a + 2b + 2c < 1$;
(vi) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, \frac{t_3+t_4}{2}, t_5, t_6\}$, where $a \in [0, 1)$;
(vii) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - L \min\{t_3, t_4, t_5, t_6\}$, where $a \in [0, 1)$;
(viii) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\} - L \min\{t_3, t_4, t_5, t_6\}$, where $a \in [0, 1)$ and $L \geqslant 0$.

**Example 1.8.** The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, t_3, t_4, t_5, t_6\},$$

where $a \in [0, 1/2)$ satisfies the properties $(F_2)$ and $(F_{1a})$-$(F_{1c})$ with $h = \dfrac{a}{1-a} < 1$.

**Example 1.9.** The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 \frac{t_5 + t_6}{t_3 + t_4},$$

where $a \in (0, 1)$ satisfies the property $(F_{1a})$ with $h = a$ but does not satisfy the properties $(F_{1b})$, $(F_{1c})$ and $(F_2)$.

**Example 1.10.** The function $F \in \mathcal{F}$, given by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_3 \frac{t_5 + t_6}{t_2 + t_4},$$

where $a \in (0, 1)$ satisfies the properties $(F_{1a})$ with $h = a \in (0, 1)$ and $(F_2)$ but does not satisfy the properties $(F_{1b})$ and $(F_{1c})$.

In the sequel, we need also the following definitions.

**Definition 1.11.** Let $X$ be a non-empty set and $f, T : X \to X$. A point $x \in X$ is called a coincidence point of $f$ and $T$ if $Tx = fx$.

**Definition 1.12.** The mappings $f$ and $T$ are said to be weakly compatible if they commute at their coincidence point, that is, $Tfx = fTx$ whenever $Tx = fx$.

**Definition 1.13.** Suppose $TX \subset fX$. For every $x_0 \in X$ we consider the sequence $\{x_n\} \subset X$ defined by $fx_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, we say that $\{Tx_n\}$ is a T-f-sequence with initial point $x_0$.

**Definition 1.14.** Let $X$ be a nonempty set. If $(X, p)$ is a partial metric space and $(X, \preceq)$ is partially ordered, then $(X, p, \preceq)$ is called an ordered partial metric space. Then, $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $f, T : X \to X$ be two self-mappings, $T$ is said to be $f$-nondecreasing if $fx \preceq fy$ implies $Tx \preceq Ty$ for all $x, y \in X$. If $f$ is the identity mapping on $X$, then $T$ is nondecreasing.

Starting from the concept of partially ordered set, the existence of fixed points in ordered metric spaces was largely investigated by many researchers, some of these are Turinici [34], Ran and Reurings [29], Nieto and Rodríguez-López [22]. For more details on this topic, we also refer to [12, 13, 16, 21, 36] and references therein.

In this paper, in the setting of partial metric spaces and ordered partial metric spaces, we state and prove coincidence point and common fixed point results for self-mappings satisfying contractive conditions that are defined by an implicit relation. Our results extend and generalize some related common fixed point theorems of the literature.

## 2. Main results

The following Lemma is useful in the sequel.

**Lemma 2.1.** *Let $(X, p)$ be a partial metric space and $T, f : X \to X$ be self-mappings. Assume that there exists $F \in \mathcal{F}$ satisfying $(F_{1a})$ such that, for all $x, y \in X$, we have*

$$F(p(Tx, Ty), p(fx, fy), p(fx, Tx), p(fy, Ty), p(fx, Ty), p(fy, Tx) - p(fy, fy)) \leqslant 0. \tag{2.1}$$

*Then, for all $z \in X$ such that $fz = Tz$ we have $p(Tz, Tz) = p(fz, fz) = 0$.*

*Proof.* Assume $p(Tz, Tz) > 0$, then using (2.1) with $x = y = z$ we get

$$F(p(Tz, Tz), p(fz, fz), p(fz, Tz), p(fz, Tz), p(fz, Tz), p(fz, Tz) - p(fz, fz)) \leqslant 0.$$

This implies $F(u, v, v, u, u + v, 0) \leqslant 0$, where $u = v = p(Tz, Tz)$ and so by $(F_{1a})$ there exists $h \in [0, 1)$ such that $u \leqslant hv = hu$. It follows $u = p(Tz, Tz) = 0$. □

Our first main theorem is essentially inspired by Berinde and Vetro [10].

**Theorem 2.2.** *Let $(X, p)$ be a partial metric space and $T, f : X \to X$ be self-mappings such that $TX \subseteq fX$. Assume that there exists $F \in \mathcal{F}$ satisfying $(F_{1a})$ such that, for all $x, y \in X$, condition (2.1) holds. If $fX$ is a 0-complete subspace of $X$, then $T$ and $f$ have a coincidence point. Moreover, if $T$ and $f$ are weakly compatible and $F$ satisfies also $(F_2)$, then $T$ and $f$ have a unique common fixed point. Further, for any $x_0 \in X$, the $T$-$f$-sequence $\{Tx_n\}$ with initial point $x_0$ converges to the common fixed point.*

*Proof.* Let $x_0 \in X$ be an arbitrary point. As $TX \subseteq fX$, one can choose a $T$-$f$-sequence $\{Tx_n\}$ with initial point $x_0$. Assume $x = x_n$ and $y = x_{n+1}$ in (2.1) and denote $u := p(Tx_n, Tx_{n+1})$ and $v := p(Tx_{n-1}, Tx_n)$, then we have

$$F(u, v, v, u, p(Tx_{n-1}, Tx_{n+1}), 0) \leqslant 0.$$

By (p4) of Definition 1.1, we get

$$p(Tx_{n-1}, Tx_{n+1}) \leqslant p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1}) - p(Tx_n, Tx_n) \leqslant u + v$$

and, since $F$ is nonincreasing in the fifth variable, we have

$$F(u, v, v, u, u + v, 0) \leqslant 0$$

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and hence, by $(F_{1a})$ there exists $h \in [0,1)$ such that $u \leqslant hv$, that is

$$p(Tx_n, Tx_{n+1}) \leqslant h\, p(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

We note that (2.2) and (p2) of Definition 1.1 imply that

$$\lim_{n\to+\infty} p(Tx_n, Tx_n) \leqslant \lim_{n\to+\infty} p(Tx_n, Tx_{n+1}) \leqslant \lim_{n\to+\infty} h^n p(Tx_0, Tx_1) = 0.$$

Now, using (2.2), it is easy to show that $\{Tx_n\}$ is a Cauchy sequence. Since $fX$ is 0-complete, there exist $z, w \in X$ such that $z = fw$ and

$$0 = p(z,z) = \lim_{n\to+\infty} p(Tx_n, z) = \lim_{n\to+\infty} p(fx_n, z) = p(fw, fw). \tag{2.3}$$

From (2.3) and the inequality

$$p(fw, Tw) + p(Tx_n, Tx_n) - p(fw, Tx_n) \leqslant p(Tx_n, Tw) \leqslant p(Tx_n, fw) + p(fw, Tw),$$

we get

$$\lim_{n\to+\infty} p(Tx_n, Tw) = p(fw, Tw).$$

Now, using (2.1) with $x = x_n$ and $y = w$, we get

$$F(p(Tx_n, Tw), p(fx_n, fw), p(fx_n, Tx_n), p(fw, Tw), p(fx_n, Tw), p(fw, Tx_n) - p(fw, fw)) \leqslant 0. \tag{2.4}$$

Using the continuity of $F$, (2.3) and letting $n \to +\infty$ in (2.4), we have

$$F(p(fw, Tw), p(fw, fw), p(fw, fw), p(fw, Tw), p(fw, Tw), p(fw, fw) - p(fw, fw)) \leqslant 0,$$

that is,

$$F(p(fw, Tw), 0, 0, p(fw, Tw), p(fw, Tw) + 0, 0) \leqslant 0,$$

which, by assumption $(F_{1a})$ yields $p(fw, Tw) \leqslant 0$, and by (p2) of Definition 1.1, it follows $p(fw, Tw) = 0$, that is, $fw = Tw = z$. In this way, we showed that $T$ and $f$ have a coincidence point.

Now, we assume that $T$ and $f$ are weakly compatible, then $fz = fTw = Tfw = Tz$. We will show that $Tz = z = Tw$.

Suppose $p(Tz, Tw) > 0$ and let $x = z$ and $y = w$ in (2.1), then we obtain

$$F(p(Tz, Tw), p(fz, fw), p(fz, Tz), p(fw, Tw), p(fz, Tw), p(fw, Tz) - p(fw, fw)) \leqslant 0,$$

that is

$$F(p(Tz, Tw), p(Tz, Tw), p(Tz, Tz), 0, p(Tz, Tw), p(Tz, Tw)) \leqslant 0.$$

Now, by Lemma 2.1 we have $p(Tz, Tz) = 0$ and so from the previous inequality we obtain $F(u, u, 0, 0, u, u) \leqslant 0$, where $u = p(Tz, Tw)$, which is a contradiction by assumption $(F_2)$. This implies that $p(Tz, Tw) = 0$ and hence $fz = Tz = Tw = z$, that is, $T$ and $f$ have a common fixed point.

To prove the uniqueness of the common fixed point, it is suffices to use again the assumption $(F_2)$ and so, to avoid repetition, we omit the details. Finally, to complete the proof, we observe that for any $x_0 \in X$, the $T$-$f$-sequence $\{Tx_n\}$ with initial point $x_0$ converges to the unique common fixed point. □

If $f$ is the identity mapping on $X$, from Theorem 2.2 we obtain the following corollary.

**Corollary 2.3.** *Let $(X, p)$ be a 0-complete metric space and $T : X \to X$ be a self-mapping. Assume that there exists $F \in \mathcal{F}$ satisfying $(F_{1a})$ such that, for all $x, y \in X$, we have*

$$F(p(Tx, Ty), p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx) - p(y, y)) \leqslant 0.$$

*Then $T$ has a fixed point. Moreover, if $F$ satisfies also $(F_2)$, then $T$ has a unique fixed point. Further, for any $x_0 \in X$, the Picard sequence $\{Tx_n\}$ with initial point $x_0$ converges to the fixed point.*

In view of the constructive character of Theorem 2.2 and from (2.2) we deduce the following unifying error estimate

$$p(Tx_{n+i-1}, z) \leqslant \frac{h^i}{1-h}\, p(Tx_{n-1}, Tx_n).$$

Then, from this we get both the a priori estimate

$$p(Tx_n, z) \leqslant \frac{h^n}{1-h}\, p(Tx_0, Tx_1), \qquad n \in \mathbb{N}$$

and the a posteriori estimate

$$p(Tx_n, z) \leqslant \frac{h}{1-h}\, p(Tx_{n-1}, Tx_n), \qquad n \in \mathbb{N}$$

which play an important role in applications, i.e., consider the problem of approximating the solutions of nonlinear equations.

Now, we state and prove a common fixed point result for two self-mappings satisfying an implicit contractive condition in the setting of ordered partial metric spaces.

**Theorem 2.4.** *Let $(X, p, \preceq)$ be a 0-complete ordered metric space and $T, f : X \to X$ be self-mappings such that $TX \subseteq fX$. Assume that there exists $F \in \mathcal{F}$ satisfying $(F_{1a})$ such that, for all $x, y \in X$ with $fx \preceq fy$, we have*

$$F(p(Tx, Ty), p(fx, fy), p(fx, Tx), p(fy, Ty), p(fx, Ty), p(fy, Tx) - p(fy, fy)) \leqslant 0. \tag{2.5}$$

*If the following conditions hold:*

*(i) there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$;*
*(ii) $T$ is $f$-nondecreasing;*
*(iii) for a nondecreasing sequence $\{fx_n\} \subseteq X$ converging to $fw \in X$, we have $fx_n \preceq fw$ for all $n \in \mathbb{N}$ and $fw \preceq ffw$,*

*then $T$ and $f$ have a coincidence point in $X$. Moreover, if $T$ and $f$ are weakly compatible and $F$ satisfies $(F_2)$, then $T$ and $f$ have a common fixed point. Further, for any $x_0 \in X$, the $T$-$f$-sequence $\{Tx_n\}$ with initial point $x_0$ converges to a common fixed point.*

*Proof.* Let $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and let $\{Tx_n\}$ be a $T$-$f$-sequence with initial point $x_0$. Since $fx_0 \preceq Tx_0$ and $Tx_0 = fx_1$, we have $fx_0 \preceq fx_1$. As $T$ is $f$-nondecreasing we get that $Tx_0 \preceq Tx_1$. Continuing this process we obtain

$$fx_0 \preceq Tx_0 = fx_1 \preceq Tx_1 = fx_2 \preceq \cdots \preceq Tx_n = fx_{n+1} \preceq \cdots.$$

In what follows we will suppose that $p(Tx_n, Tx_{n+1}) > 0$ for all $n \in \mathbb{N}$. In fact, if $Tx_n = Tx_{n+1}$ for some $n$, then $fx_{n+1} = Tx_n = Tx_{n+1}$ and so $x_{n+1}$ is a coincidence point for $T$ and $f$ and the result is proved. As $fx_n \preceq fx_{n+1}$ for all $n \in \mathbb{N}$, if we take $x = x_n$ and $y = x_{n+1}$ in (2.5) and denote $u := p(Tx_n, Tx_{n+1})$ and $v := p(Tx_{n-1}, Tx_n)$, we get

$$F(u, v, v, u, p(Tx_{n-1}, Tx_{n+1}), 0) \leqslant 0.$$

By (p4) of Definition 1.1, we have

$$p(Tx_{n-1}, Tx_{n+1}) \leqslant p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1}) - p(Tx_n, Tx_n) \leqslant u + v$$

and, since $F$ is nonincreasing in the fifth variable, we get

$$F(u, v, v, u, u+v, 0) \leqslant 0$$

and hence, in view of assumption $(F_{1a})$, there exists $h \in [0, 1)$ such that $u \leqslant hv$, that is

$$p(Tx_n, Tx_{n+1}) \leqslant h\, p(Tx_{n-1}, Tx_n). \tag{2.6}$$

By (2.6), we deduce that $\{Tx_n\}$ is a Cauchy sequence. Now, since $(X, p)$ is 0-complete, there exist $z, w \in X$ such that $z = fw$ and

$$0 = p(z, z) = \lim_{n \to +\infty} p(Tx_n, z) = \lim_{n \to +\infty} p(fx_n, z) = p(fw, fw). \tag{2.7}$$

By condition $(iii)$, $fx_n \preceq fw$ for all $n \in \mathbb{N}$, if we take $x = x_n$ and $y = w$ in (2.5) we get

$$F(p(Tx_n, Tw), p(fx_n, fw), p(fx_n, Tx_n), p(fw, Tw), p(fx_n, Tw), p(fw, Tx_n) - p(fw, fw)) \leqslant 0.$$

Since

$$\lim_{n \to +\infty} p(Tx_n, Tw) = p(fw, Tw) \text{ and } \lim_{n \to +\infty} p(Tx_n, Tx_{n+1}) = 0,$$

using the continuity of $F$, (2.7) and letting $n \to +\infty$ we obtain

$$F(p(fw, Tw), 0, 0, p(fw, Tw), p(fw, Tw), 0) \leqslant 0$$

which, by assumption $(F_{1a})$, yields $p(fw, Tw) \leqslant 0$, and by $(p2)$ of Definition 1.1, it follows $p(fw, Tw) = 0$, that is, $fw = Tw$. In this way, we showed that $T$ and $f$ have a coincidence point.

If $T$ and $f$ are weakly compatible we can also show that $z$ is a common fixed point for $T$ and $f$. In fact, as $fz = fTw = Tfw = Tz$, by condition $(iii)$, we have that $fw \preceq ffw = fz$.

Now, for $x = w$ and $y = z$ in (2.5), we get

$$F(p(Tw, Tz), p(fw, fz), p(fw, Tw), p(fz, Tz), p(fw, Tz), p(fz, Tw) - p(fw, fw)) \leqslant 0.$$

Since $p(Tz, Tz) = p(fz, fz) = 0$ by Lemma 2.1, assumption $(F_2)$ implies that $d(Tz, Tw) = 0$ and hence $fz = Tz = Tw = z$, that is, $T$ and $f$ have a common fixed point. As in the proof of Theorem 2.2, to conclude we have only to observe that, for any $x_0 \in X$, the $T$-$f$-sequence $\{Tx_n\}$ with initial point $x_0$ converges to a common fixed point. □

If we add some hypotheses to Theorem 2.4, we are ready to prove the uniqueness of the common fixed point. Precisely, we give the following result.

**Theorem 2.5.** *Let all the conditions of Theorem 2.4 be satisfied. If the following conditions hold:*

*(iv) for all $x, y \in fX$ there exists $v_0 \in X$ such that $fv_0 \preceq x$, $fv_0 \preceq y$;*
*(v) $F$ satisfies $(F_{1c})$,*

*then $T$ and $f$ have a unique common fixed point.*

*Proof.* Let $z, w$ be two common fixed points of $T$ and $f$ with $z \neq w$. If $z$ and $w$ are comparable, say $z \preceq y$. Then for $x = z$ and $y = w$ in (2.5), we get

$$F(p(Tz, Tw), p(fz, fw), p(fz, Tz), p(fw, Tw), p(fz, Tw), p(fw, Tz) - p(fw, fw)) \leqslant 0,$$

which is a contradiction by assumption $(F_2)$ and so $z = w$.

If $z$ and $w$ are not comparable, then there exists $v_0 \in X$ such that $fv_0 \preceq fz = z$ and $fv_0 \preceq fw = w$.

As $T$ is $f$-nondecreasing, from $fv_0 \preceq fz$ we get that

$$fv_1 = Tv_0 \preceq Tz = fz.$$

Continuing this process, we obtain

$$fv_{n+1} = Tv_n \preceq Tz = fz \quad \text{for all } n \in \mathbb{N}.$$

Then, for $x = v_n$ and $y = z$ in (2.5) we have

$$F(p(Tv_n, Tz), p(fv_n, fz), p(fv_n, Tv_n), p(fz, Tz), p(fv_n, Tz), p(fz, Tv_n) - p(fz, fz)) \leqslant 0,$$

that is

$$F(p(Tv_n, Tz), p(Tv_{n-1}, Tz), p(Tv_{n-1}, Tv_n), p(fz, Tz), p(Tv_{n-1}, Tz), p(Tz, Tv_n)) \leqslant 0.$$

Denote $u := p(Tv_n, Tz)$ and $v := p(Tv_{n-1}, Tz)$. As $F$ is nonincreasing in the third variable, we get

$$F(u, v, u+v, 0, v, u) \leqslant 0.$$

By assumption $(F_{1c})$, there exists $h \in [0, 1)$ such that $u \leqslant hv$, that is

$$p(Tv_n, Tz) \leqslant h\, p(Tv_{n-1}, Tz) \quad \text{for all } n \in \mathbb{N}.$$

This implies that $p(Tv_n, Tz) = p(Tv_n, z) \to 0$ as $n \to +\infty$.

With similar arguments, we deduce that $p(Tv_n, w) \to 0$ as $n \to +\infty$. Hence

$$0 < p(w, z) \leqslant p(w, Tv_n) + p(Tv_n, z) - p(Tv_n, Tv_n) \to 0$$

as $n \to +\infty$, which is a contradiction. Thus $T$ and $f$ have a unique common fixed point. □

If $f$ is the identity mapping on $X$, from Theorems 2.4 and 2.5, we deduce the following results of fixed point for a self-mapping.

**Corollary 2.6.** *Let $(X, p, \preceq)$ be a 0-complete ordered metric space and $T : X \to X$ be a self-mapping. Assume that there exists $F \in \mathcal{F}$ satisfying $(F_{1a})$ such that, for all $x, y \in X$ with $x \preceq y$, we have*

$$F(p(Tx, Ty), p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx) - p(y, y)) \leqslant 0. \tag{2.8}$$

*If the following conditions hold:*

*(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*
*(ii) $T$ is nondecreasing;*
*(iii) for a nondecreasing sequence $\{x_n\} \subseteq X$ converging to $w \in X$, we have $x_n \preceq w$ for all $n \in \mathbb{N}$,*

*then $T$ has a fixed point in $X$. Further, for any $x_0 \in X$, the Picard sequence $\{Tx_n\}$ with initial point $x_0$ converges to a fixed point.*

**Corollary 2.7.** *Let all the conditions of Corollary 2.6 be satisfied. If the following conditions hold:*

*(iv) $F$ satisfies $(F_2)$;*
*(v) for all $x, y \in X$ there exists $v_0 \in X$ such that $v_0 \preceq x$, $v_0 \preceq y$;*
*(vi) $F$ satisfies $(F_{1c})$,*

*then $T$ has a unique common fixed point.*

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