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Korovkin type approximation theorem via lacunary equi-statistical convergence in fuzzy spaces

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Abstract

In the present paper, we establish relations between equi-statistical convergence and lacunary equi-statistical convergence of sequences of fuzzy number valued functions. We make an effort to prove Korovkin type approximation theorem via lacunary equi-statistical convergence in fuzzy spaces. Further, we study rates of lacunary equi-statistical fuzzy convergence by using fuzzy modulus of continuity.

Keywords: Fuzzy number, lacunary equi-statistical convergence, Korovkin type approximation theorem, fuzzy rate, fuzzy positive linear operator.

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1. Introduction

The concept of fuzzy numbers and arithmetic operations on these numbers were first introduced and investigated by Zadeh [32] in 1965. Subsequently, many authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. Since 1951, when the concept of statistical convergence was introduced independently by Fast [12] and Steinhaus [30], the statistical convergence has been further studied by Connor [9], Fridy [13], Miller and Orhan [18], Balcerzak et al. [5] and others. The statistical convergence for a sequence of fuzzy numbers has been studied by several authors. The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but Nanda [24], Kumar et al. [17], Mursaleen and Basarir [23] extended the idea to apply to sequences of fuzzy numbers. Savas [29] in 2001, introduced statistical convergence for a sequence of fuzzy numbers. Later, Aytar et al. [4] extended the concept of statistical superior limit and inferior limit to statistically bounded sequence of fuzzy numbers. To know more about statistical convergence of fuzzy numbers one can refer to Basarir and Mursaleen [6], Colak et al. [8], and Raj et al. [28]. Aktuglu and Gezer [1] studied classical notion of lacunary equi-statistical convergence of positive linear operators.

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Later on, Edely et al. [10] proved some Korovkin type approximation theorems using the concept of λ-statistical convergence. Mursaleen and Alotaibi [22] proved Korovkin type approximation theorem for a function of two variables by using the notion of statistical A-summability. They have studied the rate of statistical A-summablility of positive linear operators. Hazarika and Esi [15] studied ideal summability and a Korovkin type approximation Theorem. A lot of developments have been made in this area, one may refer to [7, 11, 14, 16, 19–21, 26, 31].

Motivated by the above cited results, we investigate the notion of equi-statistical convergence, lacunary statistically uniform convergence, lacunary equi-statistical convergence of sequences of fuzzy numbers. We also give a Korovkin type approximation theorem using the notion of lacunary equi-statistical convergence in fuzzy spaces. Further, we have calculated rates of lacunary equi-statistical fuzzy convergence with the help of fuzzy modulus of continuity.

A fuzzy number is a function $u : \mathbb{R} \to [0, 1]$, which satisfies the following conditions:

(i) u is normal, i.e., there exists an x_0 such that $u(x_0) = 1$;

(ii) u is convex, i.e., for $x, y \in \mathbb{R}$ and $0 \leqslant \tau \leqslant 1$,
$$u(\tau x + (1-\tau)y) \geqslant \min\{u(x), u(y)\};$$

(iii) u is upper semi-continuous;

(iv) The closure of the set $\text{supp}(u)$ is compact, where $\text{supp}(u) = \{x \in \mathbb{R} : u(x) > 0\}$ and it is denoted by $[u]^0$.

The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathbb{F}}$. Let $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ and the α-level set is $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geqslant \alpha\}, (0 < \alpha \leqslant 1)$. The set $[u]^\alpha$ is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathbb{F}}$ and $\lambda \in \mathbb{R}$, it is positive to define uniquely the sum $u \oplus v$ and the product $u \odot v$ as follows:
$$[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha \text{ and } [\lambda \odot u]^\alpha = \lambda [u]^\alpha.$$

Now, denote the interval $[u]^\alpha$ by $[u_-^{(\alpha)}, u_+^{(\alpha)}]$, where $u_-^{(\alpha)} \leqslant u_+^{(\alpha)}$ and $u_-^{(\alpha)}, u_+^{(\alpha)} \in \mathbb{R}$ for $\alpha \in (0, 1]$. The partial ordering relation on $\mathbb{R}_{\mathbb{F}}$ is defined as follows
$$u \preceq v \Leftrightarrow u_-^{(\alpha)} \leqslant v_-^{(\alpha)} \text{ and } u_+^{(\alpha)} \leqslant v_+^{(\alpha)} \text{ for all } 0 < \alpha \leqslant 1.$$

The Hausdörff distance [25] between two fuzzy numbers is a function $d : \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}$ defined by
$$d(u, v) = \sup_{\alpha \in (0,1]} \max \left\{ \left| u_-^{(\alpha)} - v_-^{(\alpha)} \right|, \left| u_+^{(\alpha)} - v_+^{(\alpha)} \right| \right\}.$$

In this case, $(\mathbb{R}_{\mathbb{F}}, d)$ is a complete metric space [27]. Let $f, g : [a, b] \to \mathbb{R}_{\mathbb{F}}$ be fuzzy number valued functions. Then, the distance between f and g is given by
$$d^*(f, g) = \sup_{x \in [a,b]} \sup_{\alpha \in (0,1]} \max \left\{ \left| f_-^{(\alpha)} - g_-^{(\alpha)} \right|, \left| f_+^{(\alpha)} - g_+^{(\alpha)} \right| \right\}.$$

An increasing non-negative integer sequence $\theta = (k_r)$ with $k_0 = 0$ and $k_r - k_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and q_r denote the ratio $\frac{k_r}{k_{r-1}}$.

Let (X_k) be a sequence of fuzzy numbers. Then (X_k) is called statistically convergent to a fuzzy number X_0, if for every $\epsilon > 0$,
$$\lim_n \frac{1}{n} |\{k \leqslant n : d(X_k, X_0) \geqslant \epsilon\}| = 0.$$

Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistical convergent to the fuzzy number X_0, if for every $\epsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r : d(X_k, X_0) \geqslant \epsilon\}| = 0.$$

Now, assume that $f_r : [a, b] \to \mathbb{R}_{\mathbb{F}}, r \in \mathbb{N}$ be a sequence of fuzzy number valued functions and $f : [a, b] \to \mathbb{R}_{\mathbb{F}}$ be a fuzzy number valued function.

Definition 1.1. Let θ be a lacunary sequence. The sequence of fuzzy number valued functions (f_r) is said to be lacunary statistical pointwise convergent to f, if for every $\epsilon > 0$, and $x \in [a, b]$,

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : d^*(f_k(x), f(x)) \geqslant \epsilon\}| = 0.$$

In this case, we denote it by

$$f_r \to f \ \ (\theta\text{-stat}).$$

Definition 1.2. Let θ be a lacunary sequence. The sequence of fuzzy number valued functions (f_r) is said to be lacunary statistical uniformly convergent to f, if for every $\epsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : \sup_{x\in[a,b]} d^*(f_k(x), f(x)) \geqslant \epsilon\}| = 0.$$

In this case, we denote it by

$$f_r \rightrightarrows f \ \ (\theta\text{-stat}).$$

Definition 1.3. A sequence of fuzzy number valued functions (f_r) is said to be equi-statistical convergent to f, if for every $\epsilon > 0$,

$$\lim_{r\to\infty} s_{r,\epsilon}(x) = 0$$

uniformly with regards to $x \in [a, b]$, where

$$s_{r,\epsilon}(x) = \frac{1}{r}|\{k \leqslant r : d^*(f_k(x), f(x)) \geqslant \epsilon\}|.$$

In this case, we denote it by

$$f_r \twoheadrightarrow f \ \ (\text{equi-stat}).$$

Definition 1.4. Let θ be a lacunary sequence. The sequence of fuzzy number valued functions (f_r) is said to be lacunary equi-statistical convergent to f, if for every $\epsilon > 0$,

$$\lim_{r\to\infty} N_{r,\epsilon}(x) = 0$$

uniformly with regards to $x \in [a, b]$, where

$$N_{r,\epsilon}(x) = \frac{1}{h_r}|\{k \in I_r : d^*(f_k(x), f(x)) \geqslant \epsilon\}|.$$

In this case, we denote it by

$$f_r \twoheadrightarrow f \ \ (\theta\text{-equistat}).$$

Lemma 1.5. $f_r \rightrightarrows f \ \ (\theta\text{-}stat) \Rightarrow f_r \twoheadrightarrow f \ \ (\theta\text{-}equistat) \Rightarrow f_r \to f \ \ (\theta\text{-}stat)$.

The following examples show that, in general, the inverse implications do not hold true.

Example 1.6. Let θ be a lacunary sequence and for any $x \in [0,1]$. Consider a fuzzy-number-valued function f and the sequence of fuzzy-number-valued functions (f_r) are defined as follows:

$$f_r(x)(s) = \begin{cases} \mu_k(x)(s), & k = n^2,\ n = 1,2,3,\ldots, \\ \nu_k(x)(s), & k \neq n^2,\ n = 1,2,3,\ldots, \end{cases}$$

where,

$$\mu_k(x)(s) = \begin{cases} 0, & s \in (-\infty, k-1) \cup (k+1, +\infty), \\ s-k+1, & k-1 \leqslant s \leqslant k, \\ -s+k+1, & k < s \leqslant k+1. \end{cases}$$

and

$$\nu_k(x)(s) = \begin{cases} 0, & s \in (-\infty, \frac{2kx}{1+k^2x^2} - 1) \cup (\frac{2kx}{1+k^2x^2} + 1, +\infty), \\ s + \frac{(1-kx)^2}{1+k^2x^2}, & \frac{2kx}{1+k^2x^2} - 1 \leqslant s \leqslant \frac{2kx}{1+k^2x^2}, \\ \frac{(1+kx)^2}{1+k^2x^2} - s, & \frac{2kx}{1+k^2x^2} < s \leqslant \frac{2kx}{1+k^2x^2} + 1. \end{cases}$$

Then $f_r \to f$ (θ-stat), but $f_r \twoheadrightarrow f$ (θ-equistat) does not hold true.

Example 1.7. Let θ be a lacunary sequence and for any $x \in [0,1]$. Consider a fuzzy-number-valued function $f(x) = 0$ and the sequence of fuzzy-number-valued functions $f_r(x)$ are defined as follows:

$$f_r(x)(s) = \begin{cases} \mu_k(x)(s), & x \in [\frac{1}{2^n}, \frac{1}{2^{n-1}} - \frac{1}{2^{n+1}}], \\ \nu_k(x)(s), & x \in [\frac{1}{2^{n-1}} - \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}], \\ 0, & x \neq [\frac{1}{2^n}, \frac{1}{2^{n-1}}]. \end{cases}$$

where,

$$\mu_k(x)(s) = \begin{cases} 0, & s \in (-\infty, 2^{n+1}(x-\frac{1}{2^n})-1) \cup (2^{n+1}(x-\frac{1}{2^n})+1), +\infty), \\ s - 2^{n+1}(x-\frac{1}{2^n})+1, & 2^{n+1}(x-\frac{1}{2^n})-1 \leqslant s \leqslant 2^{n+1}(x-\frac{1}{2^n}), \\ -s + 2^{n+1}(x-\frac{1}{2^n})+1, & 2^{n+1}(x-\frac{1}{2^n}) \leqslant s \leqslant 2^{n+1}(x-\frac{1}{2^n})+1. \end{cases}$$

and

$$\nu_k(x)(s) = \begin{cases} 0, & s \in (-\infty, 2^{n+1}(x-\frac{1}{2^n})-1) \cup (2^{n+1}(x-\frac{1}{2^n})+1), +\infty), \\ s + 2^{n+1}(x-\frac{1}{2^n})+1, & -2^{n+1}(x-\frac{1}{2^n})-1 \leqslant s \leqslant -2^{n+1}(x-\frac{1}{2^n}), \\ -s - 2^{n+1}(x-\frac{1}{2^n})+1, & -2^{n+1}(x-\frac{1}{2^n}) \leqslant s \leqslant -2^{n+1}(x-\frac{1}{2^n})+1. \end{cases}$$

Then $f_r \twoheadrightarrow f$ (θ-equistat), but $f_r \rightrightarrows f$ (θ-equistat) does not hold true.

2. Some inclusion results

Theorem 2.1. *Let θ be a lacunary sequence. Then $f_r \twoheadrightarrow f$ (θ-equistat) implies $f_r \twoheadrightarrow f$ (equi-stat) if* $\limsup_r q_r < \infty$.

Proof. Suppose that $\limsup_r q_r < \infty$ and $f_r \twoheadrightarrow f$ (θ-equistat), then there exists a positive number $K > 0$ such that $q_r < K$. By the definition, for given $\epsilon > 0$, $N_{r,\epsilon}(x)$ converges uniformly with regards to $x \in [a,b]$, i.e.,

$$\lim_{r\to\infty} N_{r,\epsilon}(x) = 0.$$

In other words, there is a positive integer $r_0 \in \mathbb{N}$, such that

$$N_{r,\epsilon}(x) < \epsilon, \tag{2.1}$$

for all $r > r_0$. Now let $M = \max\{h_1, \dots, h_{r_0}\}$ and let n be any integer satisfying $k_{r-1} < n \leqslant k_r$, then we have

$$\begin{aligned} s_{n,\epsilon}(x) &= \frac{1}{n}|\{k \leqslant n : d^*(f_k(x), f(x)) \geqslant \epsilon\}| \\ &\leqslant \frac{1}{k_{r-1}}|\{k \leqslant k_r : d^*(f_k(x), f(x)) \geqslant \epsilon\}| \\ &= \frac{1}{k_{r-1}}\left\{\sum_{l=1}^{r_0} h_l N_{l,\epsilon}(x) + \sum_{l=r_0+1}^{r} h_l N_{l,\epsilon}(x)\right\} \leqslant \frac{Mr_0}{k_{r-1}} + \frac{1}{k_{r-1}}\sum_{l=r_0+1}^{r} h_l N_{l,\epsilon}(x). \end{aligned}$$

By using inequality (2.1) we have

$$s_{n,\epsilon}(x) \leqslant \frac{Mr_0}{k_{r-1}} + \frac{\epsilon(k_r - k_{r_0})}{k_{r-1}} \leqslant \frac{Mr_0}{k_{r-1}} + \epsilon q_r \leqslant \frac{Mr_0}{k_{r-1}} + \epsilon K$$

and the results follow immediately. □

Theorem 2.2. *Let θ be a lacunary sequence. Then $f_r \twoheadrightarrow f$ (equi-stat) implies $f_r \twoheadrightarrow f$ (θ-equistat) if $1 < \liminf_r q_r$.*

Proof. Suppose that $\liminf_r q_r > 1$ and $f_r \twoheadrightarrow f$. Then there exists $\gamma > 0$ such that $q_r \geqslant 1+\gamma$ for large r, which implies that

$$\frac{h_r}{k_r} \geqslant \frac{\gamma}{1+\gamma}.$$

Then for every $\epsilon > 0$ and for large r, we have

$$\begin{aligned} s_{k_r,\epsilon}(x) &= \frac{1}{k_r}|\{k \leqslant k_r : d^*(f_k(x), f(x)) \geqslant \epsilon\}| \\ &\geqslant \frac{1}{k_r}|\{k \in I_r : d^*(f_k(x), f(x)) \geqslant \epsilon\}| \\ &\geqslant \frac{\gamma}{(\gamma+1)h_r}|\{k \in I_r : d^*(f_k(x), f(x)) \geqslant \epsilon\}| \\ &\geqslant \frac{\gamma}{(\gamma+1)h_r} N_{r,\epsilon}(x), \end{aligned}$$

which proves the theorem. □

Theorem 2.3. *Let θ be a lacunary sequence. Then $f_r \twoheadrightarrow f$ (equi-stat) and $f_r \twoheadrightarrow f$ (θ-equistat) imply each other if and only if $1 < \liminf_r q_r \leqslant \limsup_r q_r < \infty$.*

Proof. Combining Theorems 2.1 and 2.2 we get the desired result. □

3. Korovkin-type theorem and rates of Lacunary equi-statistical fuzzy convergence

In this section, we use the concept of lacunary equi-statistical convergence to prove fuzzy Korovkin-type approximation theorem. A fuzzy number valued function $f : [a, b] \to \mathbb{R}_F$ is said to be fuzzy continuous at $x_0 \in [a, b]$ whenever $x_k \to x_0$, then $d(f(x_k), f(x_0)) \to 0$ as $k \to \infty$. In other words, we can say that on interval $[a, b]$, f is fuzzy continuous if it is fuzzy continuous at every point $x \in [a, b]$ and we denote the space of all fuzzy continuous functions on the interval $[a, b]$ by $B_F[a, b]$. In this case, $B_F[a, b]$ is only a cone, not a vector space. Now let $\xi : B_F[a, b] \to B_F[a, b]$ be an operator. Then, we say that ξ is fuzzy linear if, for every $\zeta_1, \zeta_2 \in \mathbb{R}$, $f_1, f_2 \in B_F[a, b]$, and $x \in [a, b]$,

$$\xi(\zeta_1 \odot f_1 \oplus \zeta_2 \odot f_2; x) = \zeta_1 \odot \xi(f_1; x) \oplus \zeta_2 \odot \xi(f_2; x).$$

Also ξ is called fuzzy positive linear operator if it is fuzzy linear and

$$\xi(f;x) \leqslant \xi(g;x)$$

for any $f, g \in B_{F[a,b]}$ and all $x \in [a,b]$ with

$$f(x) \leqslant g(x).$$

In this paper, we use the test function e_i which is given by $e_i(x) = x^i, i = 0,1,2$. Then, we have the following theorem.

Theorem 3.1. *Let θ be a lacunary sequence and let $\{\xi_m\}$ be a sequence of fuzzy positive linear operators from $B_F[a,b]$ into itself. Suppose that there exists a corresponding sequence $\{\bar{\xi}_m\}$ of positive linear operators from $C[a,b]$ (the space of all continuous real functions on $[a,b]$) into itself with the property*

$$\{\xi_m(f;x)\}_{\pm}^{(\alpha)} = \bar{\xi}_m(f_{\pm}^{(\alpha)};x), \tag{3.1}$$

for all $x \in [a,b]$, $f \in B_F[a,b]$ and $m \in \mathbb{N}$ (see [2]). Further assume that

$$\bar{\xi}_m(e_i,x) \twoheadrightarrow e_i(x) \ \ (\theta\text{-}equistat), \tag{3.2}$$

where $e_i(x) = x^i, i = 0,1,2$. Then for all $f \in B_F[a,b]$

$$\xi_m(f,x) \twoheadrightarrow f \ \ (\theta\text{-}equistat). \tag{3.3}$$

Proof. Suppose $f \in B_F[a,b]$, $x \in [a,b]$ and $\alpha \in (0,1]$. Since $f_{\pm}^{(\alpha)} \in C[a,b]$, for every $\epsilon > 0$, there exists a number $\rho > 0$ such that $|f_{\pm}^{(\alpha)}(y) - f_{\pm}^{(\alpha)}(x)| < \epsilon$ whenever $|y-x| < \rho$. Then for all $y \in [a,b]$, we have

$$|f_{\pm}^{(\alpha)}(y) - f_{\pm}^{(\alpha)}(x)| \leqslant \epsilon + 2R_{\pm}^{(r)} \frac{(y-x)^2}{\rho^2},$$

where $R_{\pm}^{(\alpha)} = \|f_{\pm}^{(\alpha)}\|$. Now, by using the positivity and linearity of the operators $\bar{\xi}_m$, we get

$$\begin{aligned}
|\bar{\xi}_m(f_{\pm}^{(\alpha)};x) - f_{\pm}^{(\alpha)}(x)| &\leqslant \bar{\xi}_m(|f_{\pm}^{(\alpha)}(y) - f_{\pm}^{(\alpha)}(x)|;x) + R_{\pm}^{(\alpha)}|\bar{\xi}_m(e_0;x) - e_0(x)| \\
&\leqslant \epsilon + (\epsilon + R_{\pm}^{(\alpha)})|\bar{\xi}_m(e_0;x) - e_0(x)| + \frac{2R_{\pm}^{(\alpha)}}{\rho^2}|\bar{\xi}_m((y-x)^2;x)| \\
&\leqslant \epsilon + (\epsilon + R_{\pm}^{(\alpha)} + \frac{2\sigma^2 R_{\pm}^{(\alpha)}}{\rho^2})|\bar{\xi}_m(e_0;x) - e_0(x)| \\
&\quad + \frac{4\sigma R_{\pm}^{(\alpha)}}{\rho^2}|\bar{\xi}_m(e_1;x) - e_1(x)| + \frac{2R_{\pm}^{(\alpha)}}{\rho^2}|\bar{\xi}_m(e_2;x) - e_2(x)| \\
&\leqslant \epsilon + U_{\pm}^{(\alpha)} \sum_{i=0}^{2} |\bar{\xi}_m(e_i;x) - e_i(x)|,
\end{aligned}$$

where $\sigma = \max\{|a|,|b|\}$. Also suppose $U_{\pm}^{(\alpha)} = \max\left\{\epsilon + R_{\pm}^{(\alpha)} + \frac{2\sigma^2 R_{\pm}^{(\alpha)}}{\rho^2}, \frac{4\sigma R_{\pm}^{(\alpha)}}{\rho^2}, \frac{2R_{\pm}^{(\alpha)}}{\rho^2}\right\}$. Now by taking supremum over $x \in [a,b]$, the above inequality becomes

$$\|\bar{\xi}_m(f_{\pm}^{(\alpha)};x) - f_{\pm}^{(\alpha)}(x)\| \leqslant \epsilon + U_{\pm}^{(\alpha)}\{\|\bar{\xi}_m(e_0) - e_0\| + \|\bar{\xi}_m(e_1) - e_1\| + \|\bar{\xi}_m(e_2) - e_2\|\}. \tag{3.4}$$

Now, by property (3.1), we have

$$d^*(\xi_m(f),f) = \sup_{x\in[a,b]} d((\xi_m(f;x),f(x)) = \sup_{x\in[a,b]} \sup_{\alpha\in(0,1]} \max\left\{\left|\bar{\xi}_m(f_-^{(\alpha)};x) - f_-^{(\alpha)}(x)\right|, \left|\bar{\xi}_m(f_+^{(\alpha)};x) - f_+^{(\alpha)}(x)\right|\right\}$$

$$= \sup_{\alpha\in(0,1]} \max\left\{\left\|\bar{\xi}_m(f_-^{(\alpha)}) - f_-^{(\alpha)}\right\|, \left\|\bar{\xi}_m(f_+^{(\alpha)}) - f_+^{(\alpha)}\right\|\right\}.$$

Now, combine the above inequality with (3.4), we get

$$d^*(\xi_m(f), f) \leqslant \epsilon + S\{\|\bar{\xi}_m(e_0) - e_0\| + \|\bar{\xi}_m(e_1) - e_1\| + \|\bar{\xi}_m(e_2) - e_2\|\},$$

where $S = \sup_{\alpha\in(0,1]} \max\{U_-^{(\alpha)}, U_+^{(\alpha)}\}$. Now for given $t > 0$, choose $0 < \epsilon < t$ and we define the following sets as:

$$V = \{m \in \mathbb{N} : d^*(\xi_m(f), f) \geqslant s\}, \qquad V_0 = \{m \in \mathbb{N} : \|\bar{\xi}_m(e_0) - e_0\| \geqslant \frac{t-\epsilon}{3S}\},$$

$$V_1 = \{m \in \mathbb{N} : \|\bar{\xi}_m(e_1) - e_1\| \geqslant \frac{t-\epsilon}{3S}\} \qquad V_2 = \{m \in \mathbb{N} : \|\bar{\xi}_m(e_2) - e_2\| \geqslant \frac{t-\epsilon}{3S}\}.$$

Then, we have

$$V \subseteq V_0 \cup V_1 \cup V_2. \tag{3.5}$$

Also, define the following fuzzy number valued functions

$$V_{r,t} = \frac{1}{h_r}|\{m \in \mathbb{N} : d^*(\xi_m(f), f) \geqslant t\}| \quad \text{and} \quad V_{r,t}^i = \frac{1}{h_r}|\{m \in \mathbb{N} : d^*(\bar{\xi}_m(e_i) - e_i) \geqslant \frac{t-\epsilon}{3S}\}|,$$

where $i = 0, 1, 2$. Then, by the monotonocity and (3.5), we get

$$V_{r,t} \leqslant \sum_{i=o}^{2} V_{r,t}^i.$$

By taking limit as $r \to \infty$ and using (3.2), we obtain (3.3). This completes the proof. □

We are now giving an example to justify the usefulness of our version of approximation theorem.

Example 3.2. Let us consider the sequences of fuzzy Bernstein-type polynomials ([2])

$$V_k^F(f; x) = \bigoplus_{i=0}^{k} \binom{k}{i} x^i (1-x)^{k-i} \odot f\left(\frac{i}{k}\right),$$

where $f \in B_F[a, b], x \in [0, 1]$ and $k \in \mathbb{N}$. This is a fuzzy positive linear operator, so we write

$$\left\{V_k^F(f; x)\right\}_{\pm}^{\alpha} = \tilde{V}_k(f_{\pm}^{\alpha}; x) = \sum_{i=0}^{k} \binom{k}{i} x^i (1-x)^{k-i} f_{\pm}^{\alpha}\left(\frac{i}{k}\right),$$

where $f_{\pm}^{\alpha} \in C[0, 1]$. Also, we observe that

$$\tilde{V}_k(e_0; x) = 1, \quad \tilde{V}_k(e_1; x) = x, \quad \tilde{V}_k(e_2; x) = x^2 + \frac{x - x^2}{k}.$$

Now we define the sequences of fuzzy Bernstein-type polynomials as

$$L_k^F(f; x) = (1 + f_k(x)) \odot V_k^F(f; x), \tag{3.6}$$

where $f \in B_F[a, b], x \in [0, 1]$ and $f_k(x)$ is given in Example 1.7. In this case, we write

$$\left\{L_k^F(f; x)\right\}_{\pm}^{\alpha} = \tilde{L}_k(f_{\pm}^{\alpha}; x) = (1 + f_k(x)) \sum_{i=0}^{k} \binom{k}{i} x^i (1-x)^{k-i} f_{\pm}^{\alpha}\left(\frac{i}{k}\right),$$

where $f_{\pm}^{\alpha} \in C[0,1]$. Then, we observe that

$$\tilde{L}_k(e_0;x) = (1+f_k(x)), \quad \tilde{L}_k(e_1;x) = (1+f_k(x))x, \quad \tilde{L}_k(e_2;x) = (1+f_k(x))\left(x^2 + \frac{x-x^2}{k}\right).$$

Since

$$f_k \twoheadrightarrow f = 0 \ \ (\theta\text{-equistat}),$$

so that we have

$$\tilde{L}_k(e_i;x) \twoheadrightarrow e_i(x) \ \ (\theta\text{-equistat})$$

for each $i = 0,1,2$. So, by Theorem 3.1, we immediately see that

$$L_k^F(f;x) \twoheadrightarrow f \ \ (\theta\text{-equistat})$$

for all $f \in B_F[a,b]$. Thus, we observe that Theorem 3.1 holds true for our operators defined by (3.6). However, the sequence (f_k) is not statistically uniform convergent in the usual sense, so we conclude that the earlier works of Anastassiou [2] and Anastassiou and Duman [3] does not holds good for our operators defined by (3.6). Hence our Theorem 3.1 is stronger than the theorem proved by Anastassiou [2] and Anastassiou and Duman [3].

Definition 3.3. A sequence (g_r) of fuzzy number valued functions is said to be lacunary equi-statistical convergent to a fuzzy number valued function function g with rate $0 < \beta < 1$ if for each $\epsilon > 0$, we have

$$\lim_r \frac{N_{r,\epsilon}(x)}{r^{-\beta}} = 0$$

uniformly in [a,b] and therefore we write it by

$$g_r - g = o(r^{-\beta}) \ \ (\theta\text{-equistat}).$$

Lemma 3.4. *Let (b_r) and (c_r) be two sequences of fuzzy number valued functions in $B_F[a,b]$, such that $b_r - b = o(r^{-\beta_1})(\theta$-equistat$)$ and $c_r - c = o(r^{-\beta_2})$ $(\theta$-equistat$)$. Then, we have*

(i) $(b_r \pm c_r) - (b \pm c) = o(r^{-\beta})$ $(\theta$-*equistat*$)$;
(ii) $(b_r - b)((c_r - c) = o(r^{-\beta})$ $(\theta$-*equistat*$)$;
(iii) $\zeta(b_r - b) = o(r^{-\beta_1})$ *for any scalar* ζ,

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof.

(i) Suppose that $b_r - b = o(r^{-\beta_1})$ $(\theta$-equistat$)$ and $c_r - c = o(r^{-\beta_2})$ $(\theta$-equistat$)$. Now by using above assumption we consider following sets for all $\epsilon > 0$ and $x \in [a,b]$,

$$N_{r,\epsilon}(x) = \frac{1}{h_r}|\{m \in I_r : d^*((b_m \pm c_m)(x), (b \pm c)(x)) \geqslant \epsilon\}|,$$
$$N^1_{r,\epsilon}(x) = \frac{1}{h_r}|\{m \in I_r : d^*(b_m(x), b(x)) \geqslant \frac{\epsilon}{2}\}|,$$
$$N^2_{r,\epsilon}(x) = \frac{1}{h_r}|\{m \in I_r : d^*(c_m(x), c(x)) \geqslant \frac{\epsilon}{2}\}|.$$

By the definitions, we observe that

$$\frac{N_{r,\epsilon}(x)}{r^{-\beta}} \leqslant \frac{N^1_{r,\epsilon}(x)}{r^{-\beta}} + \frac{N^2_{r,\epsilon}(x)}{r^{-\beta}} \leqslant \frac{N^1_{r,\epsilon}(x)}{r^{-\beta_1}} + \frac{N^2_{r,\epsilon}(x)}{r^{-\beta_2}}$$

uniformly in [a,b]. By taking limit $r \to \infty$ and this completes the proof of (i). The other cases can be proved in the similar manner. □

Now, the modulus of continuity of $g \in B_F[a,b]$ is given by

$$z(g;\varphi) = \sup_{x,y\in[a,b],|x-y|\leqslant\varphi} d(g(x),g(y)),$$

for any $0 < \varphi \leqslant b-a$ and satisfies

$$z(g,|x-y|) \leqslant \left(1+\frac{|x-y|}{\varphi}\right) z(g;\varphi). \tag{3.7}$$

Therefore, we have the following result:

Theorem 3.5. *Let* θ *be a lacunary sequence and let* $\{\xi_m\}$ *be a sequence of fuzzy positive linear operators from* $B_F[a,b]$ *into itself. Suppose that there exists a corresponding sequence* $\{\bar{\xi}_m\}$ *of positive linear operators from* $C[a,b]$ *into itself with the property*

$$\{\xi_m(g;x)\}_{\pm}^{(\alpha)} = \bar{\xi}_m(g_{\pm}^{(\alpha)};x),$$

for all $x \in [a,b]$, $g \in B_F[a,b]$ *and* $m \in \mathbb{N}$. *Further suppose that*

(i) $\bar{\xi}_m(e_0) - e_0 = o(r^{-\beta_1})$ (θ-*equistat*);
(ii) $z(g, \sqrt{\|\bar{L}_m((y-x)^2;x)\|}) = o(r^{-\beta_2})$ (θ-*equistat*),

then

$$\xi_m(g;x) - g = o(r^{-\beta}) \ (\theta\text{-}equistat),$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. By using Theorem 3 of Anastassiou [2], for each $m \in \mathbb{N}$ and $f \in B_F[a,b]$, we have

$$d^*(\xi_m(g),g) \leqslant K\|\bar{\xi}_m(e_0)-e_0\| + \|\bar{\xi}_m(e_0)-e_0\| z(g,\sqrt{\|\bar{\xi}_m((y-x)^2;x)\|}),$$

where $K = d^*(g, X_{\{0\}})$ and $X_{\{0\}}$ is the neutral element for $\oplus$. Thus, we have

$$d^*(\xi_m(g),g) \leqslant K\|\bar{\xi}_m(e_0)-e_0\| + \|\bar{\xi}_m(e_0)-e_0\| z(g,\sqrt{\|\bar{\xi}_m((y-x)^2;x)\|}) + 2z(g,\sqrt{\|\bar{\xi}_m((y-x)^2;x)\|}).$$

By using conditions (i) and (ii), inequality (3.7), and Lemma 3.4 one can get the result. □

4. Conclusion

In this article, first we have established inclusion relations between equi-statistical convergence and lacunary equi-statistical convergence of sequences of fuzzy number valued functions. Then, we provide a Korovkin-type approximation theorem for fuzzy positive linear operators by means of lacunary equi-statistical convergence. An illustrative example is provided with the help of fuzzy Bernstein operators which shows the significance of our approximation theorem. Further, we have studied the rate of lacunary equi-statistical fuzzy convergence by using fuzzy modulus of continuity. Researchers may also study similar results for sequences of fuzzy numbers by using statistical deferred Euler summability.

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