J. Math. Computer Sci., 26 (2022), 41–66

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On the domain of Cesàro matrix defined by weighted means in $\ell_{t(.)}$, and its pre-quasi ideal with some applications

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Abstract

In this article, we have constructed the sequence space $(\Xi(p,r,t))_{\upsilon}$ by the domain of Cesàro matrix defined by weighted means in Nakano sequence space $\ell_{(t_l)}$, where $t=(t_l)$ and $r=(r_l)$ are sequences of positive reals, and $\upsilon(f)=\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}$, with $f=(f_z)\in\Xi(p,r,t)$. Some geometric and topological actions of $(\Xi(p,r,t))_{\upsilon}$, the multiplication maps stand-in on $(\Xi(p,r,t))_{\upsilon}$, and the eigenvalues distribution of operator ideal formed by $(\Xi(p,r,t))_{\upsilon}$ and s-numbers are discussed. We offer the existence of a fixed point of Kannan contraction operator improvised on these spaces. It is curious that various numerical experiments are introduced to present our results. Moreover, a few gilded applications to the existence of solutions of non-linear difference equations are examined.

Keywords: Cesàro matrix, weighted means, s-numbers, multiplication operator, minimum space, pre-quasi ideal, Kannan contraction operator.

2020 MSC: 46B10, 46B15, 47B10, 46C05, 46E05, 46E15, 46E30.

1. Introduction

The theory of variable exponent function spaces contemplatively depend on the boundedness of the Hardy-Littlewood maximal operator. Which investigates its approach in differential equations, optimization, and approximation. The next conventions throughout the article will be used, if species are preowned we will give them.

Conventions 1.1.

$$N = \{0,1,2,\dots\}.$$

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doi: 10.22436/jmcs.026.01.05

Received: 2021-05-08 *Revised:* 2021-06-14 *Accepted:* 2021-08-10

$\mathcal{C}$: The space of all complex numbers.
$[a]$: The integral part of real number a.
R : The set of real numbers.
$\mathcal{C}^N$: The space of all sequences of complex numbers.
R^{+N} : The space of all sequences of positive reals.
ℓ_∞ : The space of bounded sequences of complex numbers.
ℓ_r : The space of r-absolutely summable sequences of complex numbers.
c_0 : The space of null sequences of complex numbers.
e_l $= (0, 0, \dots, 1, 0, 0, \dots)$, as 1 lies at the l^{th} coordinate, for all $l \in N$.
$\mathcal{F}$: The space of each sequences with finite non-zero coordinates.
$\mathfrak{I}$: The space of all sets with finite number of elements.
$\mathfrak{I}_\nearrow$: The space of all monotonic increasing sequences of positive reals.
$\mathfrak{I}_\searrow$: The space of all monotonic decreasing sequences of positive reals.
$\mathbb{B}$: The ideal of all bounded linear mappings between any arbitrary Banach spaces.
$\mathfrak{F}$: The ideal of finite rank mappings between any arbitrary Banach spaces.
$\mathcal{A}$: The ideal of approximable mappings between any arbitrary Banach spaces.
$\mathcal{K}$: The ideal of compact mappings between any arbitrary Banach spaces.
$\mathbb{B}(\mathcal{P}, \mathcal{Q})$: The space of all bounded linear mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathbb{B}(\mathcal{P})$: The space of all bounded linear mappings from a Banach space $\mathcal{P}$ into itself.
$\mathbb{F}(\mathcal{P}, \mathcal{Q})$: The space of finite rank mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathbb{F}(\mathcal{P})$: The space of finite rank mappings from a Banach space $\mathcal{P}$ into itself.
$\mathcal{A}(\mathcal{P}, \mathcal{Q})$: The space of approximable mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathcal{A}(\mathcal{P})$: The space of approximable mappings from a Banach space $\mathcal{P}$ into itself.
$\mathcal{K}(\mathcal{P}, \mathcal{Q})$: The space of compact mappings from a Banach space $\mathcal{P}$ into a Banach space $\mathcal{Q}$.
$\mathcal{K}(\mathcal{P})$: The space of compact mappings from a Banach space $\mathcal{P}$ into itself.

Definition 1.2 ([35]). A map $s : \mathbb{B}(\mathcal{P}, \mathcal{Q}) \to [0, \infty)^N$ is named an s-number, if the sequence $(s_x(H))_{x=0}^{\infty}$, for all $H \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, verifies the next conditions:

(a) $\|H\| = s_0(H) \geqslant s_1(H) \geqslant s_2(H) \geqslant \cdots \geqslant 0$, with $H \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$;
(b) $s_{x+y-1}(H_1 + H_2) \leqslant s_x(H_1) + s_y(H_2)$, with $H_1, H_2 \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $x, y \in N$;
(c) $s_x(ZYH) \leqslant \|Z\| s_x(Y) \|H\|$, for every $H \in \mathbb{B}(\mathcal{P}_0; \mathcal{P})$, $Y \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $Z \in \mathbb{B}(\mathcal{Q}, \mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are any two Banach spaces;
(d) suppose $G \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $\gamma \in \mathcal{C}$, hence $s_x(\gamma G) = |\gamma| s_x(G)$;
(e) assume $\text{rank}(H) \leqslant x$, then $s_x(H) = 0$, for all $H \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$,
(f) $s_{y \geqslant x}(I_x) = 0$ or $s_{y<x}(I_x) = 1$, where I_x marks the identity map on the x-dimensional Hilbert space ℓ_2^x.

We investigate a few examples of s-numbers as follows.

(1) The x-th Kolmogorov number, $d_x(H)$, where $d_x(H) = \inf_{\dim J \leqslant x} \sup_{\|\lambda\| \leqslant 1} \inf_{\beta \in J} \|H\lambda - \beta\|$.
(2) The x-th approximation number, $\alpha_x(H)$, where $\alpha_x(H) = \inf\{ \|H - Z\| : Z \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) \text{ and } \text{rank}(Z) \leqslant x\}$.

Notations 1.3 ([13]).

$$\mathbb{B}^s_{\mathcal{E}} := \Big\{\mathbb{B}^s_{\mathcal{E}}(\mathcal{P}, \mathcal{Q});\ \mathcal{P} \text{and } \mathcal{Q} \text{ are Banach Spaces}\Big\}, \text{ where } \mathbb{B}^s_{\mathcal{E}}(\mathcal{P}, \mathcal{Q}) := \Big\{H \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((s_x(H))_{x=0}^{\infty} \in \mathcal{E}\Big\}.$$

$$\mathbb{B}^\alpha_{\mathcal{E}} := \Big\{\mathbb{B}^\alpha_{\mathcal{E}}(\mathcal{P}, \mathcal{Q});\ \mathcal{P} \text{and } \mathcal{Q} \text{ are Banach Spaces}\Big\}, \text{ where } \mathbb{B}^\alpha_{\mathcal{E}}(\mathcal{P}, \mathcal{Q}) := \Big\{H \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((\alpha_x(H))_{x=0}^{\infty} \in \mathcal{E}\Big\}.$$

$$\mathbb{B}^d_{\mathcal{E}} := \Big\{\mathbb{B}^d_{\mathcal{E}}(\mathcal{P}, \mathcal{Q})\ \mathcal{P} \text{and } \mathcal{Q} \text{ are Banach Spaces}\Big\}, \text{ where } \mathbb{B}^d_{\mathcal{E}}(\mathcal{P}, \mathcal{Q}) := \Big\{H \in \mathbb{B}(\mathcal{P}, \mathcal{Q}) : ((d_x(H))_{x=0}^{\infty} \in \mathcal{E}\Big\}.$$

Pietsch [31] defined and discussed the concept of s-numbers of linear bounded operators between Banach spaces. In [32, 34], he introduced and investigated a few geometric and topological behavior of the quasi ideals $\mathbb{B}^{\alpha}_{\ell_b}$. Makarov and Faried [23], proved for any infinite dimensional Banach spaces $\mathcal{P}$, $\mathcal{Q}$, the strictly inclusion relation of $\mathbb{B}^{\alpha}_{\ell_b}(\mathcal{P}, \mathcal{Q})$ with different powers. Yaying et al. [45], defined the sequence space, χ^t_r, whose its r-Cesàro matrix in ℓ_t, with $r \in (0, 1]$ and $1 \leqslant t \leqslant \infty$. They offered the quasi Banach ideal of type χ^t_r, with $r \in (0, 1]$ and $1 < t < \infty$. They examined its Schauder basis, $\alpha-$, $\beta-$ and $\gamma-$ duals, and bent upon conclusive matrix classes connected with this sequence space. The compact maps discussed by many authors for distinct sequence spaces for completeness, see [9–12, 20, 26, 29]. Komal et al. [22], investigated the multiplication operators acting on Cesàro sequence spaces under the Luxemburg norm. The multiplication operators acting on Cesàro second order function spaces examined by İlkhan et al. [17]. The non-absolute type sequence spaces are a generalization of the equivalent absolute type. For that there exists a great interest to examine these sequence spaces. Newly, many authors in the literature have discussed a few non-absolute type sequence spaces and presented new interesting articles, for example, see Mursaleen and Noman [27, 28], and Mursaleen and Başar [25]. Many authors have introduced and studied different sequence spaces to fill in gaps in the literature, such as Tripathy [39, 40], Altin et al. [3], Tripathy et al. [41, 42], Khan et al. [21], Hazarika and Esi [15], Raj et al. [36] and Yaying et al. [43, 44]. In view of Banach fixed point theorem [8], Kannan [19] offered an example of a class of operators with the same fixed point actions as contractions though that fails to be continuous. Ghoncheh [14] was the only one who examined Kannan maps in modular vector spaces. He showed that the existence of a fixed point of Kannan operator in complete modular spaces that have Fatou property. Bakery and Mohamed [5] introduced the theory of the pre-quasi norm on Nakano sequence space so that its variable exponent in $(0, 1]$. They examined the sufficient conditions on it under the known pre-quasi norm to construct pre-quasi Banach and closed space, and offered the Fatou property of distinct pre-quasi norms on it. More, they showed the existence of a fixed point of Kannan pre-quasi norm contraction operators on it and on the pre-quasi Banach operator ideal generated by the sequence of s-numbers which belongs to this sequence space.

Lemma 1.4 ([2]). *If* $t_a > 0$ *and* $\lambda_a, \beta_a \in \mathcal{C}$, *for all* $a \in \mathbb{N}$, *and* $\hbar = \max\{1, \sup_a t_a\}$, *hence*

$$|\lambda_a + \beta_a|^{t_a} \leqslant 2^{\hbar - 1}\left(|\lambda_a|^{t_a} + |\beta_a|^{t_a}\right). \tag{1.1}$$

The goal of this paper is organized as follows. In Section 3.1, we give the definition and some inclusion relations of the sequence space $(\Xi(p, r, t))_\upsilon$ equipped with the function υ. In Section 3.2, we explain the sufficient conditions on $\Xi(p, r, t)$ with known function υ to construct pre-modular private sequence space ($\mathfrak{pss}$). This explains that $(\Xi(p, r, t))_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. In Section 3.3, we act a multiplication operator on $(\Xi(p, r, t))_\upsilon$, and investigate the necessity and enough setup on this sequence space so that the multiplication operator is bounded, approximable, invertible, Fredholm and closed range. In Section 4, first, we discuss the enough conditions (not necessary) on $(\Xi(p, r, t))_\upsilon$, so that $\overline{\mathbb{F}} = \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$. This gives a negative answer of Rhoades [37] open problem about the linearity of s-type $(\Xi(p, r, t))_\upsilon$ spaces. Second, we introduce the setup on $(\Xi(p, r, t))_\upsilon$ such that the elements of pre-quasi ideal $\mathbb{B}^s_{\Xi(p,r,t)}$ are complete and closed. Third, we offer the enough conditions on $(\Xi(p, r, t))_\upsilon$ so that $\mathbb{B}^{\alpha}_{(\Xi(p,r,t))_\upsilon}$ is strictly contained for distinct weights and powers. We establish the setup for which the pre-quasi ideal $\mathbb{B}^{\alpha}_{(\Xi(p,r,t))_\upsilon}$ is minimum. Fourth, we introduce the conditions for which the Banach pre-quasi ideal $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$ is simple. Fifth, we give the enough conditions on $(\Xi(p, r, t))_\upsilon$ so that the class $\mathbb{B}$ which sequence of eigenvalues in $(\Xi(p, r, t))_\upsilon$ equals $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$. In Section 5, the existence of a fixed point of Kannan pre-quasi norm contraction operator on this sequence space and on its pre-quasi operator ideal constructed by $(\Xi(p, r, t))_\upsilon$ and s-numbers are confirmed. Finally, in Section 6, we light our results by a few examples and applications to the existence of solutions of non-linear difference equations. Finally, we introduce our conclusion in Section 7.

2. Definitions and preliminaries

Lemma 2.1 ([34]). *If* $M \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ *and* $M \notin \mathcal{A}(\mathcal{P},\mathcal{Q})$, *then there are operators* $Q \in \mathbb{B}(\mathcal{P})$ *and* $L \in \mathbb{B}(\mathcal{Q})$ *so that* $LMQe_x = e_x$, *for all* $x \in \mathbb{N}$.

Definition 2.2 ([34]). A Banach space $\mathcal{E}$ is named simple if the algebra $\mathbb{B}(\mathcal{E})$ includes one and only one non-trivial closed ideal.

Theorem 2.3 ([34]). *Suppose* $\mathcal{E}$ *is a Banach space with* $\dim(\mathcal{E}) = \infty$, *then*

$$\mathbb{F}(\mathcal{E}) \subsetneqq \mathcal{A}(\mathcal{E}) \subsetneqq \mathcal{K}(\mathcal{E}) \subsetneqq \mathbb{B}(\mathcal{E}).$$

Definition 2.4 ([24]). An operator $U \in \mathbb{B}(\mathcal{E})$ is named Fredholm if $\dim(\mathrm{Range}(U))^c < \infty$, $\dim(\ker(U)) < \infty$ and $\mathrm{Range}(U)$ is closed, where $(\mathrm{Range}(U))^c$ marks the complement of $\mathrm{Range}(U)$.

Definition 2.5 ([18]). A class $\mathbb{W} \subseteq \mathbb{B}$ is named an operator ideal if every vector $\mathbb{W}(\mathcal{P},\mathcal{Q}) = \mathbb{W} \cap \mathbb{B}(\mathcal{P},\mathcal{Q})$ confirms the next setup.

(i) $I_\Omega \in \mathbb{W}$, if Ω marks Banach space of one dimension.
(ii) $\mathbb{W}(\mathcal{P},\mathcal{Q})$ is a linear space on $\mathcal{C}$.
(iii) If $Q \in \mathbb{B}(\mathcal{P}_0,\mathcal{P})$, $L \in \mathbb{W}(\mathcal{P},\mathcal{Q})$ and $M \in \mathbb{B}(\mathcal{Q},\mathcal{Q}_0)$, then $MLQ \in \mathbb{W}(\mathcal{P}_0,\mathcal{Q}_0)$, where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are normed spaces.

Definition 2.6 ([13]). A function $\Psi : \mathbb{W} \to [0,\infty)$ is named a pre-quasi norm on the operator ideal $\mathbb{W}$, if it confirms the next setup:

(1) for all $H \in \mathbb{W}(\mathcal{P},\mathcal{Q})$, $\Psi(H) \geqslant 0$ and $\Psi(H) = 0 \Longleftrightarrow H = 0$;
(2) we have $E_0 \geqslant 1$ so that $\Psi(\kappa H) \leqslant E_0|\kappa|\Psi(H)$, with $H \in \mathbb{W}(\mathcal{P},\mathcal{Q})$ and $\kappa \in \mathcal{C}$;
(3) we have $G_0 \geqslant 1$ so that $\Psi(H_1 + H_2) \leqslant G_0[\Psi(H_1) + \Psi(H_2)]$, for every $H_1, H_2 \in \mathbb{W}(\mathcal{P},\mathcal{Q})$;
(4) we have $D_0 \geqslant 1$ so that if $Q \in \mathbb{B}(\mathcal{P}_0,\mathcal{P})$, $L \in \mathbb{W}(\mathcal{P},\mathcal{Q})$ and $M \in \mathbb{B}(\mathcal{Q},\mathcal{Q}_0)$ hence $\Psi(MLQ) \leqslant D_0 \|M\|\Psi(L)\|Q\|$.

Theorem 2.7 ([13]). *Each quasi norm on the ideal* $\mathbb{W}$ *is a pre-quasi norm on the ideal* $\mathbb{W}$.

Definition 2.8 ([6]). The linear space of sequences $\mathcal{E}$ is named a private sequence space ($\mathfrak{pss}$), if it satisfies the following conditions:

(1) $e_x \in \mathcal{E}$, with $x \in \mathbb{N}$;
(2) $\mathcal{E}$ is solid, i.e., for $h = (h_x) \in \mathcal{C}^{\mathbb{N}}$, $|j| = (|j_x|) \in \mathcal{E}$ and $|h_x| \leqslant |j_x|$, with $x \in \mathbb{N}$, then $|h| \in \mathcal{E}$;
(3) $\left(\left|j_{[\frac{x}{2}]}\right|\right)_{x=0}^{\infty} \in \mathcal{E}$, if $(|j_x|)_{x=0}^{\infty} \in \mathcal{E}$.

Theorem 2.9 ([6]). *Assume the linear sequence space* $\mathcal{E}$ *is a* $\mathfrak{pss}$, *then* $\mathbb{B}^s_{\mathcal{E}}$ *is an operator ideal.*

Definition 2.10 ([6]). A subspace of the $\mathfrak{pss}$ is named a pre-modular $\mathfrak{pss}$, if there is a function $\upsilon : \mathcal{E} \to [0,\infty)$ that satisfies the following conditions:

(i) for every $j \in \mathcal{E}$, $j = \theta \Longleftrightarrow \upsilon(|j|) = 0$, and $\upsilon(j) \geqslant 0$, with θ is the zero vector of $\mathcal{E}$;
(ii) if $j \in \mathcal{E}$ and $\rho \in \mathcal{C}$, then there are $E_0 \geqslant 1$ with $\upsilon(\rho j) \leqslant |\rho| E_0 \upsilon(j)$;
(iii) $\upsilon(h + j) \leqslant G_0(\upsilon(h) + \upsilon(j))$ includes for some $G_0 \geqslant 1$, with $f, g \in \mathcal{E}$;
(iv) assume $x \in \mathbb{N}$, $|h_x| \leqslant |j_x|$, we have $\upsilon((|h_x|)) \leqslant \upsilon((|j_x|))$;
(v) the inequality, $\upsilon((|j_x|)) \leqslant \upsilon((|j_{[\frac{x}{2}]}|)) \leqslant D_0 \upsilon((|j_x|))$ verifies, for $D_0 \geqslant 1$;
(vi) $\overline{\mathcal{F}} = \mathcal{E}_\upsilon$;
(vii) we have $\eta > 0$ such that $\upsilon(\rho, 0, 0, 0, \ldots) \geqslant \eta|\rho|\upsilon(1, 0, 0, 0, \ldots)$, with $\rho \in \mathcal{C}$.

Definition 2.11 ([6]). The $\mathfrak{pss}$ $\mathcal{E}_\upsilon$ is named a pre-quasi normed $\mathfrak{pss}$, if υ confirms the setup (i)-(iii) of Definition 2.10. If $\mathcal{E}$ is complete equipped with υ, then $\mathcal{E}_\upsilon$ is named a pre-quasi Banach $\mathfrak{pss}$.

Theorem 2.12 ([6]). *Each pre-modular $\mathfrak{pss}$ $\mathcal{E}_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$.*

Theorem 2.13 ([6]). *The function Ψ is a pre-quasi norm on $\mathbb{B}^s_{(\mathcal{E})_\upsilon}$, where $\Psi(Y) = \upsilon(s_b(Y))_{b=0}^\infty$, for every $Y \in \mathbb{B}^s_{(\mathcal{E})_\upsilon}(\mathcal{P}, \mathcal{Q})$, if $(\mathcal{E})_\upsilon$ is a pre-modular $\mathfrak{pss}$.*

Definition 2.14 ([5]). A pre-quasi norm υ on $\mathcal{E}$ confirms the Fatou property, if for every sequence $\{t^a\} \subseteq \mathcal{E}_\upsilon$ with $\lim_{a\to\infty} \upsilon(t^a - t) = 0$ and each $z \in \mathcal{E}_\upsilon$, then $\upsilon(z - t) \leqslant \sup_j \inf_{a\geqslant j} \upsilon(z - t^a)$.

Definition 2.15 ([5]). A pre-quasi norm Ψ on the ideal $\mathbb{B}^s_{\mathcal{E}}$, where $\Psi(W) = \upsilon\Big((s_a(W))_{a=0}^\infty\Big)$, confirms the Fatou property if for every sequence $\{W_a\}_{a\in N} \subseteq \mathbb{B}^s_{\mathcal{E}}(Z, M)$ with $\lim_{a\to\infty} \Psi(W_a - W) = 0$ and each $V \in \mathbb{B}^s_{\mathcal{E}}(Z, M)$, then $\Psi(V - W) \leqslant \sup_a \inf_{i\geqslant a} \Psi(V - W_i)$.

Definition 2.16 ([5]). An operator $W : \mathcal{E}_\upsilon \to \mathcal{E}_\upsilon$ is named a Kannan υ-contraction, if there is $\lambda \in [0, \frac{1}{2})$, such that $\upsilon(Wz - Wt) \leqslant \lambda(\upsilon(Wz - z) + \upsilon(Wt - t))$, for every $z, t \in \mathcal{E}_\upsilon$.

A vector $z \in \mathcal{E}_\upsilon$ is named a fixed point of W, if $W(z) = z$.

Definition 2.17 ([5]). An operator $W : \mathbb{B}^s_{\mathcal{E}}(Z, M) \to \mathbb{B}^s_{\mathcal{E}}(Z, M)$ is called a Kannan Ψ-contraction, if there is $\lambda \in [0, \frac{1}{2})$, so that $\Psi(WV - WT) \leqslant \lambda(\Psi(WV - V) + \Psi(WT - T))$, for each $V, T \in \mathbb{B}^s_{\mathcal{E}}(Z, M)$.

Definition 2.18 ([5]). Assume $\mathcal{E}_\upsilon$ is a pre-quasi normed (sss), $W : \mathcal{E}_\upsilon \to \mathcal{E}_\upsilon$ and $b \in \mathcal{E}_\upsilon$. The operator W is named υ-sequentially continuous at b, if and only if, if $\lim_{a\to\infty} \upsilon(t_a - b) = 0$, then $\lim_{a\to\infty} \upsilon(Wt_a - Wb) = 0$.

Definition 2.19 ([5]). For the pre-quasi norm Ψ on the ideal $\mathbb{B}^s_{\mathcal{E}}$, where $\Psi(W) = \upsilon\Big((s_a(W))_{a=0}^\infty\Big)$, $G : \mathbb{B}^s_{\mathcal{E}}(Z, M) \to \mathbb{B}^s_{\mathcal{E}}(Z, M)$ and $B \in \mathbb{B}^s_{\mathcal{E}}(Z, M)$. The operator G is named Ψ-sequentially continuous at B, if and only if, if $\lim_{p\to\infty} \Psi(W_p - B) = 0$, then $\lim_{p\to\infty} \Psi(GW_p - GB) = 0$.

Definition 2.20 ([6]). Suppose $\omega = (\omega_k) \in \mathcal{C}^N$ and $\mathcal{E}_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. The operator $H_\omega : \mathcal{E}_\upsilon \to \mathcal{E}_\upsilon$ is named a multiplication operator on $\mathcal{E}_\upsilon$, if $H_\omega f = \Big(\omega_b f_b\Big) \in \mathcal{E}_\upsilon$, with $f \in \mathcal{E}_\upsilon$. The multiplication operator is named created by ω, if $H_\omega \in \mathbb{B}(\mathcal{E}_\upsilon)$.

Theorem 2.21 ([4]). *Suppose s-type $\mathcal{E}_\upsilon := \Big\{h = (s_x(H)) \in R^N : \; H \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$ and $\upsilon(h) < \infty\Big\}$. If $\mathbb{B}^s_{\mathcal{E}_\upsilon}$ is an operator ideal, then the following conditions are verified.*

1. *$\mathcal{F} \subset$ s-type $\mathcal{E}_\upsilon$.*
2. *Suppose $(s_x(H_1))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$ and $(s_x(H_2))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$, then $(s_x(H_1 + H_2))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$.*
3. *Assume $\lambda \in \mathcal{C}$ and $(s_x(H))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$, then $|\lambda| (s_x(H))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$.*
4. *The sequence space $\mathcal{E}_\upsilon$ is solid, i.e., if $(s_x(J))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$ and $s_x(H) \leqslant s_x(J)$, for all $x \in N$ and $H, J \in \mathbb{B}(\mathcal{P}, \mathcal{Q})$, then $(s_x(H))_{x=0}^\infty \in$ s-type $\mathcal{E}_\upsilon$.*

3. Main results

3.1. The sequence space $(\Xi(p, r, t))_\upsilon$

In this section, we introduce the definition the sequence space $(\Xi(p, r, t))_\upsilon$ equipped with the function υ, and some inclusion relations of it.

Definition 3.1. If $(p_l), (r_l), (t_l) \in R^{+N}$, the sequence space $(\Xi(p,r,t))_\upsilon$ with the function υ is defined by:

$$(\Xi(p,r,t))_\upsilon = \Big\{f = (f_k) \in \mathcal{C}^N : \upsilon(\rho f) < \infty, \text{ for some } \rho > 0\Big\},$$

where $\upsilon(f) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}$.

Theorem 3.2. *If* $(t_l) \in R^{+N} \cap \ell_\infty$, *then*

$$(\Xi(p,r,t))_\upsilon = \Big\{f = (f_k) \in \mathcal{C}^N : \upsilon(\rho f) < \infty, \textit{ for any } \rho > 0\Big\}.$$

Proof. Assume $(t_l) \in R^{+N} \cap \ell_\infty$, we have

$$\begin{aligned}(\Xi(\Delta,r))_\upsilon &= \Big\{f = (f_k) \in \mathcal{C}^N : \upsilon(\rho f) < \infty, \text{ for some } \rho > 0\Big\}\\ &= \Big\{f = (f_k) \in \mathcal{C}^N : \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} \rho r_z f_z\right|\right)^{t_l} < \infty, \text{ for some } \rho > 0\Big\}\\ &= \Big\{f = (f_k) \in \mathcal{C}^N : \inf_l \rho^{t_l}\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l} < \infty, \text{ for some } \rho > 0\Big\}\\ &= \Big\{f = (f_k) \in \mathcal{C}^N : \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l} < \infty\Big\}\\ &= \Big\{f = (f_k) \in \mathcal{C}^N : \upsilon(\rho f) < \infty, \text{ for any } \rho > 0\Big\}.\end{aligned}$$

□

Remark 3.3.

(1) If $t_z = t$, $r_z = r_1^z r_2^{l-z}$, $p_z = \frac{1}{\sum_{a=0}^{z} r_1^a r_2^{l-a}}$, for all $z \in N$, $r_1, r_2 \in (0,\infty)$ and $t \geqslant 1$, the sequence space $\Xi(p,r,t) = ces^t_{r_1,r_2}$ studied by Bakery and Mohamed [6].
(2) Suppose $t_z = t$, $r_z = r^z$, $p_z = \frac{1}{\sum_{a=0}^{z} r^a}$, for every $z \in N$, $0 < r \leqslant 1$ and $t \geqslant 1$, the sequence space $\Xi(p,r,t) = \chi^t_r$ studied by Yaying et al. [45].
(3) Assume $t_z = t$, $r_z = 1$, $p_z = \frac{1}{z+1}$, for every $z \in N$ and $t \geqslant 1$, then $\Xi(p,r,t) = ces^t$, defined and examined by Ng and Lee [30].

Theorem 3.4. *If* $(t_l) \in [1,\infty)^N \cap \ell_\infty$, *then* $(\Xi(p,r,t))_\upsilon$ *is a non-absolute type.*

Proof. By choosing $f = \Big(1,-1,0,0,0,\dots\Big)$, then $|f| = \Big(1,1,0,0,0,\dots\Big)$. We have

$$\begin{aligned}\upsilon(f) &= (p_0 r_0)^{t_0} + (p_1|r_0 - r_1|)^{t_1} + (p_2|r_0 - r_1|)^{t_2} + \cdots\\ &\neq (p_0 r_0)^{t_0} + (p_1|r_0 + r_1|)^{t_1} + (p_2|r_0 + r_1|)^{t_2} + \cdots = \upsilon(|f|).\end{aligned}$$

Then, the sequence space $(\Xi(p,r,t))_\upsilon$ is non-absolute type. □

Recall that, we name the sequence space $(\Xi(p,r,t))_\upsilon$ as generalized Cesàro sequence space defined by weighted means of non-absolute type since it is constructed by the domain of Cesàro matrix defined by weighted means in $\ell_{((t_l))}$, where the Cesàro matrix defined by weighted means, $\Lambda(r) = (\lambda_{lz}(r))$, is defined as:

$$\lambda_{lz}(r) = \begin{cases} p_l r_z, & 0 \leqslant z \leqslant l,\\ 0, & z > l.\end{cases}$$

Definition 3.5 ([7])**.** Assume $(p_l), (r_l), (t_l) \in \mathbb{R}^{+\mathbb{N}}$ and $t_l \geqslant 1$, for every $l \in \mathbb{N}$. The generalized Cesàro sequence space defined by weighted means $(ces(p,r,t))_\varphi$ is defined as:

$$(ces(p,r,t))_\varphi = \left\{f = (f_k) \in \mathcal{C}^{\mathbb{N}} : \varphi(\rho f) < \infty, \text{ for some } \rho > 0\right\},$$

where $\varphi(f) = \sum_{l=0}^{\infty} \left(p_l \sum_{z=0}^{l} r_z |f_z|\right)^{t_l}$.

Theorem 3.6. *Suppose* $(t_l) \in \mathbb{R}^{+\mathbb{N}} \cap \ell_\infty$ *with* $(p_l) \in \ell_{((t_l))}$ *and* $(p_l \sum_{z=0}^{l} r_z) \notin \ell_{(t_l)}$*, hence* $(ces(p,r,t))_\varphi \subsetneqq (\Xi(p,r,t))_\upsilon$.

Proof. Let $f \in (ces(p,r,t))_\varphi$, since

$$\sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l} \leqslant \sum_{l=0}^{\infty} \left(p_l \sum_{z=0}^{l} r_z |f_z|\right)^{t_l} < \infty.$$

Hence, $f \in (\Xi(p,r,t))_\upsilon$. If we choose $g = (\frac{(-1)^z}{r_z})_{z\in\mathcal{N}}$, one gets $g \in (\Xi(p,r,t))_\upsilon$ and $g \notin (ces(p,r,t))_\varphi$. □

3.2. Pre-modular private sequence space

We explain in this section the enough setup on $\Xi(p,r,t)$ with definite function υ to construct pre-modular $\mathfrak{pss}$. Which investigates that $\Xi(p,r,t)$ is a pre-quasi normed $\mathfrak{pss}$.

Theorem 3.7. $\Xi(p,r,t)$ *is a* $\mathfrak{pss}$*, if the next setup are confirmed.*

(f1) $(t_l) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ *with* $t_0 > 0$.
(f2) $(p_z)_{z=0}^{\infty} \in \mathbb{R}^{+\mathbb{N}} \cap \ell_{((t_l))}$.
(f3) $(r_z)_{z=0}^{\infty} \in \mathfrak{I}_\searrow$ *or,* $(r_z)_{z=0}^{\infty} \in \mathfrak{I}_\nearrow \cap \ell_\infty$ *and there exists* $C \geqslant 1$ *such that* $r_{2z+1} \leqslant C r_z$.

Proof.

(1-i) Assume $f, g \in \Xi(p,r,t)$. One obtains

$$\sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z f_z + r_z g_z\right|\right)^{t_l} \leqslant 2^{\hbar-1} \left(\sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l} + \sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z g_z\right|\right)^{t_l}\right) < \infty,$$

hence, $f + g \in \Xi(p,r,t)$.

(1-ii) Suppose $\rho \in \mathcal{C}$, $f \in \Xi(p,r,t)$ and as $(t_l) \in \mathfrak{I}_\nearrow \cap \ell_\infty$, we get

$$\sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z \rho f_z\right|\right)^{t_l} \leqslant \sup_l |\rho|^{t_l} \sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l} < \infty.$$

So, $\rho f \in \Xi(p,r,t)$. In view of setup (1-i) and (1-ii), we have $\Xi(p,r,t)$ is a linear space. As $(t_l) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ and $t_0 > 0$, one obtains

$$\sum_{l=0}^{\infty} \left(p_l \left|\sum_{z=0}^{l} r_z (e_b)_z\right|\right)^{t_l} = \sum_{l=b}^{\infty} (p_l r_b)^{t_l} \leqslant \sup_l r_b^{t_l} \sum_{l=b}^{\infty} p_l^{t_l} < \infty.$$

Therefore, $e_b \in \Xi(p,r,t)$, for every $b \in \mathbb{N}$.

(2) Let $|f_b| \leqslant |g_b|$, with $b \in N$ and $|g| \in \Xi(p,r,t)$. One has

$$\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z|f_z|\right|\right)^{t_l} \leqslant \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z|g_z|\right|\right)^{t_l} < \infty,$$

hence $|f| \in \Xi(p,r,t)$.

(3) Assume $(|f_z|) \in \Xi(p,r,t)$, with $(t_l), (r_z) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$ and there is $C \geqslant 1$ with $r_{2z+1} \leqslant Cr_z$, we get

$$\begin{aligned}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l} r_z|f_{[\frac{z}{2}]}|\right)^{t_l} &= \sum_{l=0}^{\infty}\left(p_{2l}\sum_{z=0}^{2l} r_z|f_{[\frac{z}{2}]}|\right)^{t_{2l}} + \sum_{l=0}^{\infty}\left(p_{2l+1}\sum_{z=0}^{2l+1} r_z|f_{[\frac{z}{2}]}|\right)^{t_{2l+1}} \\ &\leqslant \sum_{l=0}^{\infty}\left(p_l\left(r_{2l}|f_l| + \sum_{z=0}^{l}(r_{2z}+r_{2z+1})|f_z|\right)\right)^{t_l} + \sum_{l=0}^{\infty}\left(p_l\left(\sum_{z=0}^{l}(r_{2z}+r_{2z+1})|f_z|\right)\right)^{t_l} \\ &\leqslant 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l} r_{2z}|f_z|\right)^{t_l} + \sum_{l=0}^{\infty}\left(2Cp_l\sum_{z=0}^{l} r_z|f_z|\right)^{t_l}\right) + \sum_{l=0}^{\infty}\left(2Cp_l\sum_{z=0}^{l} r_z|f_z|\right)^{t_l} \\ &\leqslant (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar})C^{\hbar}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l} r_z|f_z|\right)^{t_l} < \infty,\end{aligned}$$

hence $(|f_{[\frac{z}{2}]}|) \in \Xi(p,r,t)$. □

In view of Theorem 2.9, we have the next Theorem.

Theorem 3.8. *Suppose the setup* (f1), (f2), *and* (f3) *is settled, then* $\mathbb{B}^{s}_{\Xi(p,r,t)}$ *is an operator ideal.*

Theorem 3.9. $(\Xi(p,r,t))_\upsilon$ *is a pre-modular* $\mathfrak{pss}$, *if the setup* (f1), (f2), *and* (f3) *is settled.*

Proof.

(i) Definitely, $\upsilon(f) \geqslant 0$ and $\upsilon(|f|) = 0 \Leftrightarrow f = \theta$.

(ii) There are $E_0 = \max\left\{1, \sup_l |\rho|^{t_l-1}\right\} \geqslant 1$ with $\upsilon(\rho f) \leqslant E_0|\rho|\upsilon(f)$, for each $f \in \Xi(p,r,t)$ and $\rho \in \mathfrak{C}$.

(iii) The inequality $\upsilon(f+g) \leqslant 2^{\hbar-1}(\upsilon(f)+\upsilon(g))$ satisfies, with $f, g \in \Xi(p,r,t)$.

(iv) Clearly, from the proof part (2) of Theorem 3.7.

(v) Obviously, the proof part (3) of Theorem 3.7, that $D_0 \geqslant (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar})C^{\hbar} \geqslant 1$.

(vi) Clearly, $\overline{\mathcal{F}} = \Xi(p,r,t)$.

(vii) One has $0 < \varpi \leqslant \sup_l |\rho|^{t_l-1}$ with $\upsilon(\rho,0,0,0,\dots) \geqslant \varpi|\rho|\upsilon(1,0,0,0,\dots)$, for all $\rho \neq 0$ and $\varpi > 0$, if $\rho = 0$. □

Theorem 3.10. *If the setup* (f1), (f2), *and* (f3) *is established, then* $(\Xi(p,r,t))_\upsilon$ *is a pre-quasi Banach* $\mathfrak{pss}$.

Proof. According to Theorem 3.9, the space $(\Xi(p,r,t))_\upsilon$ is a pre-modular $\mathfrak{pss}$. According to Theorem 2.12, the space $(\Xi(p,r,t))_\upsilon$ is a pre-quasi normed $\mathfrak{pss}$. To explain that $(\Xi(p,r,t))_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$, assume $f^a = (f^a_z)_{z=0}^{\infty}$ is a Cauchy sequence in $(\Xi(p,r,t))_\upsilon$, then for all $\varepsilon \in (0,1)$, there is $a_0 \in N$ so that for all $a, b \geqslant a_0$, one gets

$$\upsilon(f^a - f^b) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z(f^a_z - f^b_z)\right|\right)^{t_l} < \varepsilon^{\hbar}.$$

Hence, for $a, b \geqslant a_0$ and $z \in N$, we obtain $|f^a_z - f^b_z| < \varepsilon$. So, (f^b_z) is a Cauchy sequence in $\mathfrak{C}$, for fixed $z \in N$, this explains $\lim_{b\to\infty} f^b_z = f^0_z$, for fixed $z \in N$. Hence $\upsilon(f^a - f^0) < \varepsilon^{\hbar}$, for all $a \geqslant a_0$. Finally to investigate that $f^0 \in (\Xi(p,r,t))_\upsilon$, one has $\upsilon(f^0) \leqslant 2^{\hbar-1}(\upsilon(f^a - f^0) + \upsilon(f^a)) < \infty$, then $f^0 \in (\Xi(p,r,t))_\upsilon$. This explains that $(\Xi(p,r,t))_\upsilon$ is a pre-quasi Banach $\mathfrak{pss}$. □

In view of Theorem 2.21, we construct the next properties of the s-type $(\Xi(p,r,t))_{\upsilon}$.

Theorem 3.11. *Let* s-*type* $(\Xi(p,r,t))_{\upsilon} := \left\{f = (s_n(X)) \in \mathcal{C}^{\mathbb{N}} : X \in \mathbb{B}(\mathcal{P},\mathcal{Q}) \text{ and } \upsilon(f) < \infty\right\}$*. The next conditions are established*

1. *One has* s-*type* $(\Xi(p,r,t))_{\upsilon} \supset \mathcal{F}$.
2. *Suppose* $(s_r(X_1))_{r=0}^{\infty} \in$ s-*type* $(\Xi(p,r,t))_{\upsilon}$ *and* $(s_r(X_2))_{r=0}^{\infty} \in$ s-*type* $(\Xi(p,r,t))_{\upsilon}$, *then* $(s_r(X_1+X_2))_{r=0}^{\infty} \in$ s-*type* $(\Xi(p,r,t))_{\upsilon}$.
3. *Assume* $\lambda \in \mathcal{C}$ *and* $(s_r(X))_{r=0}^{\infty} \in$ s-*type* $(\Xi(p,r,t))_{\upsilon}$, *hence* $|\lambda|\,(s_r(X))_{r=0}^{\infty} \in$ s-*type* $(\Xi(p,r,t))_{\upsilon}$.
4. *The* s-*type* $(\Xi(p,r,t))_{\upsilon}$ *is solid.*

3.3. *Multiplication operators on* $(\Xi(p,r,t))_{\upsilon}$

We discuss here the necessity and enough setup on $(\Xi(p,r,t))_{\upsilon}$ in order to the multiplication operator defined on it is bounded, invertible, approximable, Fredholm and closed range.

Theorem 3.12. *Suppose* $\omega \in \mathcal{C}^{\mathbb{N}}$, *the setup* (f1), (f2), *and* (f3) *is entrenched, hence*

$$\omega \in \ell_{\infty} \Longleftrightarrow H_{\omega} \in \mathbb{B}((\Xi(p,r,t))_{\upsilon}).$$

Proof. Let $\omega \in \ell_{\infty}$. Hence, there is $\nu > 0$ so that $|\omega_b| \leqslant \nu$, for every $b \in \mathbb{N}$. Assume $f \in (\Xi(p,r,t))_{\upsilon}$, one has

$$\upsilon(H_{\omega}f) = \upsilon(\omega f) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z\omega_z f_z\right|\right)^{t_l} \leqslant \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z\nu f_z\right|\right)^{t_l} \leqslant \sup_l \nu^{t_l}\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l} = \sup_l \nu^{t_l}\upsilon(f).$$

Therefore, $H_{\omega} \in \mathbb{B}((\Xi(p,r,t))_{\upsilon})$. On the other hand, assume $H_{\omega} \in \mathbb{B}((\Xi(p,r,t))_{\upsilon})$ and $\omega \notin \ell_{\infty}$. Hence for all $b \in \mathbb{N}$, there are $x_b \in \mathbb{N}$ so that $\omega_{x_b} > b$. We get

$$\upsilon(H_{\omega}e_{x_b}) = \upsilon(\omega e_{x_b}) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z\omega_z(e_{x_b})_z\right|\right)^{t_l} = \sum_{l=x_b}^{\infty}(p_l r_{x_b}|\omega_{x_b}|)^{t_l} > \sum_{l=x_b}^{\infty}(p_l b r_{x_b})^{t_l} > b^{t_0}\upsilon(e_{x_b}).$$

Hence, $H_{\omega} \notin \mathbb{B}((\Xi(p,r,t))_{\upsilon})$. So $\omega \in \ell_{\infty}$. □

Theorem 3.13. *Suppose* $\omega \in \mathcal{C}^{\mathbb{N}}$ *and* $(\Xi(p,r,t))_{\upsilon}$ *is a pre-quasi normed* $\mathfrak{pss}$. *Hence* $\omega_b = g$, *for every* $b \in \mathbb{N}$ *and* $g \in \mathcal{C}$ *with* $|g| = 1$, *if and only if,* H_{ω} *is an isometry.*

Proof. Let $\omega_b = g$, for every $b \in \mathbb{N}$ and $g \in \mathcal{C}$ with $|g| = 1$. One obtains

$$\upsilon(H_{\omega}f) = \upsilon(\omega f) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k\omega_k f_k\right|\right)^{t_l} = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{k=0}^{l} |g| r_k f_k\right|\right)^{t_l} = \upsilon(f),$$

for every $f \in (\Xi(p,r,t))_{\upsilon}$. Therefore, H_{ω} is an isometry.

Suppose the necessity setup is entrenched and $|\omega_b| < 1$, for some $b = b_0$. We get

$$\upsilon(H_{\omega}e_{b_0}) = \upsilon(\omega e_{b_0}) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k\omega_k(e_{b_0})_k\right|\right)^{t_l} = \sum_{l=b_0}^{\infty}(p_l r_{b_0}|\omega_{b_0}|)^{t_l} < \sum_{l=b_0}^{\infty}(p_l r_{b_0})^{t_l} = \upsilon(e_{b_0}).$$

Next if $|\omega_{b_0}| > 1$, obviously $\upsilon(H_{\omega}e_{b_0}) > \upsilon(e_{b_0})$. This explains a contradiction for the two cases. Therefore, $|\omega_b| = 1$, for all $b \in \mathbb{N}$. □

Theorem 3.14. *Suppose* $\omega \in \mathcal{C}^{\mathbb{N}}$*, the setup* (f1), (f2)*, and* (f3) *is entrenched. Hence*

$$H_\omega \in \mathcal{A}((\Xi(p,r,t))_\upsilon) \Longleftrightarrow (\omega_b)_{b=0}^\infty \in c_0.$$

Proof. Let $H_\omega \in \mathcal{A}((\Xi(p,r,t))_\upsilon)$, then $H_\omega \in \mathcal{K}((\Xi(p,r,t))_\upsilon)$. Assume $\lim_{b\to\infty} \omega_b \neq 0$. Therefore, we have $\rho > 0$ such that the set $K_\rho = \{b \in \mathbb{N} : |\omega_b| \geqslant \rho\} \nsubseteq \mathfrak{I}$. Assume $\{\alpha_b\}_{b\in\mathbb{N}} \subset K_\rho$. Hence, $\{e_{\alpha_b} : \alpha_b \in K_\rho\} \in \ell_\infty$ is an infinite set in $(\Xi(p,r,t))_\upsilon$. Since

$$\begin{aligned}\upsilon(H_\omega e_{\alpha_a} - H_\omega e_{\alpha_b}) = \upsilon(\omega e_{\alpha_a} - \omega e_{\alpha_b}) &= \sum_{l=0}^\infty \left(p_l \left|\sum_{k=0}^l r_k \omega_k ((e_{\alpha_a})_k - (e_{\alpha_b})_k)\right|\right)^{t_l} \\ &\geqslant \sum_{l=0}^\infty \left(p_l \left|\sum_{k=0}^l r_k \rho ((e_{\alpha_a})_k - (e_{\alpha_b})_k)\right|\right)^{t_l} \geqslant \inf_l \rho^{t_l} \upsilon(e_{\alpha_a} - e_{\alpha_b}),\end{aligned}$$

for every $\alpha_a, \alpha_b \in K_\rho$. Then, $\{e_{\alpha_b} : \alpha_b \in K_\rho\} \in \ell_\infty$, which cannot have a convergent subsequence under H_ω. Hence $H_\omega \notin \mathcal{K}((\Xi(p,r,t))_\upsilon)$. This explains $H_\omega \notin \mathcal{A}((\Xi(p,r,t))_\upsilon)$, which indicates a contradiction. Hence, $\lim_{b\to\infty} \omega_b = 0$. On the other hand, assume $\lim_{b\to\infty} \omega_b = 0$. Therefore, for all $\rho > 0$, one has $K_\rho = \{b \in \mathbb{N} : |\omega_b| \geqslant \rho\} \subset \mathfrak{I}$. Hence, for every $\rho > 0$, we have $\dim\left(\left((\Xi(p,r,t))_\upsilon\right)_{K_\rho}\right) = \dim\left(\mathcal{C}^{K_\rho}\right) < \infty$. Therefore, $H_\omega \in \mathbb{F}\left(\left((\Xi(p,r,t))_\upsilon\right)_{K_\rho}\right)$. Assume $\omega_a \in \mathcal{C}^{\mathbb{N}}$, for all $a \in \mathbb{N}$, where

$$(\omega_a)_b = \begin{cases} \omega_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise}. \end{cases}$$

Obviously, $H_{\omega_a} \in \mathbb{F}\left(\left((\Xi(p,r,t))_\upsilon\right)_{B_{\frac{1}{a+1}}}\right)$, since $\dim\left(\left((\Xi(p,r,t))_\upsilon\right)_{B_{\frac{1}{a+1}}}\right) < \infty$, for all $a \in \mathbb{N}$. According to $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$ with $t_0 > 0$, we have

$$\begin{aligned}\upsilon((H_\omega - H_{\omega_a})f) &= \upsilon\left(\left((\omega_b - (\omega_a)_b)f_b\right)_{b=0}^\infty\right) \\ &= \sum_{l=0}^\infty \left(p_l \left|\sum_{b=0}^l r_b(\omega_b - (\omega_a)_b)f_b\right|\right)^{t_l} \\ &= \sum_{l=0, l\in K_{\frac{1}{a+1}}}^\infty \left(p_l \left|\sum_{b=0}^l r_b(\omega_b - (\omega_a)_b)f_b\right|\right)^{t_l} + \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^\infty \left(p_l \left|\sum_{b=0}^l r_b(\omega_b - (\omega_a)_b)f_b\right|\right)^{t_l} \\ &= \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^\infty \left(p_l \left|\sum_{b=0}^l r_b \omega_b f_b\right|\right)^{t_l} \\ &\leqslant \frac{1}{(a+1)^{t_0}} \sum_{l=0, l\notin K_{\frac{1}{a+1}}}^\infty \left(p_l \left|\sum_{b=0}^l r_b f_b\right|\right)^{t_l} \\ &< \frac{1}{(a+1)^{t_0}} \sum_{l=0}^\infty \left(p_l \left|\sum_{b=0}^l r_b f_b\right|\right)^{t_l} = \frac{1}{(a+1)^{t_0}} \upsilon(f).\end{aligned}$$

Hence, $\|H_\omega - H_{\omega_a}\| \leqslant \frac{1}{(a+1)^{t_0}}$. Which investigates that H_ω is a limit of finite rank maps. Therefore, $H_\omega \in \mathcal{A}((\Xi(p,r,t))_\upsilon)$. □

Theorem 3.15. *Assume* $\omega \in \mathcal{C}^{\mathbb{N}}$*, the setup* (f1), (f2)*, and* (f3) *is entrenched. Hence*

$$H_\omega \in \mathcal{K}((\Xi(p,r,t))_\upsilon) \Longleftrightarrow (\omega_b)_{b=0}^{\infty} \in c_0.$$

Proof. Evidently, since $\mathcal{A}((\Xi(p,r,t))_\upsilon) \subsetneqq \mathcal{K}((\Xi(p,r,t))_\upsilon)$. □

Corollary 3.16. *Suppose the setup* (f1), (f2)*, and* (f3) *is proved, hence* $\mathcal{K}((\Xi(p,r,t))_\upsilon) \subsetneqq \mathbb{B}((\Xi(p,r,t))_\upsilon)$.

Proof. As $\omega = (1,1,\ldots)$ creates the multiplication map I on $(\Xi(p,r,t))_\upsilon$. Which explains $I \notin \mathcal{K}((\Xi(p,r,t))_\upsilon)$ and $I \in \mathbb{B}((\Xi(p,r,t))_\upsilon)$. □

Theorem 3.17. *If* $(\Xi(p,r,t))_\upsilon$ *is a pre-quasi Banach* $\mathfrak{pss}$ *and* $H_\omega \in \mathbb{B}((\Xi(p,r,t))_\upsilon)$*, hence there are* $\alpha > 0$ *and* $\eta > 0$ *such that* $\alpha < |\omega_b| < \eta$*, with* $b \in (\ker(\omega))^c$*, if and only if,* Range(H_ω) *is closed.*

Proof. Assume the enough conditions are proved. Hence, there is $\rho > 0$ so that $|\omega_b| \geqslant \rho$, for all $b \in (\ker(\omega))^c$. To explain that Range(H_ω) is closed. Assume g is a limit point of Range(H_ω). We obtain $H_\omega f_b \in (\Xi(p,r,t))_\upsilon$, for every $b \in \mathbb{N}$ so that $\lim_{b\to\infty} H_\omega f_b = g$. Evidently, the sequence $H_\omega f_b$ is a Cauchy sequence. As $(t_l) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ with $t_0 > 0$, one gets

$$\begin{aligned}
\upsilon(H_\omega f_a - H_\omega f_b) &= \sum_{l=0}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k(\omega_k (f_a)_k - \omega_k (f_b)_k)\right|\right)^{t_l} \\
&= \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k(\omega_k (f_a)_k - \omega_k (f_b)_k)\right|\right)^{t_l} \\
&\quad + \sum_{l=0, l\notin(\ker(\omega))^c}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k(\omega_k (f_a)_k - \omega_k (f_b)_k)\right|\right)^{t_l} \\
&\geqslant \sum_{l=0, l\in(\ker(\omega))^c}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k(\omega_k (f_a)_k - \omega_k (f_b)_k)\right|\right)^{t_l} \\
&= \sum_{l=0}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k(\omega_k (u_a)_k - \omega_k (u_b)_k)\right|\right)^{t_l} \\
&> \sum_{l=0}^{\infty}\left(p_l\left|\sum_{k=0}^{l} r_k\rho((u_a)_k - (u_b)_k)\right|\right)^{t_l} \geqslant \inf_l \rho^{t_l}\upsilon\Big(u_a - u_b\Big),
\end{aligned}$$

where

$$(u_a)_k = \begin{cases} (f_a)_k, & k \in (\ker(\omega))^c, \\ 0, & k \notin (\ker(\omega))^c. \end{cases}$$

Hence, $\{u_a\}$ is a Cauchy sequence in $(\Xi(p,r,t))_\upsilon$. As $(\Xi(p,r,t))_\upsilon$ is complete, therefore, there is $f \in (\Xi(p,r,t))_\upsilon$ so that $\lim_{b\to\infty} u_b = f$. Since $H_\omega \in \mathbb{B}((\Xi(p,r,t))_\upsilon)$, one has $\lim_{b\to\infty} H_\omega u_b = H_\omega f$. Since $\lim_{b\to\infty} H_\omega u_b = \lim_{b\to\infty} H_\omega f_b = g$. Therefore, $H_\omega f = g$. Hence $g \in$ Range(H_ω). So Range(H_ω) is closed. Next, assume the necessity setup is confirmed. Hence, there is $\rho > 0$ so that $\upsilon(H_\omega f) \geqslant \rho\upsilon(f)$, with $f \in \Big((\Xi(p,r,t))_\upsilon\Big)_{(\ker(\omega))^c}$. If $K = \Big\{b \in (\ker(\omega))^c : |\omega_b| < \rho\Big\} \neq \emptyset$, hence for $a_0 \in K$, one has

$$\begin{aligned}
\upsilon(H_\omega e_{a_0}) = \upsilon\Big(\Big(\omega_b (e_{a_0})_b\Big)_{b=0}^{\infty}\Big) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{b=0}^{l} r_b\omega_b (e_{a_0})_b\right|\right)^{t_l} &< \sum_{l=0}^{\infty}\left(p_l\left|\sum_{b=0}^{l} r_b(e_{a_0})_b\rho\right|\right)^{t_l} \\
&\leqslant \sup_l \rho^{t_l}\upsilon(e_{a_0}),
\end{aligned}$$

which introduces a contradiction. So $K = \phi$, we have $|\omega_b| \geqslant \rho$, with $b \in (\ker(\omega))^c$. This proves the theorem. □

Theorem 3.18. *Suppose* $\omega \in \mathcal{C}^N$ *and* $(\Xi(p,r,t))_\upsilon$ *is a pre-quasi Banach* $\mathfrak{pss}$. *Hence, there are* $\alpha > 0$ *and* $\eta > 0$ *so that* $\alpha < |\omega_b| < \eta$, *for every* $b \in N$, *if and only if,* $H_\omega \in \mathbb{B}((\Xi(p,r,t))_\upsilon)$ *is invertible.*

Proof. Assume the enough setup are proved. Suppose $\kappa \in \mathcal{C}^N$ with $\kappa_b = \frac{1}{\omega_b}$. In view of Theorem 3.12, the operators H_ω and H_κ are bounded linear. We get $H_\omega.H_\kappa = H_\kappa.H_\omega = I$. Hence $H_\kappa = H_\omega^{-1}$. After, assume H_ω is invertible. Hence $\text{Range}(H_\omega) = \Big((\Xi(p,r,t))_\upsilon\Big)_N$. So, $\text{Range}(H_\omega)$ is closed. Therefore, by using Theorem 3.17, there is $\alpha > 0$ so that $|\omega_b| \geqslant \alpha$, for every $b \in (\ker(\omega))^c$. We have $\ker(\omega) = \emptyset$, if $\omega_{b_0} = 0$, with $b_0 \in N$, which gives $e_{b_0} \in \ker(H_\omega)$, this explains a contradiction, as $\ker(H_\omega)$ is trivial. Therefore, $|\omega_b| \geqslant \alpha$, for every $b \in N$. Since $H_\omega \in \ell_\infty$. By using Theorem 3.12, there is $\eta > 0$ so that $|\omega_b| \leqslant \eta$, for every $b \in N$. Therefore, we have $\alpha \leqslant |\omega_b| \leqslant \eta$, with $b \in N$. □

Theorem 3.19. *Suppose* $(\Xi(p,r,t))_\upsilon$ *is a pre-quasi Banach* $\mathfrak{pss}$ *and* $H_\omega \in \mathbb{B}((\Xi(p,r,t))_\upsilon)$. *Hence* H_ω *is Fredholm operator, if and only if, (i)* $\ker(\omega) \subsetneqq N \cap \mathfrak{I}$ *and (ii)* $|\omega_b| \geqslant \rho$, *with* $b \in (\ker(\omega))^c$.

Proof. Let the enough conditions be satisfied. Assume $\ker(\omega) \subsetneqq N$ is an infinite, hence $e_b \in \ker(H_\omega)$, for every $b \in \ker(\omega)$. Since e_b's are linearly independent, one obtains that $\dim(\ker(H_\omega)) = \infty$, which explains a contradiction. Hence, $\ker(\omega) \subsetneqq N$ must be finite. The setup (ii) follows from Theorem 3.17. Next, suppose the conditions (i) and (ii) are confirmed. In view of Theorem 3.17, the condition (ii) explains that $\text{Range}(H_\omega)$ is closed. The setup (i) gives that $\dim(\ker(H_\omega)) < \infty$ and $\dim((\text{Range}(H_\omega))^c) < \infty$. Hence H_ω is Fredholm. □

4. Features of pre-quasi ideal

In this section, we introduce the enough setup (not necessary) on $(\Xi(p,r,t))_\upsilon$ such that $\overline{\mathbb{F}} = \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$. This investigates a negative answer of Rhoades [37] open problem about the linearity of s-type $(\Xi(p,r,t))_\upsilon$ spaces. Secondly, for which conditions on $(\Xi(p,r,t))_\upsilon$, are $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$ closed and complete? Thirdly, we explain the enough setup on $(\Xi(p,r,t))_\upsilon$ such that $\mathbb{B}^\alpha_{(\Xi(p,r,t))_\upsilon}$ is strictly contained for different weights and powers. We offer the setup so that $\mathbb{B}^\alpha_{(\Xi(p,r,t))_\upsilon}$ is minimum. Fourthly, we introduce the conditions so that the Banach pre-quasi ideal $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$ is simple. Fifthly, we investigate the enough conditions on $(\Xi(p,r,t))_\upsilon$ such that the space of all bounded linear operators which sequence of eigenvalues in $(\Xi(p,r,t))_\upsilon$ equals $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$.

4.1. Finite rank pre-quasi ideal

Theorem 4.1. $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}) = \overline{\mathbb{F}(\mathcal{P},\mathcal{Q})}$, *suppose the setup* (f1), (f2), *and* (f3) *is established. But the converse is not necessarily true.*

Proof. To investigate that $\overline{\mathbb{F}(\mathcal{P},\mathcal{Q})} \subseteq \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$. As $e_l \in (\Xi(p,r,t))_\upsilon$, for every $l \in N$ and $(\Xi(p,r,t))_\upsilon$ is a linear space. Let $Z \in \mathbb{F}(\mathcal{P},\mathcal{Q})$, one gets $(s_l(Z))_{l=0}^\infty \in \mathcal{F}$. To explain that $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}) \subseteq \overline{\mathbb{F}(\mathcal{P},\mathcal{Q})}$, assume $Z \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, we obtain $(s_l(Z))_{l=0}^\infty \in (\Xi(p,r,t))_\upsilon$. Since $\upsilon(s_l(Z))_{l=0}^\infty < \infty$, let $\rho \in (0,1)$, hence there is $l_0 \in N - \{0\}$ with $\upsilon((s_l(Z))_{l=l_0}^\infty) < \frac{\rho}{2^{\hbar+3}\eta d}$, for some $d \geqslant 1$, where $\eta = \max\left\{1, \sum_{l=l_0}^\infty p_l^{t_l}\right\}$. Since $s_l(Z) \in \mathfrak{I}_\searrow$, we get

$$\sum_{l=l_0+1}^{2l_0}\left(p_l\sum_{j=0}^{l} r_j s_{2l_0}(Z)\right)^{t_l} \leqslant \sum_{l=l_0+1}^{2l_0}\left(p_l\sum_{j=0}^{l} r_j s_j(Z)\right)^{t_l} \leqslant \sum_{l=l_0}^{\infty}\left(p_l\sum_{j=0}^{l} r_j s_j(Z)\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{4.1}$$

Hence there is $Y \in \mathbb{F}_{2l_0}(\mathcal{P},\mathcal{Q})$ so that $\text{rank}(Y) \leqslant 2l_0$ and

$$\sum_{l=2l_0+1}^{3l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} \leqslant \sum_{l=l_0+1}^{2l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}, \tag{4.2}$$

since $(t_l) \in \mathfrak{I}_{\nearrow} \cap \ell_\infty$, we have

$$\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{l_0} r_j\|Z-Y\|\right)^{t_l} < \frac{\rho}{2^{2\hbar+2}\eta}. \tag{4.3}$$

Therefore, one has

$$\sum_{l=0}^{l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{4.4}$$

In view of inequalities (1.1)-(4.4), one gets

$$\begin{aligned}
&d(Z,Y)\\
&= \upsilon\left(s_l(Z-Y)\right)_{l=0}^{\infty}\\
&= \sum_{l=0}^{3l_0-1}\left(p_l\sum_{j=0}^{l} r_j s_j(Z-Y)\right)^{t_l} + \sum_{l=3l_0}^{\infty}\left(p_l\sum_{j=0}^{l} r_j s_j(Z-Y)\right)^{t_l}\\
&\leqslant \sum_{l=0}^{3l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(p_{l+2l_0}\sum_{j=0}^{l+2l_0} r_j s_j(Z-Y)\right)^{t_{l+2l_0}}\\
&\leqslant \sum_{l=0}^{3l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(p_l\sum_{j=0}^{l+2l_0} r_j s_j(Z-Y)\right)^{t_l}\\
&\leqslant 3\sum_{l=0}^{l_0}\left(p_l\sum_{j=0}^{i} r_j\|Z-Y\|\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(p_l\left(\sum_{j=0}^{2l_0-1} r_j s_j(Z-Y) + \sum_{j=2l_0}^{l+2l_0} r_j s_j(Z-Y)\right)\right)^{t_l}\\
&\leqslant 3\sum_{l=0}^{l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} + 2^{\hbar-1}\left[\sum_{l=l_0}^{\infty}\left(p_l\sum_{j=0}^{2l_0-1} r_j s_j(Z-Y)\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(p_l\sum_{j=2l_0}^{l+2l_0} r_j s_j(Z-Y)\right)^{t_l}\right]\\
&\leqslant 3\sum_{l=0}^{l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} + 2^{\hbar-1}\left[\sum_{l=l_0}^{\infty}\left(p_l\sum_{j=0}^{2l_0-1} r_j\|Z-Y\|\right)^{t_l} + \sum_{l=l_0}^{\infty}\left(p_l\sum_{j=0}^{l} r_{j+2l_0} s_{j+2l_0}(Z-Y)\right)^{t_l}\right]\\
&\leqslant 3\sum_{l=0}^{l_0}\left(p_l\sum_{j=0}^{l} r_j\|Z-Y\|\right)^{t_l} + 2^{\hbar-1}\sup_{l=l_0}^{\infty}\left(\sum_{j=0}^{2l_0-1} r_j\|Z-Y\|\right)^{t_l}\sum_{l=l_0}^{\infty} p_l^{t_l} + 2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(p_l\sum_{j=0}^{l} r_j s_j(Z)\right)^{t_l} < \rho.
\end{aligned}$$

On the other hand, one has a negative example as $I_4 \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, where $r = (0,0,0,0,1,1,\dots)$ and $t = (1,2,1,2,\dots)$, but $(t_l) \notin \mathfrak{I}_{\nearrow}$. This shows the proof. □

4.2. Banach and closed pre-quasi ideal

Theorem 4.2. *If the setup* (f1), (f2), *and* (f3) *is established, hence* $\left(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}, \Psi\right)$ *is a pre-quasi Banach ideal, where* $\psi(X) = \upsilon\left(\left(s_l(X)\right)_{l=0}^{\infty}\right)$.

Proof. As $(\Xi(p,r,t))_\upsilon$ is a pre-modular $\mathfrak{pss}$, hence from theorem 2.13, Ψ is a pre-quasi norm on $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$. Suppose $(X_b)_{b\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$. As $\mathbb{B}(\mathcal{P},\mathcal{Q}) \supseteq \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, one obtains

$$\Psi(X_a - X_b) = \sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l} r_z s_z(X_a - X_b)\right)^{t_l} \geqslant \left(p_0 r_0\,\|X_a - X_b\|\right)^{t_0},$$

hence $(X_b)_{b\in N}$ is a Cauchy sequence in $\mathbb{B}(\mathcal{P},\mathcal{Q})$. Since $\mathbb{B}(\mathcal{P},\mathcal{Q})$ is a Banach space, then there is $X\in\mathbb{B}(\mathcal{P},\mathcal{Q})$ with $\lim_{b\to\infty}\|X_b-X\|=0$. Since $(s_l(X_b))_{l=0}^{\infty}\in(\Xi(p,r,t))_{\upsilon}$, for every $b\in N$. According to Definition 2.10 setup (ii), (iii), and (v), one gets

$$\begin{aligned}\Psi(X)=\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_z(X)\right)^{t_l}&\leqslant 2^{\hbar-1}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_{[\frac{z}{2}]}(X-X_b)\right)^{t_l}+2^{\hbar-1}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_{[\frac{z}{2}]}(X_b)\right)^{t_l}\\&\leqslant 2^{\hbar-1}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z\,\|X-X_b\|\right)^{t_l}+2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_z(X_b)\right)^{t_l}<\infty.\end{aligned}$$

Therefore, $(s_l(X))_{l=0}^{\infty}\in(\Xi(p,r,t))_{\upsilon}$, then $X\in\mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$. □

Theorem 4.3. *Assume $\mathcal{P}$, $\mathcal{Q}$ are normed spaces, the setup* (f1), (f2), *and* (f3) *is satisfied, hence* $\left(\mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}},\Psi\right)$ *is a pre-quasi closed ideal, where* $\Psi(X)=\upsilon\left((s_l(X))_{l=0}^{\infty}\right)$.

Proof. As $(\Xi(p,r,t))_{\upsilon}$ is a pre-modular $\mathfrak{pss}$, by using theorem 2.13, Ψ is a pre-quasi norm on $\mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}}$. Assume $X_b\in\mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$, for every $b\in N$ and $\lim_{b\to\infty}\Psi(X_b-X)=0$. As $\mathbb{B}(\mathcal{P},\mathcal{Q})\supseteq\mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$, we have

$$\Psi(X-X_b)=\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_z(X-X_b)\right)^{t_l}\geqslant\left(p_0r_0\,\|X-X_b\|\right)^{t_0},$$

hence $(X_b)_{b\in N}$ is a convergent sequence in $\mathbb{B}(\mathcal{P},\mathcal{Q})$. Since $(s_l(X_b))_{l=0}^{\infty}\in(\Xi(p,r,t))_{\upsilon}$, for every $b\in N$. In view of Definition 2.10 and setup (ii), (iii), and (v), one has

$$\begin{aligned}\Psi(X)=\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_z(X)\right)^{t_l}&\leqslant 2^{\hbar-1}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_{[\frac{z}{2}]}(X-X_b)\right)^{t_l}+2^{\hbar-1}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_{[\frac{z}{2}]}(X_b)\right)^{t_l}\\&\leqslant 2^{\hbar-1}\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z\,\|X-X_b\|\right)^{t_l}+2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l}r_z s_z(X_b)\right)^{t_l}<\infty.\end{aligned}$$

We get $(s_l(X))_{l=0}^{\infty}\in(\Xi(p,r,t))_{\upsilon}$, so $X\in\mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}}(\mathcal{P},\mathcal{Q})$. □

4.3. Minimum pre-quasi ideal

Theorem 4.4. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{P})=\dim(\mathcal{Q})=\infty$, *and the setup* (f1), (f2), *and* (f3) *is confirmed with* $0<t_l^{(1)}<t_l^{(2)}$, $0<p_l^{(2)}\leqslant p_l^{(1)}$ *and* $0<r_l^{(2)}\leqslant r_l^{(1)}$, *for all* $l\in N$, *hence*

$$\mathbb{B}^{s}_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_{\upsilon}}(\mathcal{P},\mathcal{Q})\subsetneqq\mathbb{B}^{s}_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_{\upsilon}}(\mathcal{P},\mathcal{Q})\subsetneqq\mathbb{B}(\mathcal{P},\mathcal{Q}).$$

Proof. Let $Z\in\mathbb{B}^{s}_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_{\upsilon}}(\mathcal{P},\mathcal{Q})$, then $(s_l(Z))\in\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_{\upsilon}$. One obtains

$$\sum_{l=0}^{\infty}\left(p_l^{(2)}\sum_{z=0}^{l}r_z^{(2)}s_z(Z)\right)^{t_l^{(2)}}<\sum_{l=0}^{\infty}\left(p_l^{(1)}\sum_{z=0}^{l}r_z^{(1)}s_z(Z)\right)^{t_l^{(1)}}<\infty,$$

then $Z\in\mathbb{B}^{s}_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_{\upsilon}}(\mathcal{P},\mathcal{Q})$. Next, if we choose $(s_l(Z))_{l=0}^{\infty}$ with $\sum_{z=0}^{l}r_z^{(1)}s_z(Z)=\frac{1}{p_l^{(1)}\sqrt[t_l^{(1)}]{l+1}}$, one gets $Z\in\mathbb{B}(\mathcal{P},\mathcal{Q})$ such that

$$\sum_{l=0}^{\infty}\left(p_l^{(1)}\sum_{z=0}^{l}r_z^{(1)}s_z(Z)\right)^{t_l^{(1)}}=\sum_{l=0}^{\infty}\frac{1}{l+1}=\infty,$$

and

$$\sum_{l=0}^{\infty}\left(p_l^{(2)}\sum_{z=0}^{l} r_z^{(2)} s_z(Z)\right)^{t_l^{(2)}} \leqslant \sum_{l=0}^{\infty}\left(p_l^{(1)}\sum_{z=0}^{l} r_z^{(1)} s_z(Z)\right)^{t_l^{(2)}} = \sum_{l=0}^{\infty}\left(\frac{1}{l+1}\right)^{\frac{t_l^{(2)}}{t_l^{(1)}}} < \infty.$$

Therefore, $Z \notin \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})$ and $Z \in \mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})$.

Clearly, $\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}) \subset \mathbb{B}(\mathcal{P},\mathcal{Q})$. Next, if we put $(s_l(Z))_{l=0}^{\infty}$ such that $\sum_{z=0}^{l} r_z^{(2)} s_z(Z) = \frac{1}{p_l^{(2)} \sqrt[t_l^{(2)}]{l+1}}$, we have $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$ such that $Z \notin \mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})$. This explains the proof. □

Theorem 4.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be Banach spaces with $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, and the setup* (f1), (f2), *and* (f3) *is established with $\left(p_l \sum_{z=0}^{l} r_z\right)_{l\in N} \notin \ell_{((t_l))}$, hence $\mathbb{B}^{\alpha}_{(\Xi(p,r,t))_\upsilon}$ is minimum.*

Proof. Suppose the enough setup are confirmed. Then $(\mathbb{B}^{\alpha}_{\Xi(p,r,t)},\Psi)$, where $\Psi(Z) = \sum_{l=0}^{\infty}\left(p_l \sum_{z=0}^{l} r_z \alpha_z(Z)\right)^{t_l}$, is a pre-quasi Banach ideal. Suppose $\mathbb{B}^{\alpha}_{\Xi(p,r,t)}(\mathcal{P},\mathcal{Q}) = \mathbb{B}(\mathcal{P},\mathcal{Q})$, hence there is $\eta > 0$ with $\Psi(Z) \leqslant \eta\|Z\|$, for every $Z \in \mathbb{B}(\mathcal{P},\mathcal{Q})$. According to Dvoretzky's theorem [33], for every $b \in N$, one obtains quotient spaces $\mathcal{P}/Y_b$ and subspaces M_b of $\mathcal{Q}$ which can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \leqslant 2$ and $\|X_b\|\|X_b^{-1}\| \leqslant 2$. Let I_b be the identity operator on ℓ_2^b, T_b be the quotient operator from $\mathcal{P}$ onto $\mathcal{P}/Y_b$ and J_b is the natural embedding operator from M_b into $\mathcal{Q}$. Suppose m_z is the Bernstein numbers [31] then

$$\begin{aligned}
1 = m_z(I_b) = m_z(X_bX_b^{-1}I_bV_bV_b^{-1}) &\leqslant \|X_b\| m_z(X_b^{-1}I_bV_b)\|V_b^{-1}\| = \|X_b\| m_z(J_bX_b^{-1}I_bV_b)\|V_b^{-1}\| \\
&\leqslant \|X_b\| d_z(J_bX_b^{-1}I_bV_b)\|V_b^{-1}\| = \|X_b\| d_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\| \\
&\leqslant \|X_b\| \alpha_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\|,
\end{aligned}$$

for $0 \leqslant l \leqslant b$. We have

$$\begin{aligned}
& p_l \sum_{z=0}^{l} r_z \leqslant p_l \sum_{z=0}^{l} \|X_b\| r_z \alpha_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\| \\
& \Rightarrow \left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} \leqslant (\|X_b\|\|V_b^{-1}\|)^{t_l}\left(p_l \sum_{z=0}^{l} r_z \alpha_z(J_bX_b^{-1}I_bV_bT_b)\right)^{t_l}.
\end{aligned}$$

Hence, for some $\rho \geqslant 1$, one gets

$$\begin{aligned}
& \sum_{l=0}^{b}\left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} \leqslant \rho\|X_b\|\|V_b^{-1}\| \sum_{l=0}^{b}\left(p_l \sum_{z=0}^{l} r_z \alpha_z(J_bX_b^{-1}I_bV_bT_b)\right)^{t_l} \\
& \Rightarrow \sum_{l=0}^{b}\left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} \leqslant \rho\|X_b\|\|V_b^{-1}\|\Psi(J_bX_b^{-1}I_bV_bT_b) \\
& \Rightarrow \sum_{l=0}^{b}\left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} \leqslant \rho\eta\|X_b\|\|V_b^{-1}\|\|J_bX_b^{-1}I_bV_bT_b\| \\
& \Rightarrow \sum_{l=0}^{b}\left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} \leqslant \rho\eta\|X_b\|\|V_b^{-1}\|\|J_bX_b^{-1}\|\|I_b\|\|V_bT_b\| = \rho\eta\|X_b\|\|V_b^{-1}\|\|X_b^{-1}\|\|I_b\|\|V_b\| \leqslant 4\rho\eta.
\end{aligned}$$

Therefore, we have a contradiction, if $b \to \infty$. Then $\mathcal{P}$ and $\mathcal{Q}$ both cannot be infinite dimensional if $\mathbb{B}^{\alpha}_{\Xi(p,r,t)}(\mathcal{P},\mathcal{Q}) = \mathbb{B}(\mathcal{P},\mathcal{Q})$. This shows the proof. □

Theorem 4.6. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, *and the setup* (f1), (f2), *and* (f3) *is confirmed with* $\left(p_l \sum_{z=0}^{l} r_z\right)_{l\in\mathbb{N}} \notin \ell_{((t_l))}$, *hence* $\mathbb{B}^{d}_{\Xi(p,r,t)}$ *is minimum.*

4.4. Simple Banach pre-quasi ideal

Theorem 4.7. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, *and the setup* (f1), (f2), *and* (f3) *is confirmed with* $0 < t_l^{(1)} < t_l^{(2)}$, $0 < p_l^{(2)} \leqslant p_l^{(1)}$ *and* $0 < r_l^{(2)} \leqslant r_l^{(1)}$, *for all* $l \in \mathbb{N}$, *hence*

$$\mathbb{B}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right) = \mathcal{A}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right).$$

Proof. Let $X \in \mathbb{B}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right)$ and

$$X \notin \mathcal{A}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right).$$

Considering Lemma 2.1, there are

$$Y \in \mathbb{B}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right) \text{ and } Z \in \mathbb{B}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right)$$

with $ZXYI_b = I_b$. Therefore, for every $b \in \mathbb{N}$, we get

$$\|I_b\|_{\mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})} = \sum_{l=0}^{\infty}\left(p_l^{(1)}\sum_{z=0}^{l} r_z^{(1)} s_z(I_b)\right)^{t_l^{(1)}} \leqslant \|ZXY\|\|I_b\|_{\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})} \leqslant \sum_{l=0}^{\infty}\left(p_l^{(2)}\sum_{z=0}^{i} r_z^{(2)} s_z(I_j)\right)^{t_l^{(2)}}.$$

This contradicts Theorem 4.4. Then $X \in \mathcal{A}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right)$, which finishes the proof. □

Corollary 4.8. *Assume $\mathcal{P}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, *and the setup* (f1), (f2), *and* (f3) *is established with* $0 < t_l^{(1)} < t_l^{(2)}$, $0 < p_l^{(2)} \leqslant p_l^{(1)}$ *and* $0 < r_l^{(2)} \leqslant r_l^{(1)}$, *for all* $l \in \mathbb{N}$, *hence*

$$\mathbb{B}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right) = \mathcal{K}\left(\mathbb{B}^s_{\left(\Xi((p_l^{(2)}),(r_l^{(2)}),(t_l^{(2)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q}), \mathbb{B}^s_{\left(\Xi((p_l^{(1)}),(r_l^{(1)}),(t_l^{(1)}))\right)_\upsilon}(\mathcal{P},\mathcal{Q})\right).$$

Proof. Evidently, as $\mathcal{A} \subset \mathcal{K}$. □

Theorem 4.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be Banach spaces with* $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, *and the setup* (f1), (f2), *and* (f3) *is satisfied, hence* $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$ *is simple.*

Proof. Assume the closed ideal $\mathcal{K}(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$ includes an operator $X \notin \mathcal{A}(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$. In view of Lemma 2.1, we have $Y, Z \in \mathbb{B}(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$ with $ZXYI_b = I_b$. This gives that $I_{\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})} \in \mathcal{K}(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$. Then $\mathbb{B}(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})) = \mathcal{K}(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}))$. Hence, $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$ is simple Banach space. □

4.5. *Eigenvalues of s-type operators*

Notations 4.10.

$$(\mathbb{B}^s_{\mathcal{E}})^\rho := \left\{ (\mathbb{B}^s_{\mathcal{E}})^\rho\,(\mathcal{P},\mathcal{Q});\ \mathcal{P}\text{and } \mathcal{Q} \text{ are Banach Spaces}\right\}, \text{ where}$$

$$(\mathbb{B}^s_{\mathcal{E}})^\rho\,(\mathcal{P},\mathcal{Q}) := \left\{X \in \mathbb{B}(\mathcal{P},\mathcal{Q}) : ((\rho_l(X))_{n=0}^\infty \in \mathcal{E} \text{ and } \|X-\rho_l(X)I\| \text{ is not invertible, for all } l \in N\right\}.$$

Theorem 4.11. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be Banach spaces with* $\dim(\mathcal{P}) = \dim(\mathcal{Q}) = \infty$, *and the setup* (f1), (f2), *and* (f3) *is established with* $\inf_l \left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} > 0$, *hence*

$$\left(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}\right)^\rho(\mathcal{P},\mathcal{Q}) = \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}).$$

Proof. Suppose $X \in \left(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}\right)^\rho(\mathcal{P},\mathcal{Q})$, hence $(\rho_l(X))_{l=0}^\infty \in (\Xi(p,r,t))_\upsilon$ and $\|X-\rho_l(X)I\| = 0$, for all $l \in N$. We have $X = \rho_l(X)I$, for all $l \in N$, hence $s_l(X) = s_l(\rho_l(X)I) = |\rho_l(X)|$, for every $l \in N$. Therefore, $(s_l(X))_{l=0}^\infty \in (\Xi(p,r,t))_\upsilon$, then $X \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$.

Secondly, suppose $X \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$. Then $(s_l(X))_{l=0}^\infty \in (\Xi(p,r,t))_\upsilon$. Hence, we have

$$\sum_{l=0}^{\infty}\left(p_l\sum_{z=0}^{l} r_z s_z(X)\right)^{t_l} \geqslant \inf_l \left(p_l \sum_{z=0}^{l} r_z\right)^{t_l} \sum_{l=0}^{\infty}[s_l(X)]^{t_l}.$$

Therefore, $\lim_{l\to\infty} s_l(X) = 0$. Assume $\|X-s_l(X)I\|^{-1}$ exists, for every $l \in N$. Hence $\|X-s_l(X)I\|^{-1}$ exists and bounded, for every $l \in N$. Then, $\lim_{l\to\infty}\|X-s_l(X)I\|^{-1} = \|X\|^{-1}$ exists and bounded. As $\left(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}, \Psi\right)$ is a pre-quasi operator ideal, we get

$$I = XX^{-1} \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}) \Rightarrow (s_l(I))_{l=0}^\infty \in \Xi(p,r,t) \Rightarrow \lim_{l\to\infty} s_l(I) = 0.$$

So we have a contradiction, since $\lim_{l\to\infty} s_l(I) = 1$. Hence $\|X-s_l(X)I\| = 0$, for every $l \in N$. This gives $X \in \left(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}\right)^\rho(\mathcal{P},\mathcal{Q})$. This shows the proof. □

5. Kannan contraction operator

Theorem 5.1. *The function* $\upsilon(f) = \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}$ *establishes the Fatou property, for all* $f \in \Xi(p,r,t)$, *assuming the setup* (f1), (f2), *and* (f3) *is confirmed.*

Proof. Suppose $\{g^b\} \subseteq (\Xi(p,r,t))_\upsilon$ with $\lim_{b\to\infty}\upsilon(g^b-g) = 0$. As the space $(\Xi(p,r,t))_\upsilon$ is a pre-quasi closed space, then $g \in (\Xi(p,r,t))_\upsilon$. Hence, for all $f \in (\Xi(p,r,t))_\upsilon$, we have

$$\begin{aligned}\upsilon(f-g) &= \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(f_z-g_z)r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &\leqslant \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(f_z-g_z^b)r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}} + \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(g_z^b-g_z)r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}} \leqslant \sup_j \inf_{b\geqslant j}\upsilon(f-g^b).\end{aligned}$$

□

Theorem 5.2. *The function* $\upsilon(f) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}$ *does not establish the Fatou property, for all* $f \in \Xi(p,r,t)$*, supposing the setup* (f1), (f2)*, and* (f3) *with* $t_l > 1$*, for all* $l \in N$ *is satisfied.*

Proof. Suppose $\{g^b\} \subseteq (\Xi(p,r,t))_\upsilon$ with $\lim_{b\to\infty}\upsilon(g^b - g) = 0$. As the space $(\Xi(p,r,t))_\upsilon$ is a pre-quasi closed space, then $g \in (\Xi(p,r,t))_\upsilon$. Hence, for all $f \in (\Xi(p,r,t))_\upsilon$, we obtains

$$\begin{aligned}\upsilon(f-g) &= \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(f_z-g_z)r_z\right|\right)^{t_l} \leqslant 2^{\hbar-1}\left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(f_z-g_z^b)r_z\right|\right)^{t_l} + \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(g_z^b-g_z)r_z\right|\right)^{t_l}\right]\\ &\leqslant 2^{\hbar-1}\sup_j\inf_{b\geqslant j}\upsilon(f-g^b).\end{aligned}$$

Therefore, υ does not establish the Fatou property. □

Now, we investigate the enough setup on $(\Xi(p,r,t))_\upsilon$ under definite pre-quasi norm so that there is an unique fixed point of Kannan contraction operator.

Theorem 5.3. *Suppose the setup* (f1), (f2)*, and* (f3) *is established, and* $W : (\Xi(p,r,t))_\upsilon \to (\Xi(p,r,t))_\upsilon$ *is Kannan* υ*-contraction operator, where* $\upsilon(f) = \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}$*, for every* $f \in \Xi(p,r,t)$*, then* W *has a unique fixed point.*

Proof. Suppose $f \in \Xi(p,r,t)$, then $W^p f \in \Xi(p,r,t)$. Since W is a Kannan υ-contraction operator, we have

$$\upsilon(W^{p+1}f - W^p f) \leqslant \lambda\left(\upsilon(W^{p+1}f - W^p f) + \upsilon(W^p f - W^{p-1}f)\right)$$
$$\Rightarrow$$
$$\upsilon(W^{p+1}f - W^p f) \leqslant \frac{\lambda}{1-\lambda}\upsilon(W^p f - W^{p-1}f) \leqslant \left(\frac{\lambda}{1-\lambda}\right)^2\upsilon(W^{p-1}f - W^{p-2}f) \leqslant \cdots \leqslant \left(\frac{\lambda}{1-\lambda}\right)^p\upsilon(Wf - f).$$

Therefore, for every $p, q \in N$ with $q > p$, we have

$$\upsilon(W^p f - W^q f) \leqslant \lambda\left(\upsilon(W^p f - W^{p-1}f) + \upsilon(W^q f - W^{q-1}f)\right) \leqslant \lambda\left(\left(\frac{\lambda}{1-\lambda}\right)^{p-1} + \left(\frac{\lambda}{1-\lambda}\right)^{q-1}\right)\upsilon(Wf - f).$$

Hence, $\{W^p f\}$ is a Cauchy sequence in $(\Xi(p,r,t))_\upsilon$. Since the space $(\Xi(p,r,t))_\upsilon$ is pre-quasi Banach space. Then, there exists $g \in (\Xi(p,r,t))_\upsilon$ so that $\lim_{p\to\infty} W^p f = g$. To explain that $Wg = g$, as υ has the Fatou property, we have

$$\upsilon(Wg - g) \leqslant \sup_i\inf_{p\geqslant i}\upsilon(W^{p+1}f - W^p f) \leqslant \sup_i\inf_{p\geqslant i}\left(\frac{\lambda}{1-\lambda}\right)^p\upsilon(Wf - f) = 0,$$

hence $Wg = g$. So, g is a fixed point of W. To investigate that the fixed point is unique. Assume we have two distinct fixed points $b, g \in (\Xi(p,r,t))_\upsilon$ of W. Then, one obtains

$$\upsilon(b - g) \leqslant \upsilon(Wb - Wg) \leqslant \lambda\left(\upsilon(Wb - b) + \upsilon(Wg - g)\right) = 0.$$

Hence, $b = g$. □

Corollary 5.4. *Assume the setup* (f1), (f2)*, and* (f3) *is confirmed, and* $W : (\Xi(p,r,t))_\upsilon \to (\Xi(p,r,t))_\upsilon$ *is Kannan* υ*-contraction operator, where* $\upsilon(f) = \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}$*, for all* $f \in \Xi(p,r,t)$*, hence* W *has an unique fixed point* b *with* $\upsilon(W^p f - b) \leqslant \lambda\left(\frac{\lambda}{1-\lambda}\right)^{p-1}\upsilon(Wf - f)$.

Proof. According to Theorem 5.3, there is an unique fixed point b of W. Therefore, one obtains

$$\upsilon(W^p f - b) = \upsilon(W^p f - Wb) \leqslant \lambda\left(\upsilon(W^p f - W^{p-1} f) + \upsilon(Wb - b)\right) = \lambda\left(\frac{\lambda}{1-\lambda}\right)^{p-1} \upsilon(Wf - f).$$ □

Theorem 5.5. *Suppose the setup* (f1), (f2), *and* (f3) *is established with* $t_l > 1$, *for every* $l \in \mathbb{N}$, *and* $W : (\Xi(p,r,t))_\upsilon \to (\Xi(p,r,t))_\upsilon$, *where* $\upsilon(f) = \sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}$, *for all* $f \in \Xi(p,r,t)$. *The vector* $g \in (\Xi(p,r,t))_\upsilon$ *is the only fixed point of* W, *if the next setup are verified:*

(a) *W is Kannan υ-contraction operator;*
(b) *W is υ-sequentially continuous at $g \in (\Xi(p,r,t))_\upsilon$;*
(c) *there is $\nu \in (\Xi(p,r,t))_\upsilon$ so that the sequence of iterates $\{W^p \nu\}$ has a subsequence $\{W^{p_i}\nu\}$ converges to g.*

Proof. Suppose the enough setup are established. Let g be not a fixed point of W, then $Wg \neq g$. In view of the setup (b) and (c), one obtains

$$\lim_{p_i\to\infty} \upsilon(W^{p_i} f - g) = 0 \text{ and } \lim_{p_i\to\infty} \upsilon(W^{p_i+1} f - Wg) = 0.$$

Since the operator W is Kannan υ-contraction, we have

$$\begin{aligned} 0 < \upsilon(Wg - g) &= \upsilon\left((Wg - W^{p_i+1} f) + (W^{p_i} f - g) + (W^{p_i+1} f - W^{p_i} f)\right) \\ &\leqslant 2^{2\hbar-2}\upsilon\left(W^{p_i+1}\nu - Wg\right) + 2^{2\hbar-2}\upsilon\left(W^{p_i}\nu - g\right) + 2^{\hbar-1}\lambda\left(\frac{\lambda}{1-\lambda}\right)^{p_i-1}\upsilon(Wf - f). \end{aligned}$$

Let $p_i \to \infty$, we get a contradiction. Hence, g is a fixed point of W. To show that the fixed point g is one, assume we have two distinct fixed points $g, b \in (\Xi(p,r,t))_\upsilon$ of W. Hence, one gets

$$\upsilon(g - b) \leqslant \upsilon(Wg - Wb) \leqslant \lambda\left(\upsilon(Wg - g) + \upsilon(Wb - b)\right) = 0.$$

Therefore, $g = b$. □

Example 5.6. If $T : \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon \to \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon$, where $\upsilon(p) = \sqrt{\sum_{t=0}^{\infty}\left(\frac{\left|\sum_{x=0}^{t}\frac{x+2}{x+1}p_x\right|}{t+5}\right)^{\frac{2t+3}{t+2}}}$, with $p \in \Xi\left((\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)$ and

$$T(p) = \begin{cases} \frac{p}{4}, & \upsilon(p) \in [0,1), \\ \frac{p}{5}, & \upsilon(p) \in [1,\infty), \end{cases}$$

since for all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon$ with $\upsilon(p), \upsilon(q) \in [0,1)$, we have

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{4} - \frac{q}{4}) \leqslant \frac{1}{\sqrt[4]{27}}\left(\upsilon(\frac{3p}{4}) + \upsilon(\frac{3q}{4})\right) = \frac{1}{\sqrt[4]{27}}\left(\upsilon(Tp - p) + \upsilon(Tq - q)\right).$$

For all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon$ with $\upsilon(p), \upsilon(q) \in [1,\infty)$, one has

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{5} - \frac{q}{5}) \leqslant \frac{1}{\sqrt[4]{64}}\left(\upsilon(\frac{4p}{5}) + \upsilon(\frac{4q}{5})\right) = \frac{1}{\sqrt[4]{64}}\left(\upsilon(Tp - p) + \upsilon(Tq - q)\right).$$

For every $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $\upsilon(p) \in [0,1)$ and $\upsilon(q) \in [1,\infty)$, we obtain

$$\begin{aligned}\upsilon(Tp - Tq) = \upsilon(\frac{p}{4} - \frac{q}{5}) \leqslant \frac{1}{\sqrt[4]{27}}\upsilon(\frac{3p}{4}) + \frac{1}{\sqrt[4]{64}}\upsilon(\frac{4q}{5}) &\leqslant \frac{1}{\sqrt[4]{27}}\Big(\upsilon(\frac{3p}{4}) + \upsilon(\frac{4q}{5})\Big) \\ &= \frac{1}{\sqrt[4]{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).\end{aligned}$$

Therefore, the operator T is Kannan υ-contraction. Since υ confirms the Fatou property. In view of Theorem 5.3, the operator T has a unique fixed point $\theta \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$.

Suppose $\{p^{(a)}\} \subseteq \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $\lim_{a\to\infty}\upsilon(p^{(a)} - p^{(0)}) = 0$, where $p^{(0)} \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $\upsilon(p^{(0)}) = 1$. Since the pre-quasi norm υ is continuous, we have

$$\lim_{a\to\infty}\upsilon(Tp^{(a)} - Tp^{(0)}) = \lim_{a\to\infty}\upsilon\Big(\frac{p^{(a)}}{4} - \frac{p^{(0)}}{5}\Big) = \upsilon\Big(\frac{p^{(0)}}{20}\Big) > 0.$$

Hence, T is not υ-sequentially continuous at $p^{(0)}$. Therefore, the operator T is not continuous at $p^{(0)}$.

Let $\upsilon(p) = \sum_{t=0}^{\infty}\left(\frac{\left|\sum_{x=0}^{t}\frac{x+2}{x+1}p_x\right|}{t+5}\right)^{\frac{2t+3}{t+2}}$, for every $p \in \Xi\left((\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)$.

Since for all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $\upsilon(p), \upsilon(q) \in [0,1)$, we have

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{4} - \frac{q}{4}) \leqslant \frac{2}{\sqrt{27}}\Big(\upsilon(\frac{3p}{4}) + \upsilon(\frac{3q}{4})\Big) = \frac{2}{\sqrt{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

Let $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $\upsilon(p), \upsilon(q) \in [1,\infty)$, we have

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{5} - \frac{q}{5}) \leqslant \frac{1}{4}\Big(\upsilon(\frac{4p}{5}) + \upsilon(\frac{4q}{5})\Big) = \frac{1}{4}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

For every $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $\upsilon(p) \in [0,1)$ and $\upsilon(q) \in [1,\infty)$, we obtain

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{4} - \frac{q}{5}) \leqslant \frac{2}{\sqrt{27}}\upsilon(\frac{3p}{4}) + \frac{1}{4}\upsilon(\frac{4q}{5}) \leqslant \frac{2}{\sqrt{27}}\Big(\upsilon(\frac{3p}{4}) + \upsilon(\frac{4q}{5})\Big) = \frac{2}{\sqrt{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

Therefore, the operator T is Kannan υ-contraction and $T^r(p) = \begin{cases}\frac{p}{4^r}, & \upsilon(p) \in [0,1),\\ \frac{p}{5^r}, & \upsilon(p) \in [1,\infty).\end{cases}$ Evidently, T is υ-sequentially continuous at $\theta \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ and $\{T^r p\}$ has a subsequence $\{T^{r_j}p\}$ converging to θ. According to Theorem 5.5, the element $\theta \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ is the only fixed point of T.

Example 5.7. Let $T : \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon} \to \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$, with $\upsilon(p) = \sum_{t=0}^{\infty}\left(\frac{\left|\sum_{x=0}^{t}\frac{x+2}{x+1}p_x\right|}{t+5}\right)^{\frac{2t+3}{t+2}}$, for all $p \in \Xi\left((\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)$ and

$$T(p) = \begin{cases}\frac{1}{4}(e_1 + p), & p_0 \in (-\infty, \frac{1}{3}),\\ \frac{1}{3}e_1, & p_0 = \frac{1}{3},\\ \frac{1}{4}e_1, & p_0 \in (\frac{1}{3}, \infty).\end{cases}$$

Since for all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0, q_0 \in (-\infty, \frac{1}{3})$, we get

$$\upsilon(Tp - Tq) = \upsilon(\frac{1}{4}(p_0 - q_0, p_1 - q_1, p_2 - q_2, \dots)) \leqslant \frac{2}{\sqrt{27}}\Big(\upsilon(\frac{3p}{4}) + \upsilon(\frac{3q}{4})\Big) \leqslant \frac{2}{\sqrt{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

For all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0, q_0 \in (\frac{1}{3}, \infty)$, hence for all $\varepsilon > 0$, we have

$$\upsilon(Tp - Tq) = 0 \leqslant \varepsilon\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

For all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0 \in (-\infty, \frac{1}{3})$ and $q_0 \in (\frac{1}{3}, \infty)$, one has

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{4}) \leqslant \frac{1}{\sqrt{27}}\upsilon(\frac{3p}{4}) = \frac{1}{\sqrt{27}}\upsilon(Tp - p) \leqslant \frac{1}{\sqrt{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

Therefore, the operator T is Kannan υ-contraction. Obviously, T is υ-sequentially continuous at $\frac{1}{3}e_1 \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ and there is $p \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0 \in (-\infty, \frac{1}{3})$ such that the sequence of iterates $\{T^r p\} = \left\{\sum_{a=1}^{r}\frac{1}{4^a}e_1 + \frac{1}{4^r}p\right\}$ includes a subsequence $\{T^{r_j} p\} = \left\{\sum_{a=1}^{r_j}\frac{1}{4^a}e_1 + \frac{1}{4^{r_j}}p\right\}$ converging to $\frac{1}{3}e_1$. In view of Theorem 5.5, the operator T has one fixed point

$$\frac{1}{3}e_1 \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}.$$

Note that T is not continuous at $\frac{1}{3}e_1 \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$. Let

$$\upsilon(p) = \sqrt{\sum_{t=0}^{\infty}\left(\frac{\left|\sum_{x=0}^{t}\frac{x+2}{x+1}p_x\right|}{t+5}\right)^{\frac{2t+3}{t+2}}},$$

for all $p \in \Xi\left((\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)$. Since for all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0, q_0 \in (-\infty, \frac{1}{3})$, we have

$$\begin{aligned}\upsilon(Tp - Tq) = \upsilon(\frac{1}{4}(p_0 - q_0, p_1 - q_1, p_2 - q_2, \ldots)) &\leqslant \frac{1}{\sqrt[4]{27}}\Big(\upsilon(\frac{3p}{4}) + \upsilon(\frac{3q}{4})\Big)\\ &\leqslant \frac{1}{\sqrt[4]{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).\end{aligned}$$

For all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0, q_0 \in (\frac{1}{3}, \infty)$, hence for all $\varepsilon > 0$, one has

$$\upsilon(Tp - Tq) = 0 \leqslant \varepsilon\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

For all $p, q \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$ with $p_0 \in (-\infty, \frac{1}{3})$ and $q_0 \in (\frac{1}{3}, \infty)$, we have

$$\upsilon(Tp - Tq) = \upsilon(\frac{p}{4}) \leqslant \frac{1}{\sqrt[4]{27}}\upsilon(\frac{3p}{4}) = \frac{1}{\sqrt[4]{27}}\upsilon(Tp - p) \leqslant \frac{1}{\sqrt[4]{27}}\Big(\upsilon(Tp - p) + \upsilon(Tq - q)\Big).$$

Therefore, the operator T is Kannan υ-contraction. Since υ confirms the Fatou property, according to Theorem 5.3, the operator T has an unique fixed point $\frac{1}{3}e_1 \in \left(\Xi\left((\frac{1}{t+5})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$.

We offer the existence of a fixed point of Kannan contraction operator in the pre-quasi Banach operator ideal generated by $(\Xi(p, r, t))_{\upsilon}$ and s-numbers.

Theorem 5.8. *The pre-quasi norm* $\Psi(W) = \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}r_z s_z(W)\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}$ *does not establish the Fatou property, for every* $W \in \mathbb{B}^{s}_{(\Xi(p,r,t))_{\upsilon}}(\mathcal{P}, \mathcal{Q})$*, when the setup* (f1), (f2)*, and* (f3) *is satisfied.*

Proof. Let the conditions be confirmed and $\{W_p\}_{p\in\mathbb{N}} \subseteq \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$ with $\lim_{p\to\infty}\Psi(W_p - W) = 0$. As the space $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$ is a pre-quasi closed ideal, hence, $W \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$. Then, for all $V \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, one has

$$\begin{aligned}\Psi(V-W) &= \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z s_z(V-W)\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &\leqslant \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z s_{[\frac{z}{2}]}(V-W_i)\right|\right)^{t_l}\right]^{\frac{1}{\hbar}} + \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z s_{[\frac{z}{2}]}(W-W_i)\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &\leqslant 2^{\frac{1}{\hbar}}\sup_p \inf_{i\geqslant p}\left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z s_z(V-W_i)\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}.\end{aligned}$$

Hence, Ψ does not verify the Fatou property. □

Theorem 5.9. *Assume the setup* (f1), (f2), *and* (f3) *is established and* $G : \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q}) \to \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$, *where* $\Psi(W) = \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z s_z(W)\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}$*, for every* $W \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$*. Then* $A \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$ *is the unique fixed point of* G*, if the next setup fulfilled.*

(a) G *is Kannan* Ψ*-contraction mapping.*
(b) G *is* Ψ*-sequentially continuous at a point* $A \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$.
(c) *There is* $B \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$ *such that the sequence of iterates* $\{G^pB\}$ *has a subsequence* $\{G^{p_i}B\}$ *converging to* A.

Proof. Let the enough setup be satisfied. Assume A is not a fixed point of G, then $GA \neq A$. In view of the conditions (b) and (c), one has

$$\lim_{p_i\to\infty}\Psi(G^{p_i}B - A) = 0 \text{ and } \lim_{p_i\to\infty}\Psi(G^{p_i+1}B - GA) = 0.$$

As G is Kannan Ψ-contraction operator, we get

$$\begin{aligned}0 < \Psi(GA-A) &= \Psi\left((GA - G^{p_i+1}B) + (G^{p_i}B - A) + (G^{p_i+1}B - G^{p_i}B)\right)\\ &\leqslant 2^{\frac{1}{\hbar}}\Psi\left(G^{p_i+1}B - GA\right) + 2^{\frac{2}{\hbar}}\Psi\left(G^{p_i}B - A\right) + 2^{\frac{2}{\hbar}}\lambda\left(\frac{\lambda}{1-\lambda}\right)^{p_i-1}\Psi(GB-B).\end{aligned}$$

For $p_i \to \infty$, one obtains a contradiction. Hence, A is a fixed point of G. To prove that the fixed point A is unique, assume we have two distinct fixed points $A, D \in \mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}(\mathcal{P},\mathcal{Q})$ of G. Therefore, one gets

$$\Psi(A-D) \leqslant \Psi(GA-GD) \leqslant \lambda\Big(\Psi(GA-A) + \Psi(GD-D)\Big) = 0.$$

So, $A = D$. □

Example 5.10. Suppose $M : S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon}(\mathcal{P},\mathcal{Q}) \to S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon}(\mathcal{P},\mathcal{Q})$, where $\Psi(H) = \sqrt{\sum_{t=0}^{\infty}\left(\frac{\left|\sum_{x=0}^{t}\frac{x+1}{x+2}s_x\right|}{t+4}\right)^{\frac{2t+3}{t+2}}}$, for each $H \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon}(\mathcal{P},\mathcal{Q})$ and

$$M(H) = \begin{cases}\frac{H}{6}, & \Psi(H) \in [0,1),\\ \frac{H}{7}, & \Psi(H) \in [1,\infty).\end{cases}$$

Since for all $H_1, H_2 \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$ with $\Psi(H_1), \Psi(H_2) \in [0,1)$, we have

$$\Psi(MH_1 - MH_2) = \Psi(\frac{H_1}{6} - \frac{H_2}{6}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(\Psi(\frac{5H_1}{6}) + \Psi(\frac{5H_2}{6})\Big) = \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)\Big).$$

For all $H_1, H_2 \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$ with $\Psi(H_1), \Psi(H_2) \in [1,\infty)$, one has

$$\Psi(MH_1 - MH_2) = \Psi(\frac{H_1}{7} - \frac{H_2}{7}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{216}}\Big(\Psi(\frac{6H_1}{7}) + \Psi(\frac{6H_2}{7})\Big) = \frac{\sqrt{2}}{\sqrt[4]{216}}\Big(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)\Big).$$

For all $H_1, H_2 \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$ with $\Psi(H_1) \in [0,1)$ and $\Psi(H_2) \in [1,\infty)$, one gets

$$\Psi(MH_1 - MH_2) = \Psi(\frac{H_1}{6} - \frac{H_2}{7}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Psi(\frac{5H_1}{6}) + \frac{\sqrt{2}}{\sqrt[4]{216}}\Psi(\frac{6H_2}{7}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(\Psi(MH_1 - H_1) + \Psi(MH_2 - H_2)\Big).$$

Therefore, the operator M is Kannan Ψ-contraction and $M^r(H) = \begin{cases} \frac{H}{6^r}, & \Psi(H) \in [0,1), \\ \frac{H}{7^r}, & \Psi(H) \in [1,\infty). \end{cases}$ Evidently, M is Ψ-sequentially continuous at the zero operator $\Theta \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$ and $\{M^r H\}$ has a subsequence $\{M^{r_j} H\}$ converging to Θ. According to Theorem 5.9, the zero operator

$$\Theta \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$$

is the only fixed point of M.

Assume $\{H^{(a)}\} \subseteq S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$ is such that $\lim_{a\to\infty} \Psi(H^{(a)} - H^{(0)}) = 0$, where $H^{(0)} \in S_{\left(\Xi\left((\frac{1}{t+4})_{t=0}^{\infty},(\frac{t+1}{t+2})_{t=0}^{\infty},(\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}}$ with $\Psi(H^{(0)}) = 1$. Since the pre-quasi norm Ψ is continuous, we have

$$\lim_{a\to\infty} \Psi(MH^{(a)} - MH^{(0)}) = \lim_{a\to\infty} \Psi\Big(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\Big) = \Psi\Big(\frac{H^{(0)}}{42}\Big) > 0.$$

Hence, M is not Ψ-sequentially continuous at $H^{(0)}$. Therefore, the operator M is not continuous at $H^{(0)}$.

6. Existence of solutions of non-linear difference equations

In this section, we explore a solution in $(\Xi(p,r,t))_{\upsilon}$ to summable equations say (6.1), defined in [1, 16, 38], where the setup (f1), (f2), and (f3) is established and $\upsilon(f) = \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l} r_z f_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}$, for all $f \in \Xi(p,r,t)$.

Examine the summable equations:

$$f_z = y_z + \sum_{m=0}^{\infty} A(z,m) g(m, f_m), \tag{6.1}$$

and assume $W : (\Xi(p,r,t))_{\upsilon} \to (\Xi(p,r,t))_{\upsilon}$ is constructed by

$$W(f_z)_{z\in N} = \Big(y_z + \sum_{m=0}^{\infty} A(z,m) g(m, f_m)\Big)_{z\in N}. \tag{6.2}$$

Theorem 6.1. *The summable equation* (6.1) *holds a unique solution in* $(\Xi(p,r,t))_\upsilon$, *when* $A : \mathbb{N}^2 \to \mathbb{R}$, $g : \mathbb{N} \times \mathcal{C} \to \mathcal{C}$, $y : \mathbb{N} \to \mathcal{C}$, *assume there is* $\lambda \in \mathcal{C}$ *so that* $\sup_l |\lambda|^{\frac{t_l}{\hbar}} \in [0, \frac{1}{2})$ *and for all* $l \in \mathbb{N}$, *we have*

$$\left|\sum_{z=0}^{l}\left(\sum_{m\in\mathbb{N}} A(z,m)[g(m,f_m)-g(m,\eta_m)]\right) r_z\right| \leqslant |\lambda|\left[\left|\sum_{z=0}^{l}\left(y_z - f_z + \sum_{m=0}^{\infty} A(z,m)g(m,f_m)\right) r_z\right| + \left|\sum_{z=0}^{l}\left(y_z - \eta_z + \sum_{m=0}^{\infty} A(z,m)g(m,\eta_m)\right) r_z\right|\right].$$

Proof. Let the conditions be established. Assume the mapping $W : (\Xi(p,r,t))_\upsilon \to (\Xi(p,r,t))_\upsilon$ is defined by equation (6.2). Hence

$$\begin{aligned}\upsilon(Wf - W\eta) &= \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}(Wf_z - W\eta_z)r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &= \left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}\left(\sum_{m\in\mathbb{N}} A(z,m)[g(m,f_m)-g(m,\eta_m)]\right) r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &\leqslant \sup_l |\lambda|^{\frac{t_l}{\hbar}}\left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}\left(y_z - f_z + \sum_{m=0}^{\infty} A(z,m)g(m,f_m)\right) r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &\quad + \sup_l |\lambda|^{\frac{t_l}{\hbar}}\left[\sum_{l=0}^{\infty}\left(p_l\left|\sum_{z=0}^{l}\left(y_z - \eta_z + \sum_{m=0}^{\infty} A(z,m)g(m,\eta_m)\right) r_z\right|\right)^{t_l}\right]^{\frac{1}{\hbar}}\\ &= \sup_l |\lambda|^{\frac{t_l}{\hbar}}\left(\upsilon(Wf - f) + \upsilon(W\eta - \eta)\right).\end{aligned}$$

□

In view of Theorem 5.3, we obtain a unique solution of equation (6.1) in $(\Xi(p,r,t))_\upsilon$.

Example 6.2. Suppose the sequence space $\left(\Xi\left((\frac{1}{t+1})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon$, where $\upsilon(f) = \sqrt{\sum_{t=0}^{\infty}\left(\frac{\left|\sum_{x=0}^{t}\frac{x+2}{x+1}f_x\right|}{t+1}\right)^{\frac{2t+3}{t+2}}}$, for all $f \in \left(\Xi\left((\frac{1}{t+1})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon$. Assume the non-linear difference equations:

$$f_z = e^{-(3z+6)} + \sum_{m=0}^{\infty}(-1)^{z+m}\frac{f_{z-2}^{p}}{f_{z-1}^{q} + m^2 + 1}, \tag{6.3}$$

with $p, q, f_{-2}, f_{-1} > 0$ and suppose

$$W : \left(\Xi\left((\frac{1}{t+1})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon \to \left(\Xi\left((\frac{1}{t+1})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_\upsilon,$$

is defined by

$$W(f_z)_{z=0}^{\infty} = \left(e^{-(3z+6)} + \sum_{m=0}^{\infty}(-1)^{z+m}\frac{f_{z-2}^{p}}{f_{z-1}^{q} + m^2 + 1}\right)_{z=0}^{\infty}.$$

Evidently, there is $\lambda \in \mathcal{C}$ such that $\sup_l |\lambda|^{\frac{2l+3}{2l+4}} \in [0, \frac{1}{2})$ and for all $l \in N$, one has

$$\left|\sum_{z=0}^{l}\left(\sum_{m=0}^{\infty}(-1)^z \frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1}\left((-1)^m - (-1)^m\right)\right)\frac{z+2}{z+1}\right|$$
$$\leqslant |\lambda|\left|\sum_{z=0}^{l}\left(e^{-(3z+6)} - f_z + \sum_{m=0}^{\infty}(-1)^{z+m}\frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1}\right)\frac{z+2}{z+1}\right|$$
$$+ |\lambda|\left|\sum_{z=0}^{l}\left(e^{-(3z+6)} - \eta_z + \sum_{m=0}^{\infty}(-1)^{z+m}\frac{f_{z-2}^p}{f_{z-1}^q + m^2 + 1}\right)\frac{z+2}{z+1}\right|.$$

According to Theorem 6.1, the non-linear difference equations (6.3) contain a unique solution in $\left(\Xi\left((\frac{1}{t+1})_{t=0}^{\infty}, (\frac{t+2}{t+1})_{t=0}^{\infty}, (\frac{2t+3}{t+2})_{t=0}^{\infty}\right)\right)_{\upsilon}$.

7. Conclusion

In this article, we offer some topological and geometric properties of $(\Xi(p,r,t))_\upsilon$, of the multiplication maps acting on $(\Xi(p,r,t))_\upsilon$, of the class $\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}$, and of the class $\left(\mathbb{B}^s_{(\Xi(p,r,t))_\upsilon}\right)^\rho$. We investigate the existence of a fixed point of Kannan contraction map acting on these spaces. Some several numerical experiments are introduced to light our results. Furthermore, some successful applications to the existence of solutions of non-linear difference equations are discussed. This article has a number of advantages for researchers such as studying the fixed points of any contraction maps on this pre-quasi normed sequence space which is a generalization of the quasi normed sequence spaces, a new general space of solutions for many difference equations, the spectrum of any bounded linear operators between any two Banach spaces with s-numbers in this sequence space and note that the closed operator ideals are certain to play an important function in the principle of Banach lattices.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No.(UJ-20-110-DR). The authors, therefore, acknowledge with thanks the University technical and financial support. Also, the authors thank the anonymous referees for their constructive suggestions and helpful comments which led to significant improvement of the original manuscript of this paper.

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