



On admissible curves and their evolution equations in pseudo-Galilean space



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Abstract

The evolution equations of some forms of admissible curves in the pseudo-Galilean Space G_3^1 are investigated in this paper. In more detail, we use two separate methods to obtain coupled nonlinear partial differential equations of time evolution in terms of their curvatures. The first method studies the evolution equations for admissible curves via the frame field, while the second studies the evolution equations via the velocity vector. Then, the position vectors of the evolving curves are formulated. Also, we conduct comparative research of the evolution equations for curves in different spaces. We furthermore present some models as an application of the evolution equations of the curvature and torsion for admissible curves, confirming our theoretical results.

Keywords: Admissible curves, evolution equations, frenet frame, pseudo-Galilean space, spacelike and timelike curves.

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1. Introduction

In differential geometry, despite existing tons of studies regarding curves and surfaces and the way they evolve per time, recent studies have created a massive improvement in the evolution of the curves and surfaces. Studying evolution equations for curves and surfaces from a geometrical perspective is a core research topic in the field of differential geometry. Evolution equations have a wide range of applications in a wide range of fields, attracting the attention of a wide range of scholars, regardless of their theoretical or implementation backgrounds. They usually arise from multiple applications in many engineering and physical applications such as material science, dynamics of section boundaries in thermomechanics, computational geometry, semiconductor industry, image processing, mathematical modelling, scientific computing, and computer vision. Also, they can be investigated in a purely mathematical context [9, 17, 19, 25, 31–34]. In physics, evolution equations of curves have many critical applications such as magnetic spin chains and vortex filaments [5, 16, 21, 29]. Obtaining the evolution equations and learning how the curves evolve over time is an exciting and difficult problem that has been reviewed in a variety of spaces by several research studies, [1, 2, 18, 24]. Another related work is [3], in which Abdel-All et al. took a novel approach to the time evolution of a curve by obtaining the evolution equations of a generalized space curve. Then, Yıldız et al. studied the evolution of the generalized space curve in Minkowski Space, [35]. Several scholars have recently studied geometric flow problems for curves, [4, 20].

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In our work, the evolution of curves via the velocities of their moving frame in pseudo-Galilean space G_3^1 has been studied. Then, we acquire a combination of coupled nonlinear partial differential equations that govern the time evolution of the curvatures for the meant curves in pseudo-Galilean space G_3^1 . The solution of the coupled nonlinear partial differential equations was found by the Mathematica software system employing a package that applies tanh-sech methods [6]. To obtain curves from their curvatures we expand the Frenet equations numerically up to their position [2, 6]. Then curves and their position vectors that correspond to those solutions are shown.

The following is how the rest of the paper is organized. Section 1 contains an introduction to the work and the aim of this paper. In Section 2, we offer a short outline of the fundamentals and preliminaries for the geometry of curves in pseudo-Galilean space G_3^1 . The evolution equations of admissible curves and how to obtain them have been studied using their frame field in Section 3 and using their velocity vector in Section 4. In Section 5, we compare the evolution equations for different spaces. Finally, Section 6 concludes the paper.

2. Preliminaries on pseudo-Galilean geometry

This section covers the fundamental concepts as well as the theory of curves in pseudo-Galilean space G_3^1 ; for more details, see [11, 12]. The pseudo-Galilean geometry is one of the core geometries of Cayley-Klein with a signature metric of $(0, 0, +, -)$. The pseudo-Galilean space G_3^1 is a three-dimensional projective space in which the absolute figure is an ordered triple (σ, g, M) , where σ is the absolute plane, g is the absolute line in σ and M is the fixed hyperbolic spin of points of g [8]. If a plane contains g , it is called a pseudo-Euclidean plane; otherwise, it is called isotropic. The planes $\Pi = \text{const}$, as well as the planes σ , are pseudo-Euclidean planes. The rest of the planes are isotropic.

In pseudo-Galilean G_3^1 , the inner product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is outlined by

$$\langle a, b \rangle_{G_3^1} = \begin{cases} a_1 b_1, & \text{if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2 b_2 - a_3 b_3, & \text{if } a_1 = 0 \text{ and } b_1 = 0. \end{cases}$$

If $u_1 \neq 0$, a vector $u = (u_1, u_2, u_3)$ is said to be non-isotropic. As a consequence, all unit non-isotropic vectors have the form $u = (1, u_2, u_3)$. For isotropic vectors, $u_1 = 0$ holds true. There are four major kinds of isotropic vectors; spacelike ($u_1 = 0, u_2^2 - u_3^2 > 0$), timelike ($u_1 = 0, u_2^2 - u_3^2 < 0$) and two kinds of lightlike vectors if $(u_1 = 0, u_2 = \pm u_3)$. A non-lightlike isotropic vector is a unit vector if $u_2^2 - u_3^2 = \pm 1$ [11, 12]. See [13–15] for more information. The orthogonality of the vectors $a = (0, a_2, a_3)$ and $b = (0, b_2, b_3)$, i.e., $a \perp b$ means $a \cdot b = 0$. The pseudo-Galilean vector product of two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is defined as

$$a \wedge_{G_3^1} b = \begin{vmatrix} 0 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (0, a_3 b_1 - a_1 b_3, a_2 b_1 - a_1 b_2),$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

In pseudo-Galilean space G_3^1 , a curve Γ is defined as $\Gamma : I \rightarrow G_3^1$, $\Gamma(t) = (\zeta(t), \vartheta(t), \rho(t))$, where $I \subset \mathbb{R}$ and $\zeta(t), \vartheta(t), \rho(t) \in \mathbb{C}^3$. A given curve $\Gamma(t)$ is said to be admissible, if $\zeta'(t) \neq 0$ [11, 12]. Any admissible curve has the form $\Gamma(\zeta) = (\zeta, \vartheta(\zeta), \rho(\zeta))$, which is referred to as an admissible curve of the first type. For simplicity, we assume that $ds = d\zeta$ and $s = \zeta$ as the arc-length parameter of Γ . The following two forms of admissible curves are discussed using the absolute figure of pseudo-Galilean space G_3^1 .

2.1. The admissible curves of the first type

Allow $\Gamma : I \rightarrow G_3^1$, $I \subset \mathbb{R}$ be an admissible curve parametrized by the pseudo-Galilean invariant parameter $s = \zeta$, as mentioned previously. Then Frenet formulas are as follows [7, 12, 13]

$$\begin{pmatrix} \mathbf{T}(\zeta) \\ \mathbf{n}(\zeta) \\ \mathbf{p}(\zeta) \end{pmatrix}_\zeta = \begin{pmatrix} 0 & \kappa(\zeta) & 0 \\ 0 & 0 & \tau(\zeta) \\ 0 & \tau(\zeta) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(\zeta) \\ \mathbf{n}(\zeta) \\ \mathbf{p}(\zeta) \end{pmatrix}, \tag{2.1}$$

where,

$$\begin{cases} \mathbf{T}(\zeta) = \Gamma_\zeta = (1, \vartheta'(\zeta), \rho'(\zeta)), \\ \mathbf{n}(\zeta) = \frac{1}{\kappa}(0, \vartheta''(\zeta), \rho''(\zeta)), \\ \mathbf{p}(\zeta) = \frac{1}{\kappa}(0, \epsilon \rho''(\zeta), \epsilon \vartheta''(\zeta)), \end{cases}$$

are the vectors of the tangent, the central isotropic normal (simply it is known as principal normal or normal), and the binormal line of $\Gamma(\zeta)$, respectively. The functions $\kappa(\zeta) = \sqrt{|\vartheta''(\zeta)^2 - \rho''(\zeta)^2|}$ and $\tau(\zeta) = \frac{1}{\kappa^2} \det [\Gamma', \Gamma'', \Gamma''']$ are geometric parameters that represent, respectively, the curvature and torsion of $\Gamma(\zeta)$ and $\epsilon = \pm 1$, are chosen by criterion $\det [\mathbf{T}, \mathbf{n}, \mathbf{p}] = 1$. Through this paper, the subscripts describe partial derivatives. An admissible curve $\Gamma(\zeta)$ is timelike (resp. spacelike) if the principal normal \mathbf{n} is a spacelike (resp. timelike) vector. The normal vectors spacelike when they are $\epsilon = +1$ and timelike whenever they are $\epsilon = -1$ [7, 12]. As known, when the Frenet frame $\{\mathbf{T}, \mathbf{n}, \mathbf{p}\}$ moves along an admissible curve Γ in pseudo-Galilean space G_3^1 , the shear motion is determined by an angular velocity vector (Darboux vector), which has the equation

$$\mathbf{W}(\zeta) = \epsilon (\tau(\zeta)\mathbf{T}(\zeta) - \kappa(\zeta)\mathbf{p}(\zeta)), \tag{2.2}$$

where $\epsilon = \pm 1$ depending on the normal type as mentioned before. The Darboux vector satisfies Darboux equations which are given by

$$\frac{d\mathbf{T}}{d\zeta} = \mathbf{W}(\zeta) \wedge \mathbf{T}, \quad \frac{d\mathbf{n}}{d\zeta} = \mathbf{W}(\zeta) \wedge \mathbf{n}, \quad \frac{d\mathbf{p}}{d\zeta} = \mathbf{W}(\zeta) \wedge \mathbf{p}. \tag{2.3}$$

The related trihedron's Frenet vector is often split into two shear motions: \mathbf{p} binormal vector shear with $-\epsilon \kappa$ angular speed along absolutely the line, that is

$$\frac{d\mathbf{T}}{d\zeta} = (-\epsilon \kappa \mathbf{p}) \wedge \mathbf{T},$$

and \mathbf{n} normal vector shear with $\epsilon \tau$ angular speed along the absolute line, that is

$$\frac{d\mathbf{n}}{d\zeta} = (\epsilon \tau \mathbf{T}) \wedge \mathbf{n}.$$

2.2. The admissible curves of the second type

In this case, an admissible curve α are often written as

$$\alpha(\zeta) = (0, \vartheta(\zeta), \rho(\zeta)).$$

Then the Frenet formulae of $\alpha(\zeta)$ are given as in (2.1) with tangent, normal, and the binormal lines, respectively given by,

$$\begin{cases} \mathbf{T}(\zeta) = (1, a_2(\zeta), a_3(\zeta)), \\ \mathbf{n}(\zeta) = (0, \rho'(\zeta), \vartheta'(\zeta)), \\ \mathbf{p}(\zeta) = (0, \vartheta'(\zeta), \rho'(\zeta)), \end{cases}$$

where $\mathbf{a}(\zeta) = (1, a_2(\zeta), a_3(\zeta))$ is a unit vector field. The curvature and the torsion of α are, respectively,

$$\kappa(\zeta) = -\frac{a_2}{\rho'}, \quad \tau(\zeta) = \frac{\vartheta''}{\rho'}.$$

Also, we have the same Darboux vector as given in (2.2) which satisfies Darboux equations (2.3) too.

3. Evolution equations of admissible curves via their frame fields in pseudo-Galilean space

In pseudo-Galilean space G_3^1 , let $\Gamma(\zeta) = (\zeta, \vartheta(\zeta), \rho(\zeta))$ be an admissible curve of the first type. Then the associated trihedron (Eq. (2.1)) of the curve Γ can then be represented in a matrix form as

$$\frac{\partial \mathbf{E}}{\partial \zeta} = \mathbf{A} \mathbf{E}, \tag{3.1}$$

where,

$$\mathbf{E} = \begin{pmatrix} \mathbf{T}(\zeta) \\ \mathbf{n}(\zeta) \\ \mathbf{p}(\zeta) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \kappa(\zeta) & 0 \\ 0 & 0 & \tau(\zeta) \\ 0 & \tau(\zeta) & 0 \end{pmatrix}.$$

If $\Gamma(\zeta)$ moves with time t ; then the trihedron $\{\mathbf{T}, \mathbf{n}, \mathbf{p}\}$ and the curvatures become functions of both ζ and t , we can compute the time evolution of \mathbf{T}, \mathbf{n} , and \mathbf{p} to get

$$\frac{\partial \mathbf{E}}{\partial t} = \mathbf{F} \mathbf{E}, \tag{3.2}$$

with noting that

$$\mathbf{E} = \begin{pmatrix} \mathbf{T}(\zeta, t) \\ \mathbf{n}(\zeta, t) \\ \mathbf{p}(\zeta, t) \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 & \epsilon \Omega_3(\zeta, t) & \epsilon \Omega_2(\zeta, t) \\ 0 & 0 & -\epsilon \Omega_1(\zeta, t) \\ 0 & -\epsilon \Omega_1(\zeta, t) & 0 \end{pmatrix}.$$

The motion of the curve Γ is detected by Ω_1, Ω_2 , and Ω_3 (which are the velocities of the moving frame). One of the main results in this paper can be stated as in the following Theorem 3.1.

Theorem 3.1 (Equations of motion for an admissible curve of first type). *Let Γ be an admissible curve of the first type in pseudo-Galilean space G_3^1 . For Γ curve, the compatibility condition is satisfied between (3.1) and (3.2), i.e.,*

$$\frac{\partial}{\partial t} \frac{\partial \mathbf{E}}{\partial \zeta} = \frac{\partial}{\partial \zeta} \frac{\partial \mathbf{E}}{\partial t}, \tag{3.3}$$

and we get a system of coupled non linear partial differential equations that describe the motion equations of the curve via its frame field $\{\mathbf{T}, \mathbf{n}, \mathbf{p}\}$

$$\begin{cases} \kappa_t = \epsilon \left(-\left(\frac{\kappa \Omega_1 + \Omega_2 \zeta}{\tau} \right) \zeta + \tau \Omega_2 \right), \\ \tau_t = -\epsilon \Omega_1 \zeta. \end{cases} \tag{3.4}$$

Proof. Using the compatibility condition (eq (3.3)), which after some calculations yields the following equation:

$$\frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \mathbf{F}}{\partial \zeta} + [\mathbf{A}, \mathbf{F}] = \mathbf{0}_{3 \times 3}, \tag{3.5}$$

where,

$$[\mathbf{A}, \mathbf{F}] = \mathbf{A}\mathbf{F} - \mathbf{F}\mathbf{A}$$

is Lie bracket of \mathbf{F} and \mathbf{A} . In view of the above, Eq. (3.5) reads as follows:

$$\begin{pmatrix} 0 & \kappa_t - \epsilon (\Omega_3 \zeta + \tau \Omega_2) & -\epsilon (\kappa \Omega_1 + \Omega_2 \zeta + \tau \Omega_3) \\ 0 & 0 & \tau_t + \epsilon \Omega_1 \zeta \\ 0 & \tau_t + \epsilon \Omega_1 \zeta & 0 \end{pmatrix} = \mathbf{0}_{3 \times 3}, \tag{3.6}$$

that gives

$$\begin{cases} \kappa_t = \epsilon (\Omega_3 \zeta + \tau \Omega_2), \\ \Omega_3 = -\left(\frac{\Omega_2 \zeta + \kappa \Omega_1}{\tau} \right), \\ \epsilon \Omega_1 \zeta = -\tau_t, \end{cases} \tag{3.7}$$

which can be reduced to a set of coupled nonlinear partial differential equations in the form of (3.4). \square

By dissolving the system (3.4) with the Mathematica software program [30], we can get the equations of motion of the admissible curve Γ for a given $\Omega_1, \Omega_2, \Omega_3$ and the curvatures of the admissible curve through numerical integration of Frenet equations (1). After choosing $\{\Omega_1, \Omega_2, \Omega_3\}$ in terms of the $\{\kappa, \tau\}$ and their derivatives, the parameters $\{\kappa, \tau, \Omega_1, \Omega_2, \Omega_3\}$ are a description of the moving admissible curve, and the solutions generate the evolution of the admissible curve with time t .

If $\{\Omega_1, \Omega_2, \Omega_3\} = \{\tau, 0, \kappa\}$, then the system (3.7) above takes the form:

$$\begin{cases} \kappa_t = -\epsilon \kappa_\zeta, \\ \tau_t = -\epsilon \tau_\zeta, \end{cases}$$

which has the solution

$$\kappa = f(\zeta - \epsilon t), \quad \tau = g(\zeta - \epsilon t),$$

where f and g are arbitrary functions of ζ, t .

The following sections some alternative models for specific curvature and torsion values, as well as their representations in the pseudo-Galilean space G_3^1 .

First Model. Let Γ be a spacelike admissible curve with timelike normal. The curvature and torsion are, respectively $\kappa = \text{constant} = -1, \tau = \text{constant} = 1$, then the position vector of Γ at $t = 0$ is [28]:

$$\mathbf{r}(\zeta) = (\zeta, \int (\int [-\cosh(\zeta + A)] d\zeta) d\zeta, \int (\int [-\sinh(\zeta + A)] d\zeta) d\zeta),$$

as well as,

$$\mathbf{r}(\zeta) = (\zeta, -\cosh(\zeta + A) + B\zeta + D, -\sinh(\zeta + A) + C\zeta + F),$$

integration constants are $A, B, C, D,$ and F . Also, the representation of the evolves curve can be seen in Figure 1.

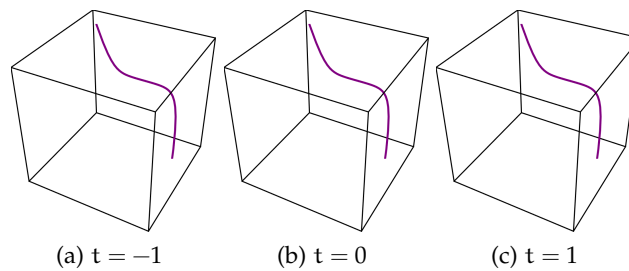


Figure 1: With curve length 20 ($-10 < \zeta < 10$).

If we set $A = B = C = D = F = 0$, we obtain the position vector as

$$\mathbf{r}(\zeta) = (\zeta, -\cosh(\zeta), -\sinh(\zeta)),$$

which has the shape shown in Figure 2 in pseudo-Galilean space G_3^1 .

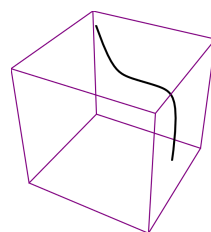


Figure 2: With curve length 20 ($-10 < \zeta < 10$).

We get the numerical representation of the curve (as in Figure 1 at $t = 0$) is identical to the representation of the curve from its position vector as in Figure 2.

Second Model. For a given Γ as above, take $\kappa = \cos(\zeta + t)$, $\tau = 2$. The position vector of the curve is then given relative to its frame from [28]:

$$\mathbf{r}(\zeta) = (\zeta, \int (\int [\cos(\zeta)\cosh(\int 2d\zeta)] d\zeta) d\zeta, \int (\int [\cos(\zeta)\sinh(\int 2d\zeta)] d\zeta) d\zeta),$$

which yields,

$$\mathbf{r}(\zeta) = (\zeta, A(\sin \zeta \sinh \zeta) + B\zeta + C, A(\cosh \zeta \sin \zeta) + D\zeta + F),$$

the integration constants $A, B, C,$ and D are used. And, the admissible curve Γ is represented by Figure 3. Let's say $A = 1$ and $B = C = D = F = 0$, the position vector will be as follows:

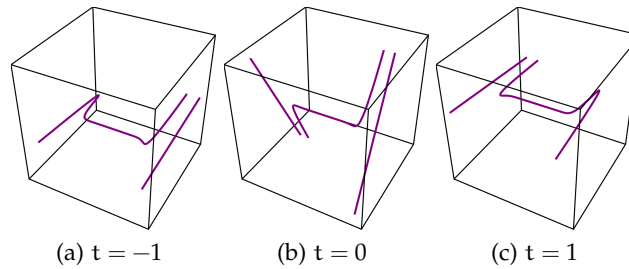


Figure 3: With curve length 20 ($-10 < \zeta < 10$).

$$\mathbf{r}(\zeta) = (\zeta, \sin \zeta \sinh \zeta, \cosh \zeta \sin \zeta),$$

which has the representation as in Figure 4.

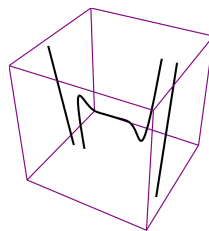


Figure 4: With curve length 20 ($-10 < \zeta < 10$).

It's worth noting that the numerical (Figure 3 b) and exact (Figure 4) representations of the curve vary slightly.

Third Model. Similarly, Consider Γ be a timelike admissible curve with spacelike normal. The curvature and the torsion of Γ are, respectively $\kappa = \cos(\zeta - t)$, $\tau = \sin(\zeta - t)$. When $t = 0$, the position vector of the curve with respect to its standard frame can be represented as [28],

$$\mathbf{r}(\zeta) = (\zeta, \int (\int [\cos(\zeta)\cosh(\int [\sin(\zeta)] d\zeta)] d\zeta) d\zeta, \int (\int [\cos(\zeta)\sinh(\int [\sin(\zeta)] d\zeta)] d\zeta) d\zeta),$$

or

$$\mathbf{r}(\zeta) = (\zeta, \frac{A}{2} \cos(\zeta)\cosh(\zeta^2) - \frac{1}{2} \cos^2(\zeta)\cosh(\zeta^2) + C\zeta + F, \frac{B}{2} \cos(\zeta)\sinh(\zeta^2) - \frac{1}{2} \cos^2(\zeta)\sinh(\zeta^2) + D\zeta + G),$$

with constants of integration A, B, C, D, F, G . The representation of the evolves curve is in Figure 5.

Now, after performing similar calculations to the ones we performed to find the evolution equations of the curvature and torsion for the admissible curve of the first type, we found that the evolution equations for the second type correspond precisely to the admissible curve of the first type if it is timelike, as stated in the next Theorem 3.2.

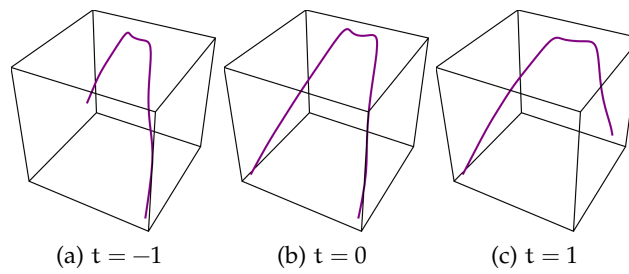


Figure 5: With curve length 20 ($-10 < \zeta < 10$).

Theorem 3.2 (Equations of motion for an admissible curve of second type). *Let $\alpha(\zeta) = (0, \vartheta(\zeta), \rho(\zeta))$, be an admissible curve of the second type in pseudo-Galilean space G_3^1 . The evolution equations for the second type correspond exactly to the admissible curve of the first type if it is timelike. These equations are:*

$$\begin{cases} \kappa_t = -\left(\frac{\kappa\Omega_1 + \Omega_2\zeta}{\tau}\right)\zeta + \tau\Omega_2, \\ \tau_t = -\Omega_1\zeta. \end{cases}$$

Remark 3.3. We note that from the previous theorem the equations of motion are the same evolution equations of the first type at $\epsilon = 1$. These equations correspond to the curve Γ being of a timelike and its principal normal \mathbf{n} of a spacelike vector.

4. Evolution equations for an admissible curve via velocity vector

Let in pseudo-Galilean G_3^1 , $\Gamma(\zeta) = (\zeta, \vartheta(\zeta), \rho(\zeta))$ be an admissible curve of the first type. The relation between motion of curves and also integrable evolution in several spaces, e.g., Euclidean, Minkowski and Galilean spaces, has been studied in [1, 2, 22, 23, 26, 27]. $\Gamma = \Gamma(\zeta, t)$ is assumed to be a point at time t on the plane curve. The velocity fields of the traditional geometrical model, $\Gamma_t(\zeta) = \frac{d\Gamma}{dt} = \mathbf{v}$, such that

$$\Gamma_t = \Lambda\mathbf{T} + \sigma\mathbf{n} + \Upsilon\mathbf{p}, \tag{4.1}$$

where

$$\begin{cases} \Lambda = \Lambda(\kappa, \kappa_s, \dots, \tau, \tau_\zeta, \dots), \\ \sigma = \sigma(\kappa, \kappa_\zeta, \dots, \tau, \tau_\zeta, \dots), \\ \Upsilon = \Upsilon(\kappa, \kappa_\zeta, \dots, \tau, \tau_\zeta, \dots), \end{cases}$$

are velocity functions of ζ, t corresponding to the tangent, normal, and binormal along the curve, i.e., it is intrinsic quantities. Also, $\Lambda, \sigma, \Upsilon$ are the velocities in the tangential, normal, and binormal directions. By using the Frenet frame (3.1), the dynamical equations of the tangent, normal, and binormal will be obtained within the next Theorem 4.1.

Theorem 4.1 (The dynamical equations via velocity vector). *In a pseudo-Galilean space G_3^1 , the dynamical equations of the curvature and torsion of the evolving curve can be expressed in terms of velocities $\Lambda, \sigma, \Upsilon$ as*

$$\begin{cases} \kappa_t = (\Lambda\kappa + \sigma_\zeta + \Upsilon\tau)\zeta + (\sigma\tau + \Upsilon_\zeta)\tau, \\ \tau_t = \left[\frac{1}{\kappa}(\Lambda\kappa + \sigma_\zeta + \Upsilon\tau)\tau + (\sigma\tau + \Upsilon_\zeta)\zeta\right]\zeta. \end{cases} \tag{4.2}$$

Proof. The condition of compatibility is specified as follows:

$$\Gamma_t\zeta = \Gamma_{\zeta t},$$

by comparing coefficients of $\mathbf{T}, \mathbf{n}, \mathbf{p}$ in the above equation, one gets

$$\begin{cases} \Lambda_\zeta = 0, \\ \epsilon \Omega_3 = \Lambda\kappa + \sigma_\zeta + \Upsilon\tau, \\ \epsilon \Omega_2 = \sigma\tau + \Upsilon\zeta. \end{cases} \tag{4.3}$$

From (3.7) and (4.3), we have:

$$\begin{cases} \epsilon \Omega_1 = -\frac{1}{\kappa} [(\Lambda\kappa + \sigma_\zeta + \Upsilon\tau)\tau + (\sigma\tau + \Upsilon\zeta)\zeta], \\ \epsilon \Omega_2 = \sigma\tau + \Upsilon\zeta, \\ \epsilon \Omega_3 = \Lambda\kappa + \sigma_\zeta + \Upsilon\tau, \end{cases}$$

and the dynamical equations of the curvature and torsion of the evolving curve can be expressed in terms of velocities $\Lambda, \sigma, \Upsilon$ as

$$\begin{cases} \kappa_t = \epsilon^2 [(\Lambda\kappa + \sigma_\zeta + \Upsilon\tau)\zeta + (\sigma\tau + \Upsilon\zeta)\tau], \\ \tau_t = \epsilon^2 [\frac{1}{\kappa} (\Lambda\kappa + \sigma_\zeta + \Upsilon\tau)\tau + (\sigma\tau + \Upsilon\zeta)\zeta]. \end{cases}$$

Since, always $\epsilon^2 = 1$, then the dynamical system (4.2) is obtained. □

We now consider, as an application, a model of curve evolution through local geometry using specific values for the intrinsic functions Λ, η, Υ without losing generality.

Model 1. Consider the admissible curve Γ moving in the space according to

$$\Gamma_t = \tau\mathbf{p},$$

where we compensated in the equation (4.1) by the values $\Lambda = \eta = 0, \Upsilon = \tau$. Accordingly, the equations of evolution (4.2) are translated into

$$\begin{cases} \kappa_t = \tau_\zeta^2 + \tau_\zeta \tau, \\ \tau_t = [\frac{1}{\kappa} (\tau^3 + \tau_\zeta \zeta)]_\zeta, \end{cases} \tag{4.4}$$

which have two solutions as follows.

First Solution. One of the solutions of the system of equations (4.4) is given from:

$$\begin{cases} \kappa = \frac{2c_2^3}{c_1} (-1 + 3 \tanh(c_1 t + c_2 \zeta + c_3))^2, \\ \tau = -2c_2 \tanh(c_1 t + c_2 \zeta + c_3), \end{cases}$$

with arbitrary real constants c_1, c_2 , and c_3 , as in Figures 6, 7, and 8.

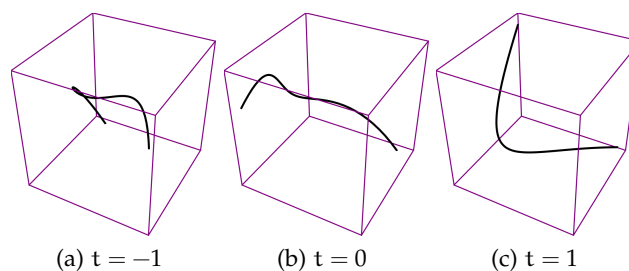


Figure 6: With curve length 20 and constants $c_1 = c_2 = -2$ and $c_3 = -1$.

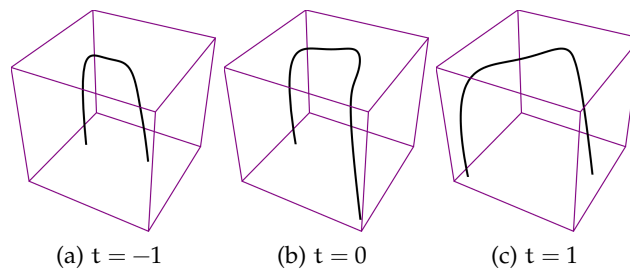


Figure 7: With curve length 20 and constants $c_1 = -0.2, c_2 = -3,$ and $c_3 = 0.1.$

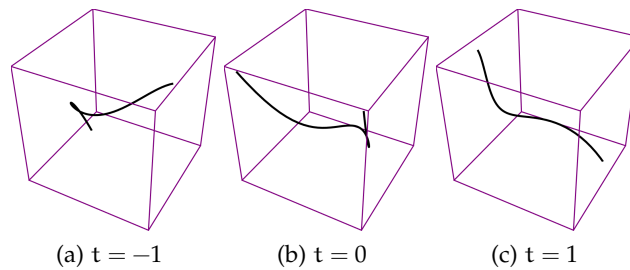


Figure 8: With curve length 20 and constants $c_1 = c_2 = -1$ and $c_3 = 0.$

Second solution. The second solution to the system (4.4) is written on the form:

$$\begin{cases} \kappa = \frac{2c_2^3}{c_1}(-1 + 3 \tanh (c_1 t + c_2 \zeta + c_3))^2, \\ \tau = 2c_2 \tanh (c_1 t + c_2 \zeta + c_3). \end{cases}$$

Likewise, graphs corresponding to some specific values of the constants $c_1, c_2,$ and c_3 can be seen through Figures 9, 10, and 11.

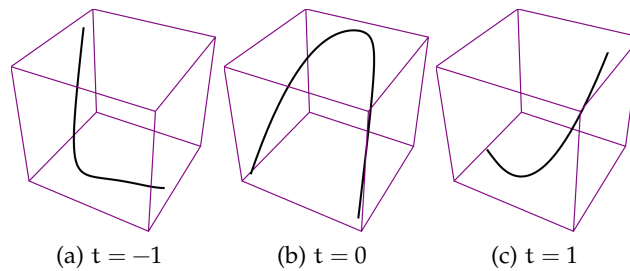


Figure 9: With curve length 20 and constants $c_1 = c_2 = -1$ and $c_3 = 0.$

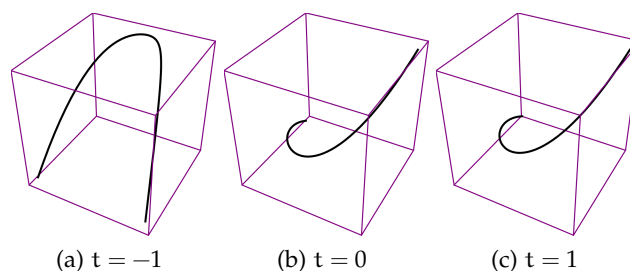


Figure 10: With curve length 20 and constants $c_1 = c_3 = -2$ and $c_2 = -1.$

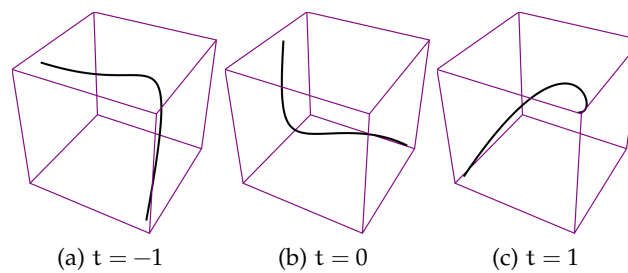


Figure 11: With curve length 20 and constants $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$.

5. Comparison between evolution equations in the different spaces

The properties of moving frames associated with curves differ from one space to another, such as the Euclidean space [2], the Minkowski space [10], the Galilean space [23] and the pseudo-Galilean space in this work. After studying the evolution equations in the pseudo-Galilean space, we found that these equations are functions in the velocities. The choice of velocities has an effect on these functions. This choice is based on space's characteristics. It is possible to choose some values of the velocities according to which the system of equations can be solved. Also, these values are not suitable for solution in another space, due to the different properties of each space. We illustrate an example by choosing values for velocities that solve equations in the Euclidean and Minkowski spaces [2], but they are not suitable for solving in the pseudo-Galilean space, which indicates that the equations of motion depend on velocities and these velocities differ in each space. We tend to conclude that the equations of motion have the following form:

- first, the evolution via Frenet frame:

$$\kappa_t = f((\Omega_1, \Omega_2, \Omega_3), \quad \tau_t = g(\Omega_1, \Omega_2, \Omega_3);$$

- second, the evolution via velocity vector field:

$$\kappa_t = h(\Lambda, \eta, \Upsilon), \quad \tau_t = w(\Lambda, \eta, \Upsilon),$$

where f, g, h , and w have different values in each space. This means the equations of motion depend on the velocities which differ according to each space.

6. Conclusion

The evolution equations for curvature and torsion for the admissible curves were obtained in the three dimensional pseudo-Galileo space G_3^1 . This was accomplished by applying compatibility conditions using the curve Frenet frame differentials for arc curve length parameter and time. Then we solved the resulting system of coupled nonlinear partial differential equations numerically using the Mathematica software program. As an important result, we observed that all evolution equations could be obtained by researchers in Euclidean and Minkowski 3-dimensional spaces via frame fields or via the velocity vector are consistent with our study in the pseudo-Galilean space G_3^1 . In these studies, they all depend on their velocities, but velocities do not have to be the same. For example, velocities values can be chosen in one of the spaces to provide a solution to the system of evolution equations that cannot be provided in another. Also, we provided some models as application for the evolution equations for admissible curves and get the solutions for these new equations. A natural extension of this work is to generate surfaces from the given evolving curves and to study the evolution equations of surfaces in the pseudo-Galilean space G_3^1 .

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