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A fixed point theorem in S_b-metric spaces

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Abstract

In this paper, we introduce an interesting extension of the S-metric spaces called S_b-metric spaces, in which we show the existence of fixed point for a self mapping defined on such spaces. We also prove some results on the topology of the S_b-metric spaces.

Keywords: Functional analysis, S_b-metric space, common fixed point.
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1. Introduction

The concept of metric spaces has been generalized in many ways. Bakhtin [2] introduced the b-metric space, in which many researchers treated the fixed point theory. Czerwick [5] extended the Banach principle contraction and its generalizations under different contractions [1, 4, 6, 7, 10, 15, 16, 17, 18] and [19].

Several authors have investigated the S-metric space and generalized many results related to the existence of fixed point, see [8, 9, 11, 12, 14] and [20]. However, no work has extended the fixed point problem from the b-metric spaces to the S-metric spaces.

Inspired by the work of Bakhtin in [2], we first introduce the S_b-metric space as a generalization of the b-metric space, and then we prove some fixed point results under different types of contractions in a complete S_b-metric space.

Recall the definitions of the b-metric space and the S-metric space.

Definition 1.1 ([2]). Let X be a nonempty set. A b-metric on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $s \geq 1$ such that the following conditions hold for all $x, y, z \in X$:

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(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.

Definition 1.2 ([13])**.** Let X be a nonempty set. An S-metric on X is a function $S : X^3 \longrightarrow [0,\infty)$ that satisfies the following conditions, for all $x,y,z,t \in X$:

(i) $S(x,y,z) = 0$ if and only if $x = y = z$,

(ii) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$.

The pair (X,S) is called an S-metric space.

Now, we give the definition of the S_b-metric space.

Definition 1.3. Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $S_b : X^3 \to [0,\infty)$ is said to be S_b-metric if and only if for all $x,y,z,t \in X$: the following conditions hold:

(i) $S_b(x,y,z) = 0$ if and only if $x = y = z$,

(ii) $S_b(x,x,y) = S_b(y,y,x)$ for all $x,y \in X$,

(iii) $S_b(x,y,z) \leq s[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)]$.

The pair (X,S_b) is called a S_b-metric space.

Remark 1.4. Note that the class of S_b-metric spaces is larger than the class of S-metric spaces. Indeed, every S-metric space is an S_b-metric space with $s = 1$. However, the converse is not always true.

Example 1.5. Let X be a nonempty set and $card(X) \geq 5$. Suppose $X = X_1 \cup X_2$ a partition of X such that $card(X_1) \geq 4$. Let $s \geq 1$. Then

$$S_b(x,y,z) = \begin{cases} 0 & \text{if } x = y = z = 0 \\ 3s & \text{if } (x,y,z) \in X_1^3 \\ 1 & \text{if } (x,y,z) \notin X_1^3 \end{cases}$$

for all $x,y,z \in X$ S_b is a S_b-metric on X with coefficient $s \geq 1$.

Proof.

i) If $x = y = z$ then $S_b(x,y,z) = 0$. Thus the first assertion of the definition of the S_b-metric space is satisfied.

ii) Let's prove the triangle inequality: $S_b(x,y,z) \leq s[S_b(x,x,t) + S_b(y,y,t) + S_b(z,z,t)]$ $(*)$.

- `Case 1:` If $(x,y,z) \notin X_1^3$. We have $S_b(x,y,z) = 1$ $S_b(x,x,t) \geq 1$, $S_b(y,y,t) \geq 1$, and $S_b(z,z,t) \geq 1$, for all $t \in X$. Thus $(*)$ is holds $(1 \leq 3s)$.

- `Case 2:` If $(x, y, z) \in X_1^3$. We distinguish two sub-cases:
  - if $t \in X_1$, $(*)$ is satisfied since $S_b(x, y, z) = S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 3s$.
  - if $t \notin X_1$, we have $S_b(x, x, t) = S_b(y, y, t) = S_b(z, z, t) = 1$ and $S_b(x, y, z) = 3s$. Then, $(*)$ holds.

□

Definition 1.6. Let (X, S_b) be an S_b-metric space and $\{x_n\}$ be a sequence in X. Then

(i) A sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S_b(x_n, x_n, z) \longrightarrow 0$ as $n \longrightarrow \infty$. In this case we write $\lim\limits_{n\longrightarrow\infty} x_n = z$.

(ii) A sequence $\{x_n\}$ is called a Cauchy sequence if and only if $S_b(x_n, x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.

(iii) (X, S_b) is said to be a complete S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} S_b(x_n, x_n, x_m) = \lim_{n,m\to\infty} S_b(x_n, x_n, x) = S_b(x, x, x).$$

(iv) Define the diameter of a subset Y of X by

$$diam(Y) := Sup\{S_b(x, y, z) \mid x, y, z \in Y\}.$$

Definition 1.7 ([3])**.**

(i) Let E be a nonempty set and $T : E \longrightarrow E$ a selfmap. We say that $x \in E$ is a fixed point of T if $T(x) = x$.

(ii) Let E be any set and $T : E \longrightarrow E$ a selfmap. For any given $x \in E$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we recall $T^n(x)$ the *nth* iterative of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \ n = 1, 2, ... \tag{1.1}$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x_0.

2. Main result

Theorem 2.1. *Let (X, S_b) be a complete S_b-metric space and T be a continuous self mapping on X satisfy*

$$S_b(Tx, Ty, Tz) \leq \psi[S_b(x, y, z)] \text{ for all } x, y, z \in X, \tag{2.1}$$

where $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ is an increasing function such that

$$\lim_{n\to\infty} \psi^n(t) = 0 \text{ for each fixed } t > 0.$$

Then T has a unique fixed point in X.

Proof. Let $x \in X$ and $\epsilon > 0$. Let n be a natural number such that $\psi^n(\epsilon) < \frac{\epsilon}{2s}$. Let $F = T^n$ and $x_k = F^k(x)$ for $k \in \mathbb{N}$. Then for $x, y \in X$ and $\alpha = \psi^n$ we have

$$\begin{aligned} S_b(Fx, Fx, Fy) &\leq \psi^n(S_b(x, x, y)) \\ &= \alpha(S_b(x, x, y)). \end{aligned}$$

Hence, for $k \in \mathbb{N}$ $S_b(x_{k+1}, x_{k+1}, x_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore, let k be such that

$$S_b(x_{k+1}, x_{k+1}, x_k) < \frac{\epsilon}{2s}.$$

Let's define the ball $B(x_k, \epsilon)$ such that for every $z \in B(x_k, \epsilon) := \{y \in X | S_b(x_k, x_k, y) \leq \epsilon\}$. Note that $x_k \in B(x_k, \epsilon)$, therefore $B(x_k, \epsilon) \neq \emptyset$. Hence, for all $z \in B(x_k, \epsilon)$ we have

$$\begin{aligned} S_b(Fz, Fz, Fx_k) &\leq \alpha(S_b(xk, xk, z)) \\ &\leq \alpha(\epsilon) = \psi^n(\epsilon) < \frac{\epsilon}{2s} < \frac{\epsilon}{s}. \end{aligned} \tag{2.2}$$

Since $S_b(Fx_k, Fx_k, Fx_k) = S_b(x_{k+1}, x_{k+1}, x_k) < \frac{\epsilon}{2s}$. Thus,

$$\begin{aligned} S_b(x_k, x_k, F_z) &\leq s[S_b(x_k, x_k, x_{k+1}) + S_b(x_k, x_k, x_{k+1}) + S_b(Fz, Fz, x_{k+1})] \\ &= s[2S_b(x_k, x_k, x_{k+1}) + S_b(Fz, Fz, x_{k+1})] \\ &\leq s[2\frac{\epsilon}{2s} + \frac{\epsilon}{s}] = \epsilon. \end{aligned}$$

Hence, F maps $B(x_k, \epsilon)$ to it self. Since $x_k \in B(x_k, \epsilon)$, we have $Fx_k \in B(x_k, \epsilon)$. By repeating this process we get

$$F^m_{x_k} \in B(x_k, \epsilon) \text{ for all } m \in \mathbb{N}.$$

That is $x_l \in B(x_k, \epsilon)$ for all $l \geq k$. Hence

$$S_b(x_m, x_m, x_l) < \epsilon \text{ for all } m, l > k.$$

Therefore $\{x_k\}$ is a Cauchy sequence and by the completeness of X, there exists $u \in X$ such that $x_k \longrightarrow u$ as $k \longrightarrow \infty$. Moreover, $u = \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} x_k = F(u)$. Thus, F has u as a fixed point.

we prove now the uniqueness of the fixed point for F. Since $\alpha(t) = \psi^n(t) < t$ for any $t > 0$, let u and u_1 be two fixed points of F.

$$\begin{aligned} S_b(u, u, u_1) &= S_b(Fu, Fu, Fu_1) \\ &\leq \psi^n(u, u, u_1) \\ &= \alpha(S_b(u, u, u_1)) \\ &\leq S_b(u, u, u_1), \end{aligned}$$

$\Longrightarrow S_b(u, u, u_1) = 0 \Longrightarrow u = u_1$ and hence, F has a unique fixed point in X.

On the other hand, $T^{nk+r}(x) = F^k(T^r(x)) \longrightarrow u$ as $k \longrightarrow \infty$. Hence, $T^m x \longrightarrow u$ as $m \longrightarrow \infty$ for every x. That is $u = \lim_{m\to\infty} Tx_m = T(u)$. Thereby, T has a fixed point. □

The following results extend the results of [4] to the S_b-metric space.

Lemma 2.2. *Let (X, S_b) be a complete S_b-metric space. Then, for every descending sequence $\{F_n\}_{n\geq 1}$ of nonempty closed subsets of X such that $diam(F_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, the intersection $\cap_{n=1}^{\infty} F_n$ contains one and only one point.*

Proof. Let x_n be any point in F_n. Because of the decrease of the sequence $\{F_n\}_{n\geq 1}$, we have $x_n, x_{n+1}, x_{n+2}, ... \in F_n$.

Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $diam(F_{n_0}) < \epsilon$. We obtain $x_{n_0}, x_{n_0+1}, x_{n_0+2}, ... \in F_{n_0}$. For $m, n \geq n_0$, we have that

$$S_b(x_n, x_n, x_m) \leq diam(F_{n_0}) < \epsilon.$$

Hence, the sequence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete S_b-metric space. Thus, it is convergent. Let $x \in X$ such that $\lim_{n\longrightarrow\infty} x_n = x$. Now, for any given n we have that $x_n, x_{n+1}, x_{n+2}, ... \in F_n$. Therefore, $x = \lim_{n\longrightarrow\infty} x_n \in \bar{F_n} = F_n$ since F_n is closed. Thus, $x \in \cap_{n=1}^{\infty} F_n$.

We now prove the uniqueness of x. If $y \in \cap_{n=1}^{\infty} F_n$ and $y \neq x$, then $S_b(x, x, y) = \alpha > 0$. There exists $n \in \mathbb{N}$ large enough such that $diam(F_n) < \alpha = S_b(x, x, y)$ which implies that $y \neq \cap_{n=1}^{\infty} F_n$, which is a contradiction. □

Definition 2.3. Let (X, S_b) be a S_b-metric space, $f : X \to \overline{\mathbb{R}}$ be a function.

- Let $x_0 \in X$, f is a lower semi continuous at x_0 if for every $\epsilon > 0$ there exists a neighborhood U of x_0 such that $f(x) > f(x_0) - \epsilon$ for all $x \in U$.
- f is said to be lower semi continuous if it is lower semi continuous at every point of X.

Theorem 2.4. *Let (X, S_b) be a complete S_b-metric space (with $s > 1$), such that the S_b-metric is continuous and let $f : X \to \overline{\mathbb{R}}$ be a a semi continuous function, proper and lower bounded mapping. Then for every $x_0 \in X$ and $\epsilon > 0$ with*

$$f(x_0) \leq \inf_{x\in X} f(x) + \epsilon,$$

there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x_\epsilon \in X$ such that:

$$i) \quad S_b(x_n, x_n, x_\epsilon) \leq \frac{\epsilon}{2^n}, \quad n \in \mathbb{N}, \tag{2.3}$$

$$ii) \quad x_n \longrightarrow x_\epsilon \text{ as } n \longrightarrow \infty, \tag{2.4}$$

$$iii) \quad f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_n, x_n, x) > f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_n, x_n, x_\epsilon), \text{ for every } x \neq x_\epsilon, \tag{2.5}$$

$$iv) \quad f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_n, x_n, x_\epsilon) \leq f(x_0) \leq \inf_{x\in X} f(x) + \epsilon. \tag{2.6}$$

Proof.

i) We consider the set

$$Tx_0 = \{x \in X | f(x) + S_b(x, x, x_0) \leq f(x_0)\}. \tag{2.7}$$

As f is a lower semi continuous mapping and $x_0 \in Tx_0$, we obtain that Tx_0 is nonempty and closed in (X, S_b) and for every $y \in Tx_0$

$$S_b(y, y, x_0) \leq f(x_0) - f(y) \leq f(x_0) - \inf_{x\in X} f(x) \leq \epsilon. \tag{2.8}$$

We choose $x_1 \in Tx_0$ such that $f(x_1) + S_b(x_1, x_1, x_0) \le \inf_{x \in Tx_0} \{f(x) + S_b(x, x, x_0)\} + \frac{\epsilon}{2s}$ and let

$$Tx_1 = \{x \in Tx_0 | f(x) + \sum_{i=0}^{1} \frac{1}{s^i} S_b(x, x, x_i) \le f(x_1) + S_b(x_1, x_1, x_0)\}. \tag{2.9}$$

Inductively, we can suppose that $x_{n-1} \in Tx_{n-2}$ was already chosen and we consider

$$Tx_{n-1} := \{x \in Tx_{n-2} | f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i) \le f(x_{n-1}) + \sum_{i=0}^{n-2} \frac{1}{s^i} S_b(x_{n-1}, x_{n-1}, x_i)\}. \tag{2.10}$$

Let $x_n \in Tx_{n-1}$ such that

$$f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i) \le \inf_{x \in Tx_{n-1}} [f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i)] + \frac{\epsilon}{2^n s^n}. \tag{2.11}$$

Define now the set

$$Tx_n := \{x \in Tx_{n-1} | f(x) + \sum_{i=0}^{n} \frac{1}{s^i} S_b(x, x, x_i) \le f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i)\}. \tag{2.12}$$

It is easy to see that the set Tx_n is nonempty and closed. Using the relations (2.11) and (2.12), we obtain for every $y \in Tx_n$

$$f(y) + \sum_{i=0}^{n} \frac{1}{s^i} S_b(y, y, x_i) \le f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i),$$

which gives

$$\begin{aligned} \frac{1}{s^n} S_b(y, y, x_n) &\le [f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i)] - [f(y) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(y, y, x_i)] \\ &\le [f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x_n, x_n, x_i)] - \inf_{x \in Tx_{n-1}} [f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} S_b(x, x, x_i)] \\ &\le \frac{\epsilon}{2^n s^n}. \end{aligned}$$

Thus, for all $y \in Tx_n$ we have

$$S_b(y, y, x_n) \le \frac{\epsilon}{2^n}. \tag{2.13}$$

ii) From (2.13), we can deduce that $S_b(y, y, x_n) \to 0$ as $n \to \infty$, so $diam(Tx_n) \to 0$. As (X, S_b) is a complete S_b-metric space and from Lemma 2.2 we have $\cap_{n=0}^{\infty} Tx_n = \{x_\epsilon\}$. Using the equations (2.8) and (2.13) we obtain that $x_\epsilon \in X$ satisfies (2.3). Therefore,

$$x_n \longrightarrow x_\epsilon \text{ as } n \longrightarrow \infty.$$

.

iii) As x_ϵ is the single intersection of all the sets Tx_n, so for all $x \neq x_\epsilon$, we have $x \notin \cap_{n=0}^{\infty} Tx_n$. Thus, there exists $m \in \mathbb{N}$ such that

$$x \in Tx_{m-1} \text{ and } x \notin Tx_m. \tag{2.14}$$

Using (2.12) and (2.14), we obtain

$$f(x) + \sum_{i=0}^{m} \frac{1}{s^i} S_b(x, x, x_i) > f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i). \tag{2.15}$$

Thereby, (2.5) holds.

iv) Using (2.14) and the definition of the set Tx_{n-1} given by (2.10), we obtain

$$f(x) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i) \leq f(x_{m-1}) + \sum_{i=0}^{m-2} \frac{1}{s^i} S_b(x_{m-1}, x_{m-1}, x_i). \tag{2.16}$$

Similarly, by applying (2.16) to x_{m-1} we have that

$$f(x_{m-1}) + \sum_{i=0}^{m-2} \frac{1}{s^i} S_b(x_{m-1}, x_{m-1}, x_i) \leq f(x_{m-2}) + \sum_{i=0}^{m-3} \frac{1}{s^i} S_b(x_{m-2}, x_{m-2}, x_i). \tag{2.17}$$

By repeating this procedure enough times, we obtain

$$f(x_0) \geq f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i).$$

Moreover, for every $q \geq m$, we have

$$f(x_0) \geq f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i} S_b(x_m, x_m, x_i) \geq f(x_q) + \sum_{i=0}^{q-1} \frac{1}{s^i} S_b(x_q, x_q, x_i) \geq f(x_\epsilon) + \sum_{i=0}^{q} \frac{1}{s^i} S_b(x_\epsilon, x_\epsilon, x_i).$$

Then, (2.6) holds. □

Next, we state this immediate consequence.

Corollary 2.5. *Let (X, S_b) be a complete S_b-metric space (with $s > 1$), such that the S_b-metric is continuous and let $f : X \to \overline{\mathbb{R}}$ be a lower semi continuous, proper and lower bounded mapping. Then for every $\epsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x^* \in X$ such that:*

i) $x_n \longrightarrow x_\epsilon$, *as* $n \longrightarrow \infty$ $x_\epsilon \in X$, (2.18)

ii) $$f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \leq \inf_{x\in X} f(x) + \epsilon, \tag{2.19}$$

iii) $$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) \geq f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \text{ for any } x \in X. \tag{2.20}$$

Theorem 2.6. *Let (X, S_b) be a complete S_b-metric space (with $s > 1$), such that the S_b-metric is continuous and let $T : X \to X$ be an operator for which there exists a lower semi continuous mapping $f : X \to \overline{\mathbb{R}}$, such that:*

i) $$S_b(u, u, v) + sS_b(u, u, Tu) \geq S_b(T_u, T_u, v), \tag{2.21}$$

ii) $$\frac{s^2}{s-1} S_b(u, u, Tu) \leq f(u) - f(Tu)), \text{ for any } u, v \in X. \tag{2.22}$$

Then T has at least one fixed point.

Proof. Assume that for all $x \in X$ we have that $Tx \neq x$. Using Corollary 2.5 for f, we obtain that, for each $\epsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \longrightarrow x_\epsilon$, as $n \longrightarrow \infty$, $x_\epsilon \in X$ and

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x, x, x_n) \geq f(x_\epsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n) \text{ for any } x \in X.$$

Since the above inequality holds for every $x \in X$, let put $x := Tx_\epsilon$ and since $Tx_\epsilon \neq x_\epsilon$, we get that

$$f(x_\epsilon) - f(Tx_\epsilon) < \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(Tx_\epsilon, Tx_\epsilon, x_n) - \sum_{n=0}^{\infty} \frac{1}{s^n} S_b(x_\epsilon, x_\epsilon, x_n). \tag{2.23}$$

Let $u = x_\epsilon$ and $v = x_n$ in (2.21), we obtain

$$S_b(x_\epsilon, x_\epsilon, x_n) + sS_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \geq S_b(Tx_\epsilon, Tx_\epsilon, x_n). \tag{2.24}$$

From (2.23) and (2.24) we have

$$\begin{aligned} f(x_\epsilon) - f(Tx_\epsilon) &< \sum_{n=0}^{\infty} \frac{s}{s^n} S_b(x_\epsilon, Tx_\epsilon, Tx_\epsilon) \\ &\leq sS_b(x_\epsilon, Tx_\epsilon, Tx_\epsilon) \sum_{n=0}^{\infty} \frac{1}{s^n} \\ &\leq \frac{s^2}{s-1} S_b(x_\epsilon, Tx_\epsilon, Tx_\epsilon). \end{aligned} \tag{2.25}$$

In (2.22) we choose $u = x_\epsilon$. Then

$$\frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \leq f(x_\epsilon) - f(Tx_\epsilon). \tag{2.26}$$

From the inequalities (2.25) and (2.26) we get that

$$\frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon) \leq f(x_\epsilon) - f(Tx_\epsilon) < \frac{s^2}{s-1} S_b(x_\epsilon, x_\epsilon, Tx_\epsilon),$$

which is a absurd. Therefore, there exists $x^* \in X$ such that $x^* \in Tx^*$. □

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