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# Joint $n$-normality of linear transformations

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**Abstract**

This paper is concerned with studying a new class of multivariable operators know as joint $n$-normal $q$-tuple of operators. Some structural properties of some members of this class are given.

**Keywords:** $n$-normal, joint $n$-normal, tensor product.

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## 1. Introduction

Along this work $H$ denotes a complex Hilbert space with inner product $\langle \mid \rangle$. $\mathcal{B}(H)$ is the algebra of all bounded linear operators on $H$.

Normal operators played a crucial role in the theory of operators. They were the basis of many extensions of families of operators. A bounded linear operator $A$ on a complex Hilbert space, $H$ is normal if $\left[A, A^*\right] := A^*A - AA^* = 0$. The class of normal operators was extended to a large classes of operators namely the classes of $n$-normal operators, $(n,m)$-normal operators and polynomially normal operators. An operator $A \in B[H]$ is

(i) $n$-normal if $A^nA^* - A^*A^n = 0$ for some positive integer $n$ ([6, 17]);

(ii) $(n,m)$-normal if $A^nA^{*m} - A^{*m}A^n = 0$ for some positive integers $n$ and $m$ ([1, 2]);

(iii) polynomially normal if there exists a polynomial $P = \sum_{k=0}^{n} a_k z^k \in \mathbb{C}[z]$ such that

$$P(A)A^* - A^*P(A) = \sum_{k=0}^{n} a_k \left(A^kA^* - A^*A^k\right) = 0, \quad ([13]).$$

For more details on these classes of operators, we invite the reader to consult the following references [1, 2, 6, 9, 10, 13, 17].

In recent years, the study of bounded operators in several variables is researched by several authors. The studies have included many classes of operators namely

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- joint $m$-symmetric tuple ([7]);
- joint normal tuple ([11, 12]);
- joint hyponormal tuple ([15]);
- joint $m$-isometries([14]);
- joint $(n_1, \dots, n_d)$-quasi-$m$-isometries ([8]);
- joint $(n_1, \dots, n_q)$-partial $m$-isometries ([3]);
- $m$-invertible $q$-tuple ([3]);
- $(m, C)$-isometric tuples ([4]).

This paper is devoted to some class of multivariable operators on the Hilbert space which is a generalization of joint normal tuple of operators. More precisely, we introduce a new class of operators which is called the class of joint $n$-normal $q$-tuple of operators.

**Definition 1.1.** An tuple $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ is said to be joint $n$-normal $q$-tuple for some positive integer $n$ if the following conditions are satisfied

$$\begin{cases} A_i A_j = A_j A_i \ \text{ for all } \ (i,j) \in \{1, \dots, q\}^2, \\ A_i^n A_i^* = A_i^* A_i^n \ \text{ for } \ i = 1, \dots, q. \end{cases}$$

*Remark* 1.2. If we take $q = 1$ in the Definition 1.1 we obtain the definition of $n$-normal single operator given in [6, 17].

It is proved in Example 2.2 that there is an operator which is joint $n$-normal tuple but not joint normal tuple, and thus, the proposed new class of operators contains the class of joint normal operators as a proper subset. Several properties of joint $n$-normal tuple are given in Section 2. In particular, we show that if $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ is an joint $n$-normal $q$-tuple and $B = (B_1, \dots, B_q) \in \mathcal{B}[H]^q$ is an joint $n$-normal $q$-tuple, then $\mathbf{AB} = (= (A_1B_1, \dots, A_qB_q)$ is an joint $n$-normal $q$-tuple, under suitable conditions. Moreover, we prove that if $\mathbf{A} = (A_1, \dots, A_q) \in B[H]^q$ is a joint $n$-normal $q$-tuple and $B = (B_1, \dots, B_d) \in \mathcal{B}[H]^d$ is a joint $n$-normal $d$-tuple, then $\mathbf{A} * \mathbf{B} \in \mathcal{B}[H]^{dq}$ is an joint $n$-normal $dq$-tuple under suitable conditions. We apply these results to obtain some properties for tensor product of joint $n$-normal $q$-tuple.

## 2. Main results

This section is devoted to the study of some properties of the new class of multivariable operators.

**Example 2.1.** Let $S \in \mathcal{B}(H)$ be an $n$-normal $q$-tuple of operators and let $\lambda = (\lambda_1, \dots, \lambda_q) \in \mathbb{C}^q$. Then the tuple $\mathbf{A} = (A_1, \dots, A_q)$ with $A_j = \lambda_j S$ for $j = 1, \dots, q$ is a joint $n$-normal $q$-tuple of operators. In fact, it is obvious that $[A_i, A_j] = 0$ for all $i, j \in \{1, \dots, q\}$. Further, for all $k \in \{1, \dots, q\}$ we have

$$A_k^n A_k^* = \left(\lambda_k S\right)^n \left(\lambda_k S\right)^* = \lambda_k^n S^n \overline{\lambda}_k S^* = \lambda_k^n \overline{\lambda_q} S^* S^n \text{ (since } S \text{ is } n\text{-normal)} = \left(\lambda_k S\right)^* \left(\lambda_k S\right)^n = A_k^* A_k^n.$$

So that $A_k$ is $n$-normal for all $k = 1, \dots, q$.

*Remark* 2.1. Every joint normal tuple is joint $n$-normal tuple for all positive integers $n$. However, the converse is not true as shown in the following example.

**Example 2.2.** Let $A = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix} \in B[\mathbb{C}^2]$ and define $\mathbf{A} = (A, \dots, A) \in B[\mathbb{C}^2]^q$. Then a direct calculation shows that $\mathbf{A}$ is joint 2-normal $q$-tuple but it is not joint normal $q$-tuple.

**Definition 2.2** ([16]). Let $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{L}(H)^q$, we will call $\mathbf{A}$ entry-wise invertible if the bounded inverse of each operator exists and in which the inverse of a tuple $\mathbf{A} = (A_1, \dots, A_q)$ is given by the tuple $\mathbf{A}^{-1} := (A_1^{-1}, \dots, A_q^{-1})$.

**Theorem 2.3.** *Let $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ be joint $n$-normal $q$-tuple, then the following statement are true.*

(1) *$\mathbf{A}^*$ is a joint $n$-normal tuple.*
(2) *If $U$ is an unitary operator, then $U^*\mathbf{A}U := (U^*A_1U, U^*A_2U, \dots, U^*A_qU)$ is joint $n$-normal tuple.*
(3) *If $\mathbf{A}$ entry-wise invertible, then $\mathbf{A}^{-1}$ is joint $n$-normal.*
(4) *$\mathbf{A}^{\mathbf{m}} := (A_1^{m_1}, \dots, A_q^{m_q})$ is a joint $n$-normal $q$-tuple for all $m = (m_1, \dots, m_q) \in \mathbb{N}^q$.*

*Proof.*

(1) We have $\mathbf{A}^* := (A_1^*, \dots, A_q^*)$. Under the assumption that $\mathbf{A}$ is joint $n$-normal tuple we get that $A_iA_j = A_jA_i$ of all $(i,j) \in \{1, \dots, q\}^2$ and $A_i^nA_i^* = A_i^*A_i^n$ for $i = 1, \dots, q$. From which it follows that

$$\begin{cases} A_i^*A_j^* = A_j^*A_i^* \ \text{ for all } \ (i,j) \in \{1, \dots, q\}^2, \\ A_i^{*n}A_i = A_iA_i^{*n} \quad \text{for } \ i = 1, \dots, q. \end{cases}$$

Therefore $\mathbf{A}^*$ is joint $n$-normal tuple.

(2) Clearly,

$$(U^*A_iU)(U^*A_jU) = U^*A_iA_jU == U^*A_jA_jU = (U^*A_jU)(U^*A_iU).$$

However

$$(U^*A_iU)^n(U^*A_iU)^* = U^*A_i^nA_i^*U = U^*A_i^*A_i^nU = (U^*A_i^*U)(U^*A_i^nU) = (U^*A_iU)^*(U^*A_iU)^n.$$

(3) We have the following implications

$$\begin{cases} A_iA_j = A_jA_i \Longrightarrow A_j^{-1}A_i^{-1} = A_i^{-1}A_j^{-1}, \ \forall \ (i,j) \in \{1, \dots, q\}^2, \\ A_k^nA_k^* = A_k^*A_k^n \Longrightarrow (A_k^{-1})^*(A_k^{-1})^n = (A_k^{-1})^n(A_k^{-1})^*, \ \forall \ k \in \{1, \dots, q\}. \end{cases}$$

Therefore $\mathbf{A}^{-1}$ is joint $n$-normal $q$-tuple.

(4) If $m_k = 1$ for all $k \in \{1, \dots, q\}$, then $[A_i^{m_i}, A_j^{m_j}] = 0$. If $m_k > 1$ for all $k \in \{1, \dots, q\}$, by taking into account [15, lemma 2.1] it follows that

$$[A_i^{m_i}, A_j^{m_j}] = \sum_{\substack{\alpha+\alpha'=m_i-1 \\ \beta+\beta'=m_j-1}} A_i^{\alpha}A_j^{\beta}[A_i, A_j]A_j^{\alpha'}A_i^{\beta'},$$

So, since $\mathbf{A}$ is a joint $n$-normal $q$-tuple, we get

$$[A_i^{m_i}, A_j^{m_j}] = \sum_{\alpha+\alpha'=m_i-1 \beta+\beta'=m_j-1} A_i^{\alpha}A_j^{\beta}[A_i, A_j]A_j^{\alpha'}A_q^{\beta'} = 0, \ \forall \ (i,j) \in \{1, \dots, q\}^2.$$

Moreover, since each $A_k$ is $n$-normal, then by referring to [17, Corollary 2.6], we obtain that $A_k^{m_k}$ is $n$-normal for all $k \in \{1, \dots, q\}$. Therefore $(A_1^{m_1}, \dots, A_q^{m_q})$ is joint $n$-normal $q$-tuple. □

**Theorem 2.4.** *Let $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$, then $\mathbf{A} - \lambda := (A_1 - \lambda_1, \dots, A_q - \lambda_q)$ is joint $n$-normal $q$-tuple for all $\lambda = (\lambda_1, \dots, \lambda_q) \in \mathbb{C}^q$ if and only if $\mathbf{A}$ is joint normal $q$-tuple.*

*Proof.* Assume that $\mathbf{A} - \lambda$ is joint $n$-normal tuple for all $\lambda = (\lambda_1, \dots, \lambda_q) \in \mathbb{C}^q$. Then we have

$$(A_i - \lambda_i)(A_j - \lambda_j) = (A_j - \lambda_j)(A_i - \lambda_i) \Longrightarrow A_iA_j = A_jA_i,; \ \forall \ (i,j) \in \{1, \dots, q\}^2.$$

However,

$$\begin{aligned}
0 &= \left(A_j-\lambda_j\right)^n\left(A_j-\lambda_j\right)^*-\left(A_j-\lambda_j\right)^*\left(A_j-\lambda_j\right)^n\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}A_j^{n-k}\lambda_j^k\left(A_j^*-\overline{\lambda_j}\right)-\left(A_j^*-\overline{\lambda_j}\right)\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_j^{n-k}\lambda_j^k\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}A_j^{n-k}A_j^*\lambda_j^k-\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_j^{n-k}\lambda_j^k\right)\overline{\lambda_j}\\
&\quad -\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_j^*A_j^{n-k}\lambda_j^k+\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}A_j^{n-k}\lambda_j^k\right)\overline{\lambda_j}\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\lambda_j^k\left(A_j^{n-k}A_j^*-A_j^*A_j^{n-k}\right)\\
&= \left(A_j^nA_j^*-A_j^*A_j^n\right)+(-1)^{n-1}n\lambda_j^{n-1}\left(A_jA_j^*-A_j^*A_j\right)+\sum_{k=1}^{n-2}(-1)^k\binom{n}{k}\left(A_j^{n-k}A_j^*-A_j^*A_j^{n-k}\right)\\
&= (-1)^{n-1}\lambda_j^{n-1}n\left(A_jA_j^*-A_j^*A_j\right)+\sum_{k=1}^{n-2}(-1)^k\binom{n}{k}\lambda_j^k\left(A_j^{n-k}A_j^*-A_j^*A_j^{n-k}\right).
\end{aligned}$$

By setting $\lambda_j=r_je^{i\phi_j}$ where $r_j>0$ and $0\leqslant\phi_j\leqslant 2\pi$ , it follows that

$$\left(A_j^*A_j-A_jA_j^*\right)=\frac{(-1)^n}{n\left(r_je^{i\phi_j}\right)^{n-1}}\sum_{k=1}^{n-2}(-1)^k(\binom{n}{k}\left(r_je^{i\phi_j}\right)^k\left(A_j^{n-k}A_j^*-A_j^*A_j^{n-k}\right),$$

and so

$$\|A_j^*A_j-A_jA_j^*\|\leqslant\frac{1}{r_j^{n-1}}\sum_{k=1}^{n-2}\binom{n}{k}r_j^k\|A_j^{n-k}A_j^*-A_j^*A_j^{n-k}\|.$$

Letting $r_j\longrightarrow\infty$, we get $A_j^*A_j-A_jA_j^*=0$ for $j=1,\dots,q$. Therefore, **A** is joint normal tuple.

Conversely, assume that **A** is joint normal tuple. Then we have

$$\begin{cases} A_iA_j=A_jA_i \ \text{ for all } \ (i,j)\in\{1,\dots,q\}^2,\\ A_i^nA_i^*=A_i^*A_i^n \ \text{ for } \ i=1,\dots,q.\end{cases}$$

This means that

$$\begin{cases} \left(A_i-\lambda_i\right)\left(A_j-\lambda_j\right)=\left(A_j-\lambda_j\right)\left(A_i-\lambda_i\right) \ \text{ for all } \ (i,j)\in\{1,\dots,q\}^2,\\ \left(A_i-\lambda_i\right)^n\left(A_i-\lambda_i\right)^*=\left(A_i-\lambda_i\right)^*\left(A_i-\lambda_i\right)^n \ \text{ for } \ i=1,\dots,q.\end{cases}$$

Therefore, $\mathbf{A}-\lambda=(A_1-\lambda_1,\dots,A_q-\lambda_q)$ is joint $n$-normal tuple. □

**Proposition 2.5.** *Let* $\mathbf{A}=(A_1,\dots,A_q)\in\mathcal{B}[H]^q$ *such is joint* $n$*-normal tuple and joint* $(n+1)$*-normal* $q$*-tuple. Then* **A** *is joint* $(n+2)$*-normal tuple.*

*Proof.* Under the assumptions that joint $n$-normal tuple and joint $(n+1)$-normal tuple, it follows that $A_iA_j=A_jA_i$ for all $i,j$ and $A_j^n\left(A_jA_j^*-A_j^*A_j\right)=0$. Therefore

$$A_j^{n+2}A_j^*-A_j^*A_j^{n+2}=0,\ j=1,\dots,q.$$

□

*Remark* 2.6. As a immediate consequence of Proposition 2.5, if **A** is both joint $n$-normal $q$-tuple and joint $(n+1)$-normal $q$-tuple, then it is joint $k$-normal for all $k\geqslant n$. In particular if **A** is both joint 2-normal $q$-tuple and joint 3-normal $q$-tuple, then it is joint $k$-normal $q$-tuple for all $k\geqslant 2$.

**Proposition 2.7.** *Let* $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ *such is joint* $n$*-normal tuple. If each* $A_j$ *is a partial isometry for* $j = 1, \dots, q$*, then* $\mathbf{A}$ *is joint* $(n+1)$*-normal* $q$*-tuple.*

*Proof.* Since for each $k = 1, \dots, q$, $A_k$ is $n$-normal and partial isometry, if follows by applying [5, Theorem 2.4] that $A_k$ is $(n+1)$-normal. The desired result follows immediately. □

**Theorem 2.8.** *Let* $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ *be joint* $n$*-normal* $q$*-tuple. Then* $\mathbf{A}^{mn} := (A_1^{mn}, \dots, A_q^{mn})$ *is joint normal* $q$*-tuple for all positive integer.*

*Proof.* From the assumption that $\mathbf{A}$ is a joint $n$-normal $q$-tuple, it follows that

$$\begin{cases} A_i^{mn} A_j^{mn} = A_j^{mn} A_i^{mn} \quad \text{for all } (i,j) \in \{1, \dots, q\}^2, \\ A_i^{mn} A_i^* = \left(A_i^n\right)^m A_i^* = A_i^* A_i^{mn}, \ \forall \ i = 1, \dots, q. \end{cases}$$

Therefore $\mathbf{A}^{nm}$ is joint normal $q$-tuple. □

Let $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ and $\mathbf{B} = (B_1, \dots, B_q) \in \mathcal{B}[H]^q$. Denote by $\mathbf{AB} = (A_1B_1, \dots, A_qB_q)$ and $\mathbf{A} + \mathbf{B} = (A_1 + B_1, \dots, A_q + B_q)$.

*Remark* 2.9. It was observed by the author Kaplansky in [18] that if $A$ and $B$ are normal operators it may be possible that $AB$ is normal while $BA$ is not. However, he showed that if $A$ and $AB$ are normal, then $BA$ is normal if and only if $B$ commutes with $AA^*$.

**Proposition 2.10.** *Let* $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ *and* $\mathbf{B} = (B_1, \dots, B_q) \in \mathcal{B}[H]^q$ *be two joint* $n$*-normal* $q$*-tuples of operators. The following statements are true.*

(1) *If* $[A_i, B_j] = 0, \ \forall i, j \in \{1, \dots, q\}$ *and* $[A_k, B_k^*] = 0$ *for all* $k \in \{1, \dots, q\}$*, then* $\mathbf{AB}$ *and* $\mathbf{BA}$ *are joint* $n$*-normal* $q$*-tuple.*
(2) *If* $[A_i, B_j] = 0, \ \forall i, j \in \{1, \dots, q\}$ *and* $A_k B_k = A_k B_k^* = 0$ *for all* $k \in \{1, \dots, q\}$*, then* $\mathbf{A} + \mathbf{B}$ *is joint* $n$*-normal* $q$*-tuple.*

*Proof.*

(1) We have for all $i, j \in \{1, \dots, q\}$,

$$\begin{aligned} [A_iB_i, A_jB_j] = A_iB_iA_jB_j - A_jB_jA_iB_i &= A_iA_jB_iB_j - A_jA_iB_jB_i \\ &= A_iA_jB_iB_j - A_iA_jB_jB_i = A_iA_j(B_iB_j - B_jB_i) = A_iA_j[B_i, B_j] = 0. \end{aligned}$$

Furthermore, let $k \in \{1, \dots, q\}$, we have

$$(A_kB_k)^* \left(A_kB_k\right)^n = B_k^* A_k^* A_k^n B_k^n = B_k^* A_k^n A_k^* B_k^n = B_k^* A_k^n B_k^n A_k^* = A_k^n B_k^n B_k^* A_k^* = \left(A_kB_k\right)^n \left(A_kB_k\right)^*.$$

Therefore, $\mathbf{AB}$ is a joint $n$-normal $q$-tuple. Similarly, we show that $\mathbf{BA}$ is a joint $n$-normal $q$-tuple.

(2) For all $(i, j) \in \{1, \dots, q\}^2$, one can see that

$$[A_i + B_i, A_j + B_j] = [A_i, A_j] + [B_i, B_j] + [A_i, B_j] + [B_i, A_j] = 0.$$

Besides, for $k \in \{1, 2, \dots, q\}$, we get

$$\begin{aligned} \left(A_k + B_k\right)^* \left(A_k + B_k\right)^n &= \left(A_k^* + B_k^*\right) \left( \sum_{j=0}^{n} \binom{n}{j} A_k^j B_k^{n-j} \right) \\ &= \left(A_k^* + B_k^*\right)\left(A_k^n + B_k^n\right) \\ &= \left(A_k^* A_k^n + A_k^* B_k^n + B_k^* A_k^n + B_k^* B_k^n \right. \\ &= A_k^n A_k^* + B_k^n B_k^* \end{aligned}$$

$$
\begin{aligned}
&= \left(A_k^n + B_k^n\right)\left(A_k + B_k\right)^* \\
&= \left(\sum_{j=0}^{n} \binom{n}{j} A_k^j B_k^{n-j}\right)\left(A_k + B_k\right)^* \\
&= \left(A_k + B_k\right)^n \left(A_k + B_k\right)^*.
\end{aligned}
$$

Therefore, $\mathbf{A}+\mathbf{B}$ is joint $n$-normal $q$-tuple. □

**Corollary 2.11.** *Let $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ and $\mathbf{B} = (B_1, \dots, B_m) \in \mathcal{B}[H]^m$ be two joint $n$-normal $q$-tuples of operators such that $[A_i, B_j] = [A_i, B_j^*] = 0$, $\forall (i, j) \in \{1, 2..., d\} \times \{1, \dots, m\}$. Then*

$$\mathbf{A} * \mathbf{B} := (A_1B_1, \dots, A_1B_m, A_2B_1, \dots, A_2B_m, \dots, A_qB_1, \dots, A_qB_m)$$

*is joint $n$-normal $(qm)$-tuple of operators.*

*Proof.* By noting $\widetilde{\mathbf{A}} = (A_1, \dots, A_1, A_2, \dots, A_2, \dots, A_q, \dots, A_q)$ and $\widetilde{\mathbf{B}} = (B_1, \dots, B_m, B_1, \dots, B_m, B_1, \dots, B_m)$, we get $\mathbf{A} * \mathbf{B} = \widetilde{\mathbf{A}}\widetilde{\mathbf{B}}$ and so the proof follows by applying Proposition 2.10. □

**Proposition 2.12.** *Let $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[\mathcal{H}]^q$ be commuting tuple of operators. For $n \in \mathbb{N}$, set $X = (A_1^n + A_1^*, \dots, A_q^n + A_q^*)$ and $Y = (A_1^n - A_1^*, \dots, A_q^n - A_q^*)$. Then $\mathbf{A}$ is joint $n$-normal $q$-tuple if and only if $[X, Y] = 0$.*

*Proof.* Obviously, $A_iA_j = A_jA_i \quad \forall\, (i, j) \in \{1, \dots, q\}^2$. On the other hand

$$
\begin{aligned}
\left[X, Y\right] = 0 &\Longleftrightarrow XY - YX = 0 \\
&\Longleftrightarrow \left(A_k^n + A_k^*\right)\left(A_k^n - A_k^*\right) - \left(A_k^n - A_k^*\right)\left(A_k^n + A_k^*\right) = 0 \\
&\Longleftrightarrow A_k^{2n} - A_k^nA_k^* + A_k^*A_k^n - A_k^{*2} - \left(A_k^{2n} + A_k^nA_k^* - A_k^*A_k^n - A_k^{*2}\right) \\
&\Longleftrightarrow A_k^nA_k^* - A_k^*A_k^n = 0, \ \forall\, k \in \{1, \dots, q\}.
\end{aligned}
$$

Hence, the result is proved. □

The following proposition shows that the class of joint $n$-normal $q$-tuple is closed in $\mathcal{B}[H]^q$ .

**Proposition 2.13.** *The class of joint $n$-normal $q$-tuple is a closed subset of $\mathcal{B}[H]^q$.*

*Proof.*

**Step 1.** consider $(A_k)_k \subset \mathcal{B}[H]$ be a sequence of $n$-normal single operators such that $\|A_k - A\| \to 0$, as $k \to +\infty$ for $A \in \mathcal{B}[H]$. Then we have

$$
\begin{cases}
\|A_k^* - A^*\| = \|A_k - A\| \to 0, \text{ as } k \to +\infty, \\
\|A_k^n - A^n\| \to 0, \text{ as } k \to +\infty.
\end{cases}
$$

However, we have

$$
\begin{aligned}
\|A_k^nA_k^* - A^nA^*\| &= \|A_k^nA_k^* - A_k^nA^* + A_k^nA^* - A^nA^*\| \\
&\leqslant \|A_k^nA_k^* - A_k^nA^*\| + \|A_k^nA^* - A^nA^*\| \\
&\leqslant \|A_k^n(A_k^* - A^*)\| + \|(A_k^n - A^n)A^*\| \\
&\leqslant \|A_k^n\|\|A_k - A\| + \|A^*\|\|A_k^n - A^n\| \\
&\leqslant \|A_k^n - A^n\|\Big(\|A_k - A\| + \|A\|\Big) + \|A^n\|\Big(\|A_k - A\|\Big).
\end{aligned}
\tag{2.1}
$$

Hence the limiting case of (2.1) shows that,

$$A^nA^* = \lim_{k \to +\infty} A_k^nA_k^*.$$

Similarly we can also obtain $A^*A^n = \lim_{k \to +\infty} A_k^*A_k^n$. Since $A_k$ is $n$-normal, it follows that $A^nA^* = A^*A^n$. Therefore, $A$ is $n$-normal.

**Step 2.** Let $(\mathbf{A}_k)_k = (A_1(k), \ldots, A_q(k))_k$ be a sequence of joint $n$-normal $q$-tuple of operators in $\mathcal{B}[H]^q$ such that

$$\|\mathbf{A}_k - \mathbf{A}\| = \left(\sum_{j=1}^{q} \|A_j(k) - A_j\|^2\right)^{1/2} \longrightarrow 0, \text{ as } k \longrightarrow \infty,$$

where $\mathbf{A} = (A_1, \ldots, A_q) \in \mathcal{B}[H]^q$.

It is obvious that for each $j \in \{1, \ldots, q\}$ we have

$$\lim_{k \to +\infty} \|A_j(k) - A_j\| = 0. \tag{2.2}$$

Since $A_j(k)^n A_j(k)^* = A_j(k)^* A_j(k)^n$ for each $j = 1, \ldots, q$, it follows by taking into account **Step 1,** that

$$A_j^n A_j^* = A_j^* A_j^n, \ \forall\, j \in \{1, \ldots, q\}.$$

Moreover, for all $i, j \in \{1, \ldots, q\}$ and $k \in \mathbb{N}$, we see that

$$\begin{aligned}
\|A_i(k)A_j(k) - A_iA_j\| &= \|A_i(k)\big(A_j(k) - A_j\big) + \big(A_i(k) - A_i\big)A_j\| \\
&\leqslant \|A_i(k)\|\|A_j(k) - A_j\| + \|A_i(k) - A_i\|\|A_j\| \\
&\leqslant \big(\|A_i(k) - A_i\| + \|A_i\|\big)\|A_j(k) - A_j\| + \|A_i(k) - A_i\|\|A_j\|.
\end{aligned}$$

Hence, in view of (2.2), we obtain

$$\|A_i(k)A_j(k) - A_iA_j\| \longrightarrow 0, \text{ as } n \to +\infty, \ \forall (i, j) \in \{1, \ldots, q\}^2.$$

On the other hand, since $\{\mathbf{A}_k\}_k = \{(A_1(k), \ldots, A_q(k))\}_k$ is a sequence of joint $n$-normal $q$-tuple, then

$$[A_i(k), A_j(k)] = 0, \ \ \forall\, (i, j) \in \{1, \ldots, q\}^2 \text{ and } k \in \mathbb{N}.$$

So, we immediately get

$$[A_i, A_j] = 0, \ \ \forall\, (i, j) \in \{1, 2, \ldots, q\}^2.$$

Therefore, $\mathbf{A}$ is joint $n$-normal $q$-tuple. □

Let $\mathbf{A} = (A_1, \ldots, A_q) \in \mathcal{B}[H]^q$ and $\mathbf{B} = (B_1, \ldots, B_q) \in \mathcal{B}[H]^q$. We denote by

$$\mathbf{A} \otimes \mathbf{B} = \big(A_1 \otimes B_1, \ldots, A_q \otimes B_q\big).$$

In [6] it was observed that If $A_1, A_2 \in \mathcal{B}[H]$, then $A_1 \otimes A_2$ is $n$-normal if and only if $A_1$ and $A_2$ are $n$-normal. The following theorem studied the tensor product of two joint $n$-normal $q$-tuple.

**Theorem 2.14.** *Let $\mathbf{A} = (A_1, \ldots, A_q) \in \mathcal{B}[H]^q$ and $\mathbf{B} = (B_1, \ldots, B_q) \in \mathcal{B}[H]^q$ are two joint $n$-normal $q$-tuple, then $\mathbf{A} \otimes \mathbf{B}$ is joint $n$-normal $q$-tuple.*

*Proof.* Under the condition $\mathbf{A} = \big(A_1, \ldots, A_q\big)$ and $\mathbf{B} = \big(B_1, \ldots, B_q\big)$ are joint $n$-normal $q$-tuples, we can obtained that for all $(i, j) \in \{1, \ldots, q\}^2$

$$\begin{aligned}
\big[(A_i \otimes B_i), (A_j \otimes B_j)\big] &= \big[(A_i \otimes B_i)(A_j \otimes B_j) - (A_j \otimes B_j)(A_i \otimes B_i)\big] \\
&= \big(A_iA_j \otimes B_iB_j\big) - \big(A_jA_i \otimes B_jB_i\big) = \big(A_jA_i \otimes B_jB_i\big) - \big(A_jA_i \otimes B_jB_i\big) = 0.
\end{aligned}$$

Moreover, for all $k \in \{1, \ldots, q\}$, we have

$$\big(A_k \otimes B_k\big)^n \big(A_k \otimes B_k\big)^* = A_k^n A_k^* \otimes B_k^n B_k^* = A_k^* A_k^n \otimes B_k^* B_k^n = \big(A_k \otimes B_k\big)^* \big(A_k \otimes B_k\big)^n.$$

□

The following example shows that the converse of the above theorem need not to be hold in general.

**Example 2.3.** Let $A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}[\mathbb{C}^2]$ and $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{B}[\mathbb{C}^2]$. By elementary calculation we have $A_1 \otimes A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $A_2 \otimes A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Set $\mathbf{A} = \big(A_1, A_2\big)$ and $\mathbf{A} \otimes \mathbf{A} = \big(A_1 \otimes A_1, A_2 \otimes A_2)\big)$. We observe that $\mathbf{A}$ is not joint normal 2-tuple since $A_1A_2 \neq A_2A_1$ . However $\mathbf{A} \otimes \mathbf{A}$ is joint normal 2-tuple.

The following theorem illustrates the conditions under which the converse of Theorem 2.14 is true.

**Theorem 2.15.** *Let* $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ *and* $\mathbf{B} = (B_1, \dots, B_q) \in \mathcal{B}[H]^q$ *such that*

$$\begin{cases} A_iA_j > 0 \ \ \textit{and} \ \ B_iB_j > 0, \ \ i,j = 1,\dots,q, \\ \|A_iA_j\| = \|A_jA_i\| \ \ \textit{and} \ \ \|B_iB_j\| = \|B_jB_i\|, \ i,j = 1,\dots,q. \end{cases}$$

*If* $\mathbf{A} \otimes \mathbf{B}$ *is joint* $n$*-normal* $q$*-tuple, then* $\mathbf{A}$ *and* $\mathbf{B}$ *are joint* $n$*-normal* $q$*-tuples.*

*Proof.* From the condition that $\mathbf{A} \otimes \mathbf{B}$ is joint $n$-normal $q$-tuple, and taking into account Theorem 2.8 it follows that

$$\Big(\mathbf{A} \otimes \mathbf{B}\Big)^n = \Big(\big(A_1 \otimes B_1\big)^n, \dots, \big(A_q \otimes B_q\big)^n\Big) = \Big(A_1^n \otimes B_1^n, \dots A_q^n \otimes B_q^n\Big)$$

is joint normal $q$-tuple. This means that $\big(A_k \otimes B_k\big)^n = A_k^n \otimes B_k^n$ is normal for each $k = 1, \dots, q$. By ([19]) it is well known that

$$A_k^n \otimes B_k^n \ \text{ is normal if and only if } \ A_k^n \text{ and } B_k^n \text{ are normal operators.}$$

However $A_k^n$ being normal, implies that $A_k$ is $n$-normal. Similarly, $B_k^n$ being normal implies that $B_k$ is $n$-normal. Therefore

$$\big[A_k^n, A_k^*\big] = \big[B_k^n, B_k^*\big] = 0, \ \text{ for each } \ k = 1, \dots, q.$$

On the other hand, the joint $n$-normality of $\mathbf{A} \otimes \mathbf{B}$ implies that

$$A_iA_j \otimes B_iB_j = A_jA_i \otimes B_jB_i, \ \ \forall\, (i,j) \in \{1, \dots, q\}^2.$$

Since $A_iA_j$ and $B_iB_j$ are positive for all $i,j = 1,\dots,q$ we have by [19, Proposition 2.2] that there exists a constant $c_{ij} > 0$ such that

$$A_iA_j = c_{ij}A_jA_i \text{ and } \ B_iB_j = c_{ij}^{-1}B_jB_i \ \text{ for } i,j = 1,\dots,q.$$

However

$$\|A_iA_j\| = c_{ij}\|A_jA_i\| \Longrightarrow c_{ij} = 1, \ \ \forall\, i,j.$$

Hence,

$$A_iA_j = A_jA_i \text{ and } \ B_iB_j = B_jB_i \ \text{ for } i,j = 1,\dots,q.$$

Consequently, $\mathbf{A}$ and $\mathbf{B}$ are joint $n$-normal $q$-tuples. □

**Theorem 2.16.** *Let* $\mathbf{A} = (A_1, \dots, A_q) \in \mathcal{B}[H]^q$ *such that* $\ker(A_k) = \{0\}$ *for* $k = 1, \dots, q$. *If* $\mathbf{A}$ *is joint* $n$*-normal and joint* $m$*-normal* $q$*-tuple for some positive integer* $n$ *and* $m$*, then,* $\mathbf{A}$ *is joint* $(\max\{n,m\} - \min\{n,m\})$*-normal* $q$*-tuple. In particular, if* $\mathbf{A}$ *is joint* $n$*-normal and joint* $(n+1)$*-normal, then* $\mathbf{A}$ *is joint normal* $q$*-tuple.*

*Proof.* Obviously, $[A_i, A_j] = 0$ for all $(i, j) \in \{1, \ldots, q\}^2$. Moreover for each $k = 1, \ldots, q$ we have

$$\begin{cases} A_k^n A_k^* - A_k^* A_k^n = 0, \\ A_k^m A_k^* - A_k^* A_k^m = 0. \end{cases}$$

Now by considering that $n \geqslant m$, we get

$$\begin{aligned} A_k^n A_k^* - A_k^* A_k^n = 0 &\Longrightarrow A_k^m \left( A_k^{n-m} A_k^* - A_k^* A_k^{n-m} \right) = 0 \\ &\Longrightarrow A_k^{n-m} A_k^* - A_k^* A_k^{n-m} = 0, \end{aligned}$$

and therefore $\mathbf{A}$ is joint $(n-m)$-normal $q$-tuple. □

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