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Interval-valued vector optimization problems involving generalized approximate convexity

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Abstract

Interval-valued functions have been recently used to accommodate data inexactness in optimization and decision theory. In this paper, we consider the case of interval-valued vector optimization problems, and derive their relationships to interval variational inequality problems, of both Stampacchia and Minty types. Using the concept of interval approximate convexity, we establish necessary and sufficient optimality conditions for local strong quasi and approximate efficient solutions.

Keywords: Interval-valued vector optimization, generalized approximate LU-e-convexity, interval vector variational inequalities, efficient solutions.

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1. Introduction

In various real-life problems in engineering and economic, the occurrence of imprecision in the data which is taken from measurements or observations is inevitable. Therefore, in reason of simplicity and confidentiality, one has to consider an objective function taking values as real intervals. In the literature, numerous examples can be found where imprecision in real-life applications is modeled by the help of a mathematical tool [7, 9, 15]. Recently, many researchers studied interval-valued vector optimization problems. For instance, in order to characterize solutions of interval-valued programming problems, the Karush-Kuhn-Tucker optimality conditions were obtained in [16–18]. Wolfe and Mond-Weir-type duality were investigated for these problems in [10] where weak and strong duality results were provided.

On the other hand, a considerable and growing interest has been centered about studying the relationship between vector optimization problems and vector variational inequalities. In particular, many results providing optimality conditions in terms of vector variational inequalities were proven for both smooth and nonsmooth vector-valued objective functions; see [1, 2] for a current state-of-the-art. With respect to interval-valued objective functions, Zhang et al. [18, 19] studied optimality conditions under the assumption of LU-convexity introduced by Wu [16] as an extension of convexity for real-valued functions.

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In this work, we adapt the concepts of approximate convexity [14] and generalized approximate convexity [3] to interval-valued vector functions. Afterwards, we will use these concepts as a tool to establish optimality conditions for interval-valued vector optimization problems in terms of Stampacchia and Minty vector variational inequalities using two solutions types: local strong quasi and approximate efficient solutions.

The layout of this article is as follows. First, we recall in Section 2 some preliminary definitions. In Section 3, basic properties and arithmetic for intervals are introduced. In Section 4, sufficient optimality conditions characterizing local strong quasi efficient solutions for interval-valued vector optimization problems are established. In Section 5, sufficient and necessary optimality conditions are proved under generalized approximate convexity assumptions using the concept of approximate vector variational inequalities. Finally, we provide an example in Section 6 illustrating the main results, and conclude our work in Section 7.

2. Preliminaries

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+$ be its nonnegative orthant defined as follows

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n,\ x_i \geqslant 0,\ i = 1, \ldots n\},$$

and X be a nonempty set in $\mathbb{R}^n$. For all $x_0 \in \mathbb{R}^n$ and $\delta > 0$ we denote by $B(x_0, \delta)$ the ball of radius δ and center x_0. For any $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ in X, we say that $x < y$ if $x_i < y_i$ for all $i = 1, 2, \ldots, n$, $x \leqslant y$ if $x_i \leqslant y_i$ for all $i = 1, 2, \ldots, n$ except at least one index for which the inequality holds strict, and $x \leqq y$ if $x_i \leqslant y_i$ for all $i = 1, 2, \ldots, n$.

Definition 2.1 ([4])**.** A function $\phi : X \to \mathbb{R}$ is said to be locally Lipschitz at $x_0 \in X$, if there are positive constants k and δ satisfying for all $x, y \in B(x_0, \delta) \cap X$

$$|\phi(x) - \phi(y)| \leqslant k\|x - y\|.$$

It is said to be locally Lipschitz on X if it is so at each $x_0 \in X$.

Definition 2.2 ([4])**.** Let $f : X \to \mathbb{R}^m$ be a vector valued function such that its components $f_i : X \to \mathbb{R}, i = 1, 2, \ldots, m$ are locally Lipschitz on X.

(i) If $m = 1$, then Clarke's generalized subdifferential of f at $x \in X$ is defined as

$$\partial f(x) := \{y \in \mathbb{R}^n : f^\circ(x; v) \geqslant \langle y, v\rangle,\ \forall v \in \mathbb{R}^n\},$$

where $f^\circ(x; v)$ is Clarke's generalized directional derivative of f along $v \in \mathbb{R}^n$ at $x \in X$, which is defined as

$$f^\circ(x, v) := \limsup_{\substack{t \downarrow 0 \\ y \to x}} \frac{f(y + tv) - f(y)}{t}.$$

(ii) If $m > 1$, then we define Clarke's generalized Jacobian of f at $x \in X$ to be the Cartesian product set

$$\partial f(x) := \partial f_1(x) \times \partial f_2(x) \times \ldots \times \partial f_m(x).$$

Let us recall the notions of approximate convexity which are provided in [3, 6, 14].

Definition 2.3. Let $e > 0$. A function $f : X \to \mathbb{R}$ is called

(i) approximate e-convex at $x_0 \in X$, if there exists $\delta > 0$ such that for all $x, y \in B(x_0, \delta) \cap X$,

$$f(x) - f(y) \geqslant \langle \xi, x - y\rangle - e\|x - y\|, \quad \text{for all } \xi \in \partial f(y);$$

(ii) approximate pseudo e-convex of type I at $x_0 \in X$, if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta) \cap X$,
$$f(x) - f(y) < -e\|x-y\| \quad \Rightarrow \quad \langle \xi, x-y \rangle < 0, \ \forall \xi \in \partial f(y);$$
(iii) approximate pseudo e-convex of type II at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta) \cap X$,
$$f(x) - f(y) < 0 \quad \Rightarrow \quad \langle \xi, x-y \rangle + e\|x-y\| < 0, \ \forall \xi \in \partial f(y);$$
(iv) approximate quasi e-convex of type I at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta) \cap X$,
$$\exists \xi \in \partial f(y) : \ \langle \xi, x-y \rangle - e\|x-y\| > 0 \quad \Rightarrow \quad f(x) > f(y);$$
(v) approximate quasi e-convex of type II at $x_0 \in X$ if there exists $\delta > 0$ such that for any $x, y \in B(x_0, \delta) \cap X$,
$$\exists \xi \in \partial f(y) : \ \langle \xi, x-y \rangle > 0 \quad \Rightarrow \quad f(x) - f(y) > e\|x-y\|.$$

Remark 2.4. The relationship between the above concepts of convexity can be summarized as follows.

1. If f is approximate pseudo (resp. quasi) e-convex of type II at $x_0 \in X$, then f is approximate pseudo (resp. quasi) e-convex of type I at x_0.
2. It is easy to see that any approximate e-convex function at x_0 is approximate pseudo e-convex function of type I and approximate quasi e-convex function of type I at x_0.
3. There is no relation between approximate pseudo e-convex functions of type II and approximate quasi e-convex functions of type II and approximate e-convex functions (see [6]).

3. Interval-valued vector functions

We first recall some basic arithmetic operations on real intervals, which are the same for general sets (for more details on the topic of interval analysis, we refer to [12, 13]). Let us denote by $\mathbb{IR}$ the class of all closed intervals in $\mathbb{R}$, and let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be in $\mathbb{IR}$. The sum and product are defined by

$$A + B := \{a + b : a \in A, \ b \in B\} = [a^L + b^L, a^U + b^U], \quad A \times B := \{ab : a \in A, \ b \in B\} = [\min S, \max S],$$

where $S := \{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$. It is worth mentioning that any real number a can be regarded as a closed interval $A_a = [a, a]$ and for that the sum $a + B$ means $A_a + B$.

From the above operations, we can define the multiplication of an interval with a real number α as

$$\alpha A := \{\alpha a : a \in A\} = \begin{cases} [\alpha a^L, \alpha a^U], & \text{if } \alpha \geqslant 0, \\ [\alpha a^U, \alpha a^L], & \text{if } \alpha < 0. \end{cases}$$

Note the special case of $-A = \{-a : a \in A\} = [-a^U, -a^L]$. Henceforth, the difference of two sets is defined by
$$A - B := A + (-B) = [a^L - b^U, a^U - b^L].$$
On the other hand, an order relation can be defined for intervals. We write $A \preceq_{LU} B$ if $a^L \leqslant b^L$ and $a^U \leqslant b^U$. Also, we say $A \prec_{LU} B$ if $A \preceq_{LU} B$ and $A \neq B$; that is, we have either $a^L < b^L$ and $a^U \leqslant b^U$, $a^L \leqslant b^L$ and $a^U < b^U$, or $a^L < b^L$ and $a^U < b^U$.

Following the partial order defined above, A and B are said to be comparable if $A \preceq_{LU} B$ or $A \succeq_{LU} B$. Henceforth, these two intervals are not comparable means either $A \subseteq B$ or $A \supseteq B$. We call $A = (A_1, \dots, A_n)$ an interval-valued vector if for each $k = 1, \dots, n$ we have $A_k = [a_k^L, a_k^U]$ is a closed interval. We consider two interval-valued vectors $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ such that A_k and B_k are comparable for all $k = 1, \dots, n$. We write

- $A \preceq_{LU} B$ if $A_k \preceq_{LU} B_k$ for all $k = 1, \ldots, n$;
- $A \prec_{LU} B$ if $A_k \preceq_{LU} B_k$ for all $k = 1, \ldots, n$, and $A_j \prec_{LU} B_j$ for at least one j.

An interval-valued function $f : X \to \mathbb{IR}$ is defined by $f(x) = [f^L(x), f^U(x)]$ for each $x \in X$, where f^L and f^U are two real-valued functions on X satisfying $f^L(x) \leqslant f^U(x)$. If $f_1, \ldots, f_m : X \to \mathbb{IR}$ are m interval-valued functions, then we call $f = (f_1, \ldots, f_m) : X \to \mathbb{IR}^m$ an interval-valued vector function.

Definition 3.1 ([19])**.** An interval-valued function $f = [f^L, f^U] : X \to \mathbb{IR}$ is called locally Lipschitz at $x_0 \in X$ with respect to the Hausdorff metric if there exist $L > 0$ and $\delta > 0$ such that for any $x, y \in B(x_0, \delta) \cap X$ one has

$$d_H(f(x), f(y)) \leqslant L\|x - y\|,$$

where $d_H(f(x), f(y))$ is the Hausdorff metric between $f(x)$ and $f(y)$ defined by

$$d_H(f(x), f(y)) = \max\{| f(x)^L - f(y)^L |, | f(x)^U - f(y)^U |\}.$$

f is locally Lipschitz on X if it is so at any $x_0 \in X$.

Proposition 3.2 ([19])**.** *If $f = [f^L, f^U] : X \to \mathbb{IR}$ is locally Lipschitz on X, then both f^L and f^U are locally Lipschitz on X (as real-valued functions).*

From now on, we take $e = (e_1, \ldots, e_m) \in \int(\mathbb{R}^m_+)$. The following concept of generalized approximate LU-e-convexity for nonsmooth interval-valued vector functions can be introduced.

Definition 3.3. Let $f = (f_1, \ldots, f_m) : X \to \mathbb{IR}^m$ such that for each $k \in \{1, \ldots, m\}$ one has $f_k = [f_k^L, f_k^U]$ is a locally Lipschitz interval-valued function.

(i) f_k is approximate LU-e_k-convex (respectively approximate pseudo LU-e_k-convex of type II, approximate quasi LU-e_k-convex of type II) at $x_0 \in X$, if and only if both f_k^L and f_k^U are approximate e_k-convex (respectively approximate pseudo e_k-convex of type II, approximate quasi e_k-convex of type II) at x_0.

(ii) f is approximate LU-e-convex (respectively approximate pseudo LU-e-convex of type II, approximate quasi LU-e-convex of type II) at $x_0 \in X$, if and only if for all $k \in \{1, \ldots, m\}$ we have f_k is approximate e_k-convex (respectively approximate pseudo e_k-convex, approximate quasi e_k-convex) at x_0.

Remark 3.4. From Remark 2.4, it is easy to check that there is no relation between approximate pseudo LU-e-convex functions of type II and approximate quasi LU-e-convex functions of type II and approximate e-convex functions.

Example 3.5. Let $f = [f^L, f^U]$ such that for all $x \in \mathbb{R}$ one has $f^L(x) = x^3 - 2x^2$ and $f^U(x) = x^3 - x^2$. An easy calculation shows that, for $e > 0$, the interval-valued function f is approximate LU-e-convex and approximate quasi LU-e-convex of type II at $\bar{x} = 0$, but there does not exist any $\delta > 0$ such that f is approximate pseudo LU-e-convex of type II at $\bar{x} = 0$.

Example 3.6. Consider the following interval-valued function $f = [f^L, f^U]$ such that

$$f^L(x) = \begin{cases} x^2 + x, & x \geqslant 0, \\ x^2, & x < 0, \end{cases} \quad \text{and} \quad f^U(x) = \begin{cases} x^2 + x + 1, & x \geqslant 0, \\ x^2 + 1, & x < 0. \end{cases}$$

We can easily show that, for $e = 2$, f is approximate LU-e-convex and approximate pseudo LU-e-convex of type II at $\bar{x} = 0$, but there does not exist any $\delta > 0$ such that f is approximate quasi LU-e-convex of type II at $\bar{x} = 0$.

To introduce the interval-valued vector optimization problem (IVOP), we consider in what follows an interval-valued multiobjective function $f = (f_1, \dots, f_m)$. Each component objective function $f_k = [f_k^L, f_k^U]$ is a locally Lipschitz interval-valued function defined on the nonempty feasible set $X \subseteq \mathbb{R}^n$. Our optimization problem is written as:

$$\min_{x \in X} f(x) = (f_1(x), f_2(x), \dots, f_m(x)). \tag{IVOP}$$

Assume we are given a vector $\bar{x}$ as a feasible solution to (IVOP).

Definition 3.7 ([17])**.** The vector $\bar{x}$ is said to be

i) an efficient solution of (IVOP) if there is no $x \in X$ with $f(x) \prec_{LU} f(\bar{x})$;
ii) a strong efficient solution of (IVOP) if there is no $x \in X$ with $f(x) \preceq_{LU} f(\bar{x})$.

In [8], the concepts of quasi efficient solutions of vector optimization introduced in [5] was extended to local e-quasi efficient and local strong e-quasi efficient solutions. We adopt here the latter type of solutions to our problem (IVOP).

Definition 3.8. The vector $\bar{x}$ is said to be a local (strong) e-quasi efficient solution to (IVOP) if there is $\delta > 0$ such that there is no $x \in B(\bar{x}, \delta) \cap X$ satisfying

$$f(x) + e\|x - \bar{x}\| \prec_{LU} \ (\preceq_{LU})\ f(\bar{x}).$$

We introduce the following concept of approximate efficient solutions to (IVOP), which is useful when no efficient solution exists.

Definition 3.9. The vector $\bar{x}$ is said to be an e-approximate efficient solution of (IVOP) if there is no $\delta > 0$ such that, for all $x \in B(\bar{x}, \delta) \cap X$, one has

$$f(x) + e\|x - \bar{x}\| \preceq_{LU} f(\bar{x}).$$

The following proposition shows the relation between the notion of local strong quasi efficiency and the notion of approximate efficiency.

Proposition 3.10. *If $\bar{x}$ is a local strong e-quasi efficient solution to* (IVOP)*, then $\bar{x}$ is an e-approximate efficient solution of* (IVOP).

Proof. Assume the vector $\bar{x}$ is not an e-approximate efficient solution of (IVOP). Then, there exists $\bar{\delta} > 0$ such that, for all $x \in B(\bar{x}, \bar{\delta}) \cap X$, we have

$$f(x) + e\|x - \bar{x}\| \preceq_{LU} f(\bar{x}).$$

Let $\delta > 0$. There is $x \in B(\bar{x}, \bar{\delta}) \cap B(\bar{x}, \delta) \cap X$ satisfying the above inequality. Hence $\bar{x}$ is not a local strong e-quasi efficient solution to (IVOP). □

Remark 3.11. The converse of the above proposition is not generally true as shown in the following example.

Example 3.12. Consider the following example of (IVOP):

$$\min f(x) = [f^L(x), f^U(x)]$$
$$\text{subject to} \quad x \in X = [-1, 1],$$

where

$$f^L(x) = -2x \quad \text{and} \quad f^U(x) = x^2 - 2x.$$

Let $e = 1$ and $\bar{x} = 0$. Observe first that for any $\delta > 0$ and $x \in (-\delta, 0) \cap X$, the inequality

$$f(x) + |x| = [-3x, x^2 - 3x] \preceq_{LU} f(0) = [0, 0]$$

do not hold true. Hence, we deduce that there exist no $\delta > 0$ such that, for all $x \in (-\delta, \delta) \cap X$, one has

$$f(x) + e\|x - \bar{x}\| \preceq_{LU} f(0).$$

This means that $\bar{x} = 0$ is an e-approximate efficient solution to (IVOP).

However, it is easy to check that for any $\delta > 0$ there is $x \in (0, \delta) \cap X \subset (-\delta, \delta) \cap X$, satisfying

$$f(x) + e\|x - \bar{x}\| \preceq_{LU} f(0).$$

Therefore $\bar{x}$ is not a local strong e-quasi efficient solution to (IVOP).

4. Sufficient conditions for local strong e-quasi efficient solutions

We consider the following vector variational inequalities of Stampacchia and Minty types:

$$\text{Find } \bar{x} \in X \text{ such that } \begin{cases} \langle \xi^L, x - \bar{x} \rangle_m > 0, \ \forall \xi^L \in \partial f_1^L(\bar{x}) \times \cdots \times \partial f_m^L(\bar{x}), \\ \langle \xi^U, x - \bar{x} \rangle_m > 0, \ \forall \xi^U \in \partial f_1^U(\bar{x}) \times \cdots \times \partial f_m^U(\bar{x}), \end{cases} \quad \forall x \in X. \tag{SVVI}$$

$$\text{Find } \bar{x} \in X \text{ such that } \begin{cases} \langle \zeta^L, x - \bar{x} \rangle_m > 0, \ \forall \zeta^L \in \partial f_1^L(x) \times \cdots \times \partial f_m^L(x), \\ \langle \zeta^U, x - \bar{x} \rangle_m > 0, \ \forall \zeta^U \in \partial f_1^U(x) \times \cdots \times \partial f_m^U(x), \end{cases} \quad \forall x \in X. \tag{MVVI}$$

Here $\langle \xi^L, x - \bar{x} \rangle_m = \big(\langle \xi_1^L, x - \bar{x} \rangle, \dots, \langle \xi_m^L, x - \bar{x} \rangle\big)$ (the same if we replace ξ^L by ξ^U, or ζ^L, or ζ^U).

We first present sufficient conditions for local strong e-quasi efficient solutions of (IVOP) using the approximate pseudo LU-e-convexity of type II assumption.

Theorem 4.1. *Suppose f is approximate pseudo LU-e-convex of type II at $\bar{x} \in X$. If $\bar{x}$ is a solution to (SVVI), then it is also a local strong e-quasi efficient solution to (IVOP).*

Proof. Assume $\bar{x}$ fails to be a local strong e-quasi efficient solution to (IVOP). Hence for any $\delta > 0$, there exists $x_0 \in B(\bar{x}, \delta) \cap X$ such that

$$f(x_0) + e\|x_0 - \bar{x}\| \preceq_{LU} f(\bar{x}),$$

which implies that

$$\begin{cases} f_k^L(x_0) + e_k\|x_0 - \bar{x}\| \leqslant f_k^L(\bar{x}), \\ f_k^U(x_0) + e_k\|x_0 - \bar{x}\| \leqslant f_k^U(\bar{x}), \end{cases}$$

for each $k = 1, \dots, m$. Then

$$\begin{cases} f_k^L(x_0) - f_k^L(\bar{x}) \leqslant -e_k\|x_0 - \bar{x}\| < 0, \\ f_k^U(x_0) - f_k^U(\bar{x}) \leqslant -e_k\|x_0 - \bar{x}\| < 0, \end{cases} \tag{4.1}$$

are satisfied for all $k = 1, \dots, m$. Since f is approximate pseudo LU-e-convex of type II at $\bar{x}$, then $f_k = [f_k^L, f_k^U]$ is a approximate pseudo LU-e_k-convex function of type II at $\bar{x}$ for all $k = 1, \dots, m$. Hence, f_k^L and f_k^U are both approximate pseudo LU-e_k-convex of type II functions at $\bar{x}$ for all $k = 1, \dots, m$. Then, there exists $\bar{\delta} > 0$ such that for each $x \in B(\bar{x}, \bar{\delta}) \cap X$ and $k = 1, \dots, m$,

$$\begin{cases} f_k^L(x) - f_k^L(\bar{x}) < 0 \Rightarrow \langle \xi_k^L, x - \bar{x} \rangle < -e_k\|x - \bar{x}\| \leqslant 0, & \xi_k^L \in \partial f_k^L(\bar{x}), \\ f_k^U(x) - f_k^U(\bar{x}) < 0 \Rightarrow \langle \xi_k^U, x - \bar{x} \rangle < -e_k\|x - \bar{x}\| \leqslant 0, & \xi_k^U \in \partial f_k^U(\bar{x}). \end{cases} \tag{4.2}$$

From (4.1) and (4.2), we deduce that there is $x_0 \in B(\bar{x}, \bar{\delta}) \cap X$ satisfying

$$\begin{cases} \langle \xi^L, x_0 - \bar{x} \rangle_m \leqq 0, & \xi^L \in \partial f_1^L(\bar{x}) \times \cdots \times \partial f_m^L(\bar{x}), \\ \langle \xi^U, x_0 - \bar{x} \rangle_m \leqq 0, & \xi^U \in \partial f_1^U(\bar{x}) \times \cdots \times \partial f_m^U(\bar{x}). \end{cases}$$

We conclude that $\bar{x}$ is not a solution of (SVVI). □

Remark 4.2. We can obtain the same result of the above theorem using the approximate LU-e-convexity assumption (see Theorem 5.2 in [19]).

Sufficient optimality conditions in terms of (MVVI) instead of (SVVI) requires approximate convexity assumptions to be imposed on $-f_k$ as shown in the next theorem.

Theorem 4.3. *Suppose $-f$ is approximate LU-e-convex at $\bar{x}$. If $\bar{x}$ is a solution to (MVVI), then it is a local strong e-quasi efficient solution to (IVOP).*

Proof. Assume the vector $\bar{x}$ fails to be a local strong e-quasi efficient solution to (IVOP). Hence for any $\delta > 0$ there exists $x_0 \in B(\bar{x}, \delta) \cap X$ satisfying

$$f(x_0) + e\|x_0 - \bar{x}\| \preceq_{LU} f(\bar{x}),$$

which implies that

$$\begin{cases} f_k^L(x_0) + e_k\|x_0 - \bar{x}\| \leqslant f_k^L(\bar{x}), \\ f_k^U(x_0) + e_k\|x_0 - \bar{x}\| \leqslant f_k^U(\bar{x}), \end{cases}$$

for each $k = 1, \ldots, m$. Then

$$\begin{cases} f_k^L(x_0) - f_k^L(\bar{x}) + e_k\|x_0 - \bar{x}\| \leqslant 0, \\ f_k^U(x_0) - f_k^U(\bar{x}) + e_k\|x_0 - \bar{x}\| \leqslant 0, \end{cases} \tag{4.3}$$

are satisfied for each $k = 1, \ldots, m$.

Since $-f$ is approximate LU-e-convex at $\bar{x}$, then $-f_k = [-f_k^U, -f_k^L]$ is approximate LU-e_k-convex function at $\bar{x}$ for all $k = 1, \ldots, m$. Therefore, both $-f_k^L$ and $-f_k^U$ are approximate e_k-convex functions at $\bar{x}$ for all $k = 1, \ldots, m$. Then, there is $\bar{\delta} > 0$ such that for each $x \in B(\bar{x}, \bar{\delta}) \cap X$ and $k = 1, \ldots, m$,

$$\begin{cases} (-f_k^L)(\bar{x}) - (-f_k^L)(x) \geqslant \langle \zeta_k^L, \bar{x} - x \rangle - e_k\|\bar{x} - x\|, & \forall \zeta_k^L \in \partial(-f_k^L)(x), \\ (-f_k^U)(\bar{x}) - (-f_k^U)(x) \geqslant \langle \zeta_k^U, \bar{x} - x \rangle - e_k\|\bar{x} - x\|, & \forall \zeta_k^U \in \partial(-f_k^U)(x), \end{cases}$$

which implies that

$$\begin{cases} f_k^L(x) - f_k^L(\bar{x}) \geqslant \langle \zeta_k^L, \bar{x} - x \rangle - e_k\|x - \bar{x}\|, & \forall \zeta_k^L \in \partial(-f_k^L)(x), \\ f_k^U(x) - f_k^U(\bar{x}) \geqslant \langle \zeta_k^U, \bar{x} - x \rangle - e_k\|x - \bar{x}\|, & \forall \zeta_k^U \in \partial(-f_k^U)(x). \end{cases} \tag{4.4}$$

Using (4.3), (4.4), and taking into account the fact that $\partial(-f)(x) = -\partial f(x)$, we obtain that there is $x_0 \in B(\bar{x}, \bar{\delta}) \cap X$ such that for all $\zeta_k^L \in \partial f_k^L(x_0)$ and $\zeta_k^U \in \partial f_k^U(x_0)$,

$$\begin{cases} \langle \zeta_k^L, x_0 - \bar{x} \rangle = \langle -\zeta_k^L, \bar{x} - x_0 \rangle \leqslant f_k^L(x_0) - f_k^L(\bar{x}) + e_k\|x_0 - \bar{x}\| \leqslant 0, \\ \langle \zeta_k^U, x_0 - \bar{x} \rangle = \langle -\zeta_k^U, \bar{x} - x_0 \rangle \leqslant f_k^U(x_0) - f_k^U(\bar{x}) + e_k\|x_0 - \bar{x}\| \leqslant 0. \end{cases}$$

Therefore, there is $x_0 \in B(\bar{x}, \bar{\delta}) \cap X$ satisfying

$$\begin{cases} \langle \zeta^L, x_0 - \bar{x} \rangle_m \leqq 0, & \forall \zeta^L \in \partial f_1^L(x_0) \times \cdots \times \partial f_m^L(x_0), \\ \langle \zeta^U, x_0 - \bar{x} \rangle_m \leqq 0, & \forall \zeta^U \in \partial f_1^U(x_0) \times \cdots \times \partial f_m^U(x_0). \end{cases}$$

We conclude that $\bar{x}$ does not solve (MVVI). □

The previous result still hold true if we replace the approximate convexity assumption by approximate pseudo convexity.

Theorem 4.4. *Suppose* $-f$ *is approximate pseudo* LU-e-*convex of type II at* $\overline{x}$. *If* $\overline{x}$ *solves* (MVVI), *then* $\overline{x}$ *is a local strong* e-*quasi efficient solution to* (IVOP).

Proof. The proof is similar to that of Theorem 4.3. □

Remark 4.5. The results of this section gives new sufficient conditions for local strong e-quasi efficient solutions of (IVOP) by using the concepts of (MVVI) and approximate pseudo/quasi LU-e-convexity of type II, which improve [19, Theorem 5.2].

5. Necessary and sufficient conditions for e-approximate efficient solutions

We consider the following approximate vector variational inequalities of Stampacchia and Minty type:

Find $\overline{x} \in X$ such that there is no $\delta > 0$ satisying

$$\begin{cases} \langle \xi^L, x-\overline{x}\rangle_m \leqq -e\|x-\overline{x}\|, \ \forall \xi^L \in \partial f_1^L(\overline{x}) \times \cdots \times \partial f_m^L(\overline{x}), \\ \langle \xi^U, x-\overline{x}\rangle_m \leqq -e\|x-\overline{x}\|, \ \forall \xi^U \in \partial f_1^U(\overline{x}) \times \cdots \times \partial f_m^U(\overline{x}), \end{cases} \quad \forall x \in B(\overline{x},\delta)\cap X. \tag{ASVVI}$$

Find $\overline{x} \in X$ such that there is no $\delta > 0$ satisfying

$$\begin{cases} \langle \zeta^L, x-\overline{x}\rangle_m \leqq -e\|x-\overline{x}\|, \ \forall \zeta^L \in \partial f_1^L(x) \times \cdots \times \partial f_m^L(x), \\ \langle \zeta^U, x-\overline{x}\rangle_m \leqq -e\|x-\overline{x}\|, \ \forall \zeta^U \in \partial f_1^U(x) \times \cdots \times \partial f_m^U(x), \end{cases} \quad \forall x \in B(\overline{x},\delta)\cap X. \tag{AMVVI}$$

Hereafter, if the above definition is fulfilled for a given e, then we say that $\overline{x}$ is a solution for (ASVVI) (or (AMVVI)) with respect to e.

In the following theorem, we will see that solutions to (ASVVI) are also e-approximate efficient solutions of (IVOP) when the interval-valued objective function satisfies the pseudo approximate convexity hypothesis.

Theorem 5.1. *Suppose* f *is approximate pseudo* LU-e-*convex function of type II at* $\overline{x}$. *If* $\overline{x}$ *solves* (ASVVI) *w.r.t.* e, *then* $\overline{x}$ *is an* e-*approximate efficient solution to* (IVOP).

Proof. Assume the vector $\overline{x}$ is not an e-approximate efficient solution of (IVOP). Hence, there exists $\overline{\delta} > 0$ such that, for all $x \in B(\overline{x},\overline{\delta})\cap X$, we have

$$f(x) + e\|x-\overline{x}\| \preceq_{LU} f(\overline{x}),$$

which implies that

$$\begin{cases} f_k^L(x) + e_k\|x-\overline{x}\| \leqslant f_k^L(\overline{x}), \\ f_k^U(x) + e_k\|x-\overline{x}\| \leqslant f_k^U(\overline{x}), \end{cases}$$

for each $k = 1,\dots,m$. Then

$$f_k^L(x) - f_k^L(\overline{x}) < 0 \quad \text{and} \quad f_k^U(x) - f_k^U(\overline{x}) < 0$$

hold true for any $k = 1,\dots,m$.

Since f is approximate pseudo LU-e-convex function of type II at $\overline{x}$, then $f_k = [f_k^L, f_k^U]$ is approximate pseudo LU-e_k-convex function of type II at $\overline{x}$ for all $k = 1,\dots,m$. Therefore, both f_k^L and f_k^U are approximate pseudo LU-e_k-convex functions of type II at $\overline{x}$ for all $k = 1,\dots,m$. Consequently, there exists $\delta > 0$ with $\delta < \overline{\delta}$, such that, for all $x \in B(\overline{x},\delta)\cap X$ and $k = 1,\dots,m$ one has

$$\begin{cases} \langle \xi_k^L, x-\overline{x}\rangle < -e_k\|x-\overline{x}\|, \quad \forall \xi_k^L \in \partial f_k^L(\overline{x}), \\ \langle \xi_k^U, x-\overline{x}\rangle < -e_k\|x-\overline{x}\|, \quad \forall \xi_k^U \in \partial f_k^U(\overline{x}). \end{cases} \tag{5.1}$$

From (5.1), there is $\delta > 0$ such that for all $x \in B(\overline{x}, \delta) \cap X$ one has

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \xi^L \in \partial f_1^L(\overline{x}) \times \cdots \times \partial f_m^L(\overline{x}), \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, & \forall \xi^U \in \partial f_1^U(\overline{x}) \times \cdots \times \partial f_m^U(\overline{x}). \end{cases}$$

We deduce that $\overline{x}$ cannot be a solution of (ASVVI) with respect to e. □

In the following theorem, we prove that every e-approximate efficient solution to (IVOP) is still a solution of (ASVVI) w.r.t. e in the case of approximate quasi LU-e-convexity of type II of $-f$.

Theorem 5.2. *Suppose $-f$ is approximate quasi LU-e-convex function of type II at $\overline{x}$. If $\overline{x}$ is an e-approximate efficient solution to (IVOP), then $\overline{x}$ solves (ASVVI) w.r.t. e.*

Proof. Assume that $\overline{x}$ is not a solution of (ASVVI) w.r.t. e. Hence, there is $\overline{\delta} > 0$ such that, for all $x \in B(\overline{x}, \overline{\delta}) \cap X$, $\xi^L \in \partial f_1^L(\overline{x}) \times \cdots \times \partial f_m^L(\overline{x})$ and $\xi^U \in \partial f_1^U(\overline{x}) \times \cdots \times \partial f_m^U(\overline{x})$ one has

$$\begin{cases} \langle \xi^L, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|, \\ \langle \xi^U, x - \overline{x} \rangle_m \leqq -e\|x - \overline{x}\|. \end{cases}$$

Then,

$$\begin{cases} \langle \xi_k^L, x - \overline{x} \rangle \leqslant -e_k\|x - \overline{x}\| < 0, & \forall \xi_k^L \in \partial f_k^L(\overline{x}), \\ \langle \xi_k^U, x - \overline{x} \rangle \leqslant -e_k\|x - \overline{x}\| < 0, & \forall \xi_k^U \in \partial f_k^U(\overline{x}), \end{cases}$$

hold true for all $k = 1, \ldots, m$. Consequently, from $\partial(-f)(x) = -\partial f(x)$, it follows that

$$\begin{cases} \langle -\xi_k^L, x - \overline{x} \rangle > 0, & (-\xi_k^L) \in \partial(-f_k^L)(\overline{x}), \\ \langle -\xi_k^U, x - \overline{x} \rangle > 0, & (-\xi_k^U) \in \partial(-f_k^U)(\overline{x}). \end{cases} \tag{5.2}$$

Since $-f$ is approximate quasi LU-e-convex function of type II at $\overline{x}$, then $-f_k = [-f_k^U, -f_k^L]$ is approximate quasi LU-e_k-convex function of type II at $\overline{x}$ for all $k = 1, \ldots, m$. Therefore, both $-f_k^L$ and $-f_k^U$ are approximate quasi LU-e_k-convex functions of type II at $\overline{x}$ for all $k = 1, \ldots, m$. Then, by (5.2) there is $\delta > 0$ with $\delta < \overline{\delta}$, such that, for each $x \in B(\overline{x}, \delta) \cap X$, one has

$$\begin{cases} (-f_k^L)(x) - (-f_k^L)(\overline{x}) > e_k\|x - \overline{x}\|, \\ (-f_k^U)(x) - (-f_k^U)(\overline{x}) > e_k\|x - \overline{x}\|, \end{cases}$$

hold true for all $k = 1, \ldots, m$. This yields

$$\begin{cases} f_k^L(x) + e_k\|x - \overline{x}\| < f_k^L(\overline{x}), \\ f_k^U(x) + e_k\|x - \overline{x}\| < f_k^U(\overline{x}). \end{cases}$$

Therefore there is $\delta > 0$ satisfying for each $x \in B(\overline{x}, \delta) \cap X$,

$$f(x) + e\|x - \overline{x}\| \preceq_{LU} f(\overline{x}).$$

This proves the theorem as $\overline{x}$ cannot be an e-approximate efficient solution to (IVOP). □

A direct consequence of Theorem 5.1 and Theorem 5.2 is presented in the following corollary.

Corollary 5.3. *Suppose f is approximate pseudo LU-e-convex of type II at $\overline{x} \in X$ and $-f$ is approximate quasi LU-e-convex of type II at $\overline{x}$. Then, $\overline{x}$ is an e-approximate efficient solution to (IVOP) if and only if $\overline{x}$ solves (ASVVI) w.r.t. e.*

The following theorem illustrates when a solution of (AMVVI) w.r.t. e is also an e-approximate efficient solution to (IVOP).

Theorem 5.4. *Suppose* $-f$ *is approximate pseudo* LU-e-*convex function of type II at* $\bar{x}$. *If* $\bar{x}$ *solves* (AMVVI) *w.r.t.* e, *then* $\bar{x}$ *is an* e-*approximate efficient solution to* (IVOP).

Proof. Assume that $\bar{x}$ is not an e-approximate efficient solution to (IVOP). Thus, there exists $\bar{\delta} > 0$ such that, for all $x \in B(\bar{x}, \bar{\delta}) \cap X$, we have

$$f(x) + e\|x - \bar{x}\| \preceq_{LU} f(\bar{x}),$$

which implies that

$$\begin{cases} f_k^L(x) + e_k\|x - \bar{x}\| \leqslant f_k^L(\bar{x}), \\ f_k^U(x) + e_k\|x - \bar{x}\| \leqslant f_k^U(\bar{x}), \end{cases}$$

for each $k = 1, \dots, m$. Then

$$f_k^L(x) - f_k^L(\bar{x}) \leqslant -e_k\|x - \bar{x}\| < 0 \quad \text{and} \quad f_k^U(x) - f_k^U(\bar{x}) \leqslant e_k\|x - \bar{x}\| < 0,$$

are satisfied for each $k = 1, \dots, m$. Then

$$\begin{cases} -f_k^L(\bar{x}) - (-f_k^L)(x) < 0, \\ -f_k^U(\bar{x}) - (-f_k^U)(x) < 0. \end{cases} \tag{5.3}$$

Since $-f$ is approximate pseudo LU-e-convex function of type II at $\bar{x}$, then $-f_k = [-f_k^U, -f_k^L]$ is approximate pseudo LU-e_k-convex function of type II at $\bar{x}$ for all $k = 1, \dots, m$. Therefore, both $-f_k^L$ and $-f_k^U$ are approximate pseudo LU-e_k-convex functions of type II at $\bar{x}$. Then, by (5.3) there exists $\delta > 0$ with $\delta < \bar{\delta}$, such that, for all $x \in B(\bar{x}, \delta) \cap X$,

$$\begin{cases} \langle \zeta_k^L, \bar{x} - x\rangle < -e_k\|\bar{x} - x\|, & \zeta_k^L \in \partial(-f_k^L)(x), \\ \langle \zeta_k^U, \bar{x} - x\rangle < -e_k\|\bar{x} - x\|, & \zeta_k^U \in \partial(-f_k^U)(x). \end{cases} \tag{5.4}$$

Using (5.4) and taking into account the fact that $\partial(-f)(x) = -\partial f(x)$ for all $x \in X$, we obtain

$$\begin{cases} \langle \zeta_k^L, x - \bar{x}\rangle = \langle -\zeta_k^L, \bar{x} - x\rangle \leqslant -e_k\|x - \bar{x}\|, & \zeta_k^L \in \partial f_k^L(x), \\ \langle \zeta_k^U, x - \bar{x}\rangle = \langle -\zeta_k^U, \bar{x} - x\rangle \leqslant -e_k\|x - \bar{x}\|, & \zeta_k^U \in \partial f_k^U(x). \end{cases}$$

Therefore, there exists $\delta > 0$ such that for any $x \in B(\bar{x}, \delta) \cap X$ we have

$$\begin{cases} \langle \zeta^L, x - \bar{x}\rangle_m \leqq -e\|x - \bar{x}\|, & \zeta^L \in \partial f_1^L(x) \times \cdots \times \partial f_m^L(x), \\ \langle \zeta^U, x - \bar{x}\rangle_m \leqq -e\|x - \bar{x}\|, & \zeta^U \in \partial f_1^U(x) \times \cdots \times \partial f_m^U(x). \end{cases}$$

This establishes that $\bar{x}$ is not a solution of (AMVVI) w.r.t. e. □

The next result specifies when an e-approximate efficient solution to (IVOP) is also a solution of (AMVVI) w.r.t. e.

Theorem 5.5. *Suppose* $-f$ *is approximate quasi* LU-e-*convex function of type II at* $\bar{x}$. *If* $\bar{x}$ *is an* e-*approximate efficient solution to* (IVOP), *then* $\bar{x}$ *solves* (AMVVI) *w.r.t.* e.

Proof. Assume that $\bar{x}$ is not a solution of (AMVVI) w.r.t. e. Then there is $\bar{\delta} > 0$ such that for each $x \in B(\bar{x}, \bar{\delta}) \cap X$ and $\zeta^L \in \partial f_1^L(x) \times \cdots \times \partial f_m^L(x)$, $\zeta^U \in \partial f_1^U(x) \times \cdots \times \partial f_m^U(x)$, we have

$$\begin{cases} \langle \zeta^L, x - \bar{x}\rangle_m \leqq -e\|x - \bar{x}\|, \\ \langle \zeta^U, x - \bar{x}\rangle_m \leqq -e\|x - \bar{x}\|. \end{cases}$$

Hence

$$\begin{cases} \langle \zeta_k^L, x-\bar{x}\rangle \leqslant -e_k\|x-\bar{x}\| < 0, & \zeta_k^L \in \partial f_k^L(x),\\ \langle \zeta_k^U, x-\bar{x}\rangle \leqslant -e_k\|x-\bar{x}\| < 0, & \zeta_k^U \in \partial f_k^U(x), \end{cases}$$

are satisfied for all $k = 1,\dots,m$. Consequently, from $\partial(-f)(x) = -\partial f(x)$ we deduce that

$$\begin{cases} \langle -\zeta_k^L, x-\bar{x}\rangle > 0, & (-\zeta_k^L) \in \partial(-f_k^L)(x),\\ \langle -\zeta_k^U, x-\bar{x}\rangle > 0, & (-\xi_k^U) \in \partial(-f_k^U)(x). \end{cases} \tag{5.5}$$

Since $-f$ is approximate quasi LU-e-convex function of type II at $\bar{x}$, then $-f_k = [-f_k^U, -f_k^L]$ is a locally Lipschitz and approximate quasi LU-e_k-convex function of type II at $\bar{x}$ for all $k = 1,\dots,m$. Therefore, both $-f_k^L$ and $-f_k^U$ are all approximate quasi LU-e_k-convex functions of type II at $\bar{x}$ for all $k = 1,\dots,m$. It follows from (5.5) that there exists $\delta > 0$ with $\delta < \bar{\delta}$ such that for all $x \in B(\bar{x},\delta)\cap X$,

$$\begin{cases} (-f_k^L)(x) - (-f_k^L)(\bar{x}) > e_k\|x-\bar{x}\|,\\ (-f_k^U)(x) - (-f_k^U)(\bar{x}) > e_k\|x-\bar{x}\|. \end{cases}$$

This implies that

$$\begin{cases} f_k^L(x) + e_k\|x-\bar{x}\| < f_k^L(\bar{x}),\\ f_k^U(x) + e_k\|x-\bar{x}\| < f_k^U(\bar{x}). \end{cases}$$

Thus there is $\delta > 0$ satisfying for each $x \in B(\bar{x},\delta)\cap X$,

$$f(x) + e\|x-\bar{x}\| \preceq_{LU} f(\bar{x}).$$

We conclude that $\bar{x}$ cannot be an e-approximate efficient solution to (IVOP). □

The following corollary is a direct consequence of Theorems 5.4 and 5.5.

Corollary 5.6. *Suppose* f *is approximate pseudo* LU-*e*-*convex of type II at* $\bar{x}$ *and* $-f$ *is approximate quasi* LU-*e*-*convex of type II at* $\bar{x}$. *Then,* $\bar{x}$ *is an* e-*approximate efficient solution to* (IVOP) *if and only if* $\bar{x}$ *solves* (AMVVI) *w.r.t.* e.

Remark 5.7.

i) We can show that similar results of this section can be obtained when using approximate LU-e-convexity assumptions.
ii) As the interval-valued vector optimization problems is more general than vector optimization problems, the results of this section represent a generalization of the corresponding results obtained in [6, 11].

6. Numerical example

Consider the following example of (IVOP):

$$\begin{aligned} &\min f(x) = (f_1(x), f_2(x))^T = ([f_1^L(x), f_1^U(x)], [f_2^L(x), f_2^U(x)])^T\\ &\text{such that} \quad x \in X = [-1,1], \end{aligned}$$

where

$$f_1^L(x) = \begin{cases} 2x - x^2, & x \geqslant 0,\\ 3x, & x < 0, \end{cases} \qquad f_1^U(x) = \begin{cases} x^3 + 2x, & x \geqslant 0,\\ 2.5x, & x < 0, \end{cases}$$

$$f_2^L(x) = \begin{cases} x^3 + x, & x \geqslant 0, \\ 3x, & x < 0, \end{cases} \qquad f_2^U(x) = \begin{cases} 2x^3 + x, & x \geqslant 0, \\ 1.5x, & x < 0. \end{cases}$$

Let $e = (1,1)^T$. Observe that f is an approximate pseudo LU-e-convex function of type II at $\bar{x} = 0$. It is also easy to check that for any $\delta > 0$ and $x \in (0,\delta) \cap X$, the following inequalities are not satisfied

$$(\langle \xi_1^L, x-\bar{x}\rangle, \langle \xi_2^L, x-\bar{x}\rangle)^T + e\|x-\bar{x}\| = (\xi_1^L x, \xi_2^L x)^T + (|x|,|x|)^T < 0,$$
$$(\langle \xi_1^U, x-\bar{x}\rangle, \langle \xi_2^U, x-\bar{x}\rangle)^T + e\|x-\bar{x}\| = (\xi_1^U x, \xi_2^U x)^T + (|x|,|x|)^T < 0,$$

where

$$\xi_1^L \in \partial f_1^L(0) = [2,3], \quad \xi_2^L \in \partial f_2^L(0) = [1,3],$$
$$\xi_1^U \in \partial f_1^U(0) = [2,2.5], \quad \xi_2^U \in \partial f_2^U(0) = [1,1.5].$$

Thus, there does not exist $\delta > 0$ such that, for all $x \in (-\delta,\delta) \cap X$, $\xi^L \in \partial f_1^L(\bar{x}) \times f_2^L(\bar{x})$ and $\xi^U \in \partial f_1^U(\bar{x}) \times \partial f_2^U(\bar{x})$ one has

$$\begin{cases} \langle \xi^L, x-\bar{x}\rangle_2 \leqq -e\|x-\bar{x}\|, \\ \langle \xi^U, x-\bar{x}\rangle_2 \leqq -e\|x-\bar{x}\|. \end{cases}$$

Therefore the point $\bar{x} = 0$ solves (ASVVI).

Now, since f is approximate pseudo LU-e-convex of type II at $\bar{x} = 0$, then by Theorem 5.1, $\bar{x} = 0$ should be an e-approximate efficient solution to (IVOP). Indeed, for any $\delta > 0$ and $x \in (0,\delta) \cap X$, the following inequalities are not satisfied

$$f_1(x) + \|x-\bar{x}\| = [3x - x^2, x^3 + 3x] \prec_{LU} f_1(0) = [0,0],$$

$$f_2(x) + \|x-\bar{x}\| = [x^3 + 2x, 2x^3 + 2x] \prec_{LU} f_2(0) = [0,0].$$

Hence, we deduce that there exist no $\delta > 0$ such that, for all $x \in (-\delta,\delta) \cap X$, one has

$$f(x) + e\|x-\bar{x}\| \preceq_{LU} f(0).$$

7. Conclusion

In this paper, we have introduced new optimality conditions for a vector optimization problem with interval-valued vector functions using the concept of local strong e-quasi efficiency and e-approximate efficiency hypotheses. We have established the relationships between this problem and vector variational inequality problems under the hypotheses of approximate LU-e-convexity or generalized approximate LU-e-convexity. Hence, our presented results extend and improve the corresponding main results obtained in [6, 11, 19].

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