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Topological pseudo-UP algebras

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Abstract

The aim of this paper is to study the concept of topological pseudo-UP algebra which is a pseudo-UP algebra equipped with a specific type of topology that makes the two binary operations topologically continuous. This concept is an extension of the concept of topological UP-algebra. Thereupon, we obtain many properties of topological pseudo-UP algebras.

Keywords: Topological pseudo-UP algebra, minimal open sets, T_i-spaces, pseudo-UP homomorphism.

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1. introduction

In the last five decades many mathematicians have been interested in studying topologies of classes of algebras. The topological concepts of (BCK, BCC, BE)-algebras are given in [1, 3, 4]. In 1998, Lee and Ryu investigated and presented some topological characteristics to the topological BCK-Algebras notion. In 2008, Ahn and Kwon introduced the concept of topological BCC-algebras. In 2017, Mehrshad and Golzarpoor investigated certain characteristics of uniform topology and topological BE-algebras. In this same year, Iampan [2] introduced a new class of algebras termed UP- algebras which is a generalization of KU-algebras [6] established by Prabpayak and Leerawat in 2009. Later in 2019, Satirad and Iampan [10] defined topological UP-algebras and discovered more features of this structure. In 2020, Romano introduced a generalization of UP-algebras that he called pseudo-UP algebras. Also, he studied the concepts of pseudo-UP filters and pseudo-UP ideals of pseudo-UP algebras in [8]. Furthermore, he introduced the concept of homomorphisms between pseudo-UP algebras in [9].

This paper is structured as follows. In Section 2, we present some definitions and propositions on pseudo-UP algebras and topologies which are needed to develop this paper. In Section 3, we study a pseudo-UP algebra fitted with a topology in which the two binary operations of the structure satisfy the continuity, we call this pseudo-UP algebra associated with such a topology by a topological pseudo-UP algebra and we obtain many of its properties.

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2. Preliminaries

In this section, we provide some background information and notes on the topology and pseudo-UP algebra, which are necessary for the development of this paper.

Definition 2.1 ([7]). A pseudo-UP algebra is a structure $((X, \leqslant), \cdot, *, 0)$ where $\leqslant$ is a binary operation on a set X, $\cdot$ and $*$ are two binary operations on X if X satisfies the following axioms: for all $x, y, z \in X$,

1. $y \cdot z \leqslant (x \cdot y) * (x \cdot z)$ and $y * z \leqslant (x * y) \cdot (x * z)$;
2. If $x \leqslant y$ and $y \leqslant x$ then $x = y$, (i.e. $\leqslant$ is an anti-symmetric);
3. $(y \cdot 0) * x = x$ and $(y * 0) \cdot x = x$; and
4. $x \leqslant y$ if and only if $x \cdot y = 0$ and $x \leqslant y$ if and only if $x * y = 0$.

Proposition 2.2 ([7]). *In a pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ the following statements hold: for all $x \in X$,*

1. $x \cdot 0 = 0$ *and* $x * 0 = 0$*;*
2. $0 \cdot x = x$ *and* $0 * x = x$*; and*
3. *($\leqslant$) is a reflexive (i.e.,* $x \cdot x = 0$ *and* $x * x = 0$*).*

Proposition 2.3 ([7]). *In a pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ the following statements hold: for all $x, y \in X$,*

1. $x \leqslant y \cdot x$*;*
2. $x \leqslant y * x$*.*

Proposition 2.4 ([7]). *Every pseudo-UP algebra X satisfying $x * y = x \cdot y$ is UP-algebra for all $x, y \in X$.*

Definition 2.5 ([8]). A non-empty subset $\mathcal{F}$ of a pseudo-UP algebra X is said to be a pseudo-UP filter of X if it satisfies: for all $x, y \in X$,

1. $0 \in \mathcal{F}$;
2. $x \cdot y \in \mathcal{F}$ and $x \in \mathcal{F}$ then $y \in \mathcal{F}$;
3. $x * y \in \mathcal{F}$ and $x \in \mathcal{F}$ then $y \in \mathcal{F}$.

Definition 2.6. A non-empty subset S of a pseudo-UP algebra X is called a pseudo-UP subalgebra of X if it satisfies:

1. $0 \in S$;
2. S is closed under two binary operations $\cdot$ and $*$ (i.e., $x \cdot y \in S$ and $x * y \in S$ for all $x, y \in S$).

It is clear that $\{0\}$ and X are two pseudo-UP subalgebras of X.

Example 2.7. Let $X = \{0, a, b, c\}$ with two binary operations $\cdot$ and $*$ defined in Table 1.

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	0	0

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

Table 1: A pseudo-UP subalgebra of a pseudo-UP algebra.

By easy calculation, we can check that $((X, \leqslant), \cdot, *, 0)$ is a pseudo-UP algebra and $S = \{0, b\}$ is a pseudo-UP subalgebra of X. Obviously. $S_1 = \{a, b\}$ is not a pseudo-UP subalgebra of X.

Definition 2.8 ([9]). Let $((X, \leqslant), \cdot, *, 0)$ and $((Y, \leqslant_Y), \cdot_Y, *_Y, 0_Y)$ be two pseudo-UP algebras. A map $f : X \to Y$ is a pseudo-UP homomorphism if

$$f(x \cdot y) = f(x) \cdot_Y f(y) \quad \text{and} \quad f(x * y) = f(x) *_Y f(y),$$

for all $x, y \in X$. Moreover, f is a pseudo-UP isomorphism if it is bijective.

In the remainder of this section, we recall some topological concepts from [5]. By (X, τ) or X we mean a topological space. A space X is a compact if every open cover of X has a finite subcover. Also, A space X is a connected if and only if ϕ and X are only clopen sets in τ. Let A be a subset X, the closure of A is defined by $cl(A) = \{x \in X : \forall O \in \tau \text{ such that } x \in O, O \cap A \neq \phi\}$. The set of all interior points of A defined by $int(A) = \bigcup\{O : O \in \tau \text{ and } O \subseteq A\}$. Let $f : (X, \tau) \to (Y, \tau_Y)$ be a function, then f is a continuous if the inverse image of every open set in Y is an open set X. Also, f is an open map if the image of every open set in X is an open set in Y. A topological space (X, τ) is called:

1. T_0 if for each two distinct point $x, y \in X$, there exists an open set U containing one of them but not the other;
2. T_1 if for each two distinct point $x, y \in X$, there exist two open sets U and V such that U containing x but not y and V containing y but not x;
3. T_2 if for each two distinct point $x, y \in X$, there exist two disjoint open sets U and V containing x and y, respectively.

Definition 2.9 ([10]). A UP-algebra $(X, *, 0)$ equipped with a topology τ is called a topological UP-algebra (for short TUP-algebra) if for each open set O containing $x * y$, there exist two open sets U and V containing x and y, respectively such that $U * V \subseteq O$.

3. Topological pseudo-UP algebras

In this section, we introduce the concept of a topological pseudo-UP algebras and establish some of its properties.

Definition 3.1. A pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with a topology τ is called a topological pesudo-UP algebra (for short TPUP-algebra) if for each open set O containing $x \cdot y$ and for each open set W containing $x * y$, there exist two open sets U_1 and V_1 (U_2 and V_2) containing x and y, respectively such that $U_1 \cdot V_1 \subseteq O$ and $U_2 * V_2 \subseteq W$ for all $x, y \in X$.

Lemma 3.2. *A pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with a topology τ is a TPUP-algebra if and only if for each open set O containing $x \cdot y$ and for each open set W containing $x * y$, there exist two open sets U and V containing x and y, respectively such that $U \cdot V \subseteq O$ and $U * V \subseteq W$ for all $x, y \in X$.*

Lemma 3.3. *If a pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with a topology τ is a TPUP-algebra, then for each open set O containing $x \cdot y$ and $x * y$, there exist two open sets U and V containing x and y, respectively such that $U \cdot V \subseteq O$ and $U * V \subseteq O$ for all $x, y \in X$.*

The converse of Lemma 3.3 may not be true. But whenever $x \cdot y = x * y$ for all $x, y \in X$, then the converse is also true.

Lemma 3.4. *A pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with a topology τ is a TPUP algebra, if the two binary operations $\cdot$ and $*$ are continuous (i.e., the inverse image of every open set containing either $x \cdot y$ or $x * y$ is an open set in $X \times X$ for all $x, y \in X$).*

Lemma 3.5. *If a pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with a topology τ is a TPUP-algebra and if O_1 and O_2 are two open sets containing $x \cdot y$ and $x * y$, respectively, then $(\cdot^{-1})(O_1)$ and $(*^{-1})(O_2)$ are open sets in $X \times X$ for all $x, y \in X$. Hence, $(\cdot^{-1})(O_1) \cap (*^{-1})(O_2)$ is an open set in $X \times X$.*

Example 3.6. Let $X = \{0, a, b, c\}$ with two binary operations $\cdot$ and $*$ defined in Table 2. Then $((X, \leqslant), \cdot, *, 0)$ is a pseudo-UP algebra [7]. Now, if we take the topology $\tau = \mathcal{P}(X)$ on X then it is not difficult to check that the pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with the topology τ is a TPUP-algebra.

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

$*$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	b	0	0
c	0	b	b	0

Table 2: Topological pseudo-UP algebra.

Remark 3.7. If a topological pseudo-UP algebra X satisfies $x * y = x \cdot y$ for all $x, y \in X$, then X is a topological UP-algebra.

Proposition 3.8. *Let A and B be any two subsets of a TPUP-algebra X, then the following statements hold:*

1. $cl(A) \cdot cl(B) \subseteq cl(A \cdot B)$ *and* $cl(A) * cl(B) \subseteq cl(A * B)$;
2. *if* $cl(A) \cdot cl(B)$ *and* $cl(A) * cl(B)$ *are closed sets, then* $cl(A) \cdot cl(B) = cl(A \cdot B)$ *and* $cl(A) * cl(B) = cl(A * B)$.

Proof.

1. Let $x \in cl(A) \cdot cl(B)$, $y \in cl(A) * cl(B)$ and O, W be two open sets containing x and y, respectively such that $x = a \cdot b$ and $y = a * b$ where $a \in cl(A)$ and $b \in cl(B)$. Since X is a TPUP-algebra, then there exist two open sets U and V containing a and b, respectively such that $U \cdot V \subseteq O$ and $U * V \subseteq W$. Also, we have $a \in cl(A)$ and $b \in cl(B)$, so $A \cap U \neq \phi$ and $B \cap V \neq \phi$. Suppose that $a_1 \in A \cap U$ and $b_1 \in B \cap V$, then $a_1 \cdot b_1 \in U \cdot V \subseteq O$ and $a_1 * b_1 \in U * V \subseteq W$. Therefore, $x \in cl(A \cdot B)$ and $x \in cl(A * B)$. Hence, $cl(A) \cdot cl(B) \subseteq cl(A \cdot B)$ and $cl(A) * cl(B) \subseteq cl(A * B)$.

2. Suppose that $cl(A) \cdot cl(B)$ and $cl(A) * cl(B)$ are closed sets. Since $A \cdot B \subseteq cl(A) \cdot cl(B)$ and $A * B \subseteq cl(A) * cl(B)$, then $cl(A \cdot B) \subseteq cl(cl(A) \cdot cl(B)) = cl(A) \cdot cl(B)$, $cl(A * B) \subseteq cl(cl(A) * cl(B)) = cl(A) * cl(B)$ and from part (1) we get $cl(A) \cdot cl(B) = cl(A \cdot B)$ and $cl(A) * cl(B) = cl(A * B)$. □

The following example shows that the equality in Proposition 3.8 may not be true and $cl(A) \cdot cl(B)$, $cl(A) * cl(B)$ are not closed sets in general.

Example 3.9. Let $X = \{0, a, b, c\}$ with two binary operations $\cdot$ and $*$ defined by Table 3.

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	a	b	0

Table 3: TPUP-algebra that does not satisfy the equality of the proposition 3.8.

It is clear that $((X, \leqslant), \cdot, *, 0)$ is a pseudo-UP algebra. Now, let $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{0, c\}, \{0, a, c\}, \{0, b, c\}, X\}$ then it is not difficult to check that the pseudo-UP algebra $((X, \leqslant), \cdot, *, 0)$ equipped with the topology τ is a TPUP-algebra. Moreover, let $A = \{a\}$ and $B = \{a, b\}$ then $cl(A) = \{a\}$, $cl(B) = \{a, b\}$, $cl(A \cdot B) = cl(\{0, b\}) = \{0, b, c\}$ and $cl(A * B) = cl(\{0, b\}) = \{0, b, c\}$. Therefore, $cl(A) \cdot cl(B) = \{0, b\}$ and $cl(A) * cl(B) = \{0, b\}$ are not closed sets and hence $cl(A) \cdot cl(B) \neq cl(A \cdot B)$ and $cl(A) * cl(B) \neq cl(A * B)$.

Proposition 3.10. *Let* $((X, \leqslant), \cdot, *, 0, \tau)$ *be a TPUP-algebra and* $\phi \neq W \in \tau$, *then the following statements hold.*

1. *If* $x \in W$, *then there exists an open set* U *containing* 0 *such that* $U \cdot x \subseteq W$ *and* $U * x \subseteq W$.

2. *If* $0 \in W$*, then there exists an open set* U *containing* x *such that* $U \cdot U \subseteq W$ *and* $U * U \subseteq W$*.*
3. *If* $0 \in W$*, then there exist two open sets* U *and* V *containing* x *and* 0*, respectively such that* $(U \cdot V) * V \subseteq W$ *and* $(U * V) \cdot V \subseteq W$*.*
4. *If* $0 \in W$*, then there exist two open sets* U *and* V *containing* x *and* y*, respectively such that* $U \cdot (V * U) \subseteq W$ *and* $U * (V \cdot U) \subseteq W$*.*

Proof.

1. Obvious.

2. Let $0 \in W$ and $x \in X$. Since $x \cdot x = 0 \in W$, $x * x = 0 \in W$ and X is a TPUP-algebra, then there exist two open sets G and H containing x such that $G \cdot H \subseteq W$ and $G * H \subseteq W$. Suppose that $U = G \cap H$, then U is an open set containing x. Hence, $U \cdot U \subseteq W$ and $U * U \subseteq W$.

3. Let $0 \in W$ and $x \in X$. Since $(x * 0) \cdot 0 = 0$, $(x \cdot 0) * 0 = 0$ and X is a TPUP-algebra, then there exists an open set G containing $x * 0$, $x \cdot 0$ and an open set V_1 containing 0 such that $G \cdot V_1 \subseteq W$ and $G * V_1 \subseteq W$. Again, since X is a TPUP-algebra, then there exist two open sets U and V_2 containing x and 0, respectively such that $U \cdot V_2 \subseteq G$ and $U * V_2 \subseteq G$. Let $V = V_1 \cap V_2$, then V is an open set containing 0. Therefore, $(U \cdot V) * V \subseteq G * V \subseteq W$ and $(U * V) \cdot V \subseteq G \cdot V \subseteq W$.

Let $0 \in W$ and $x, y \in X$. Since $x \leqslant y * x$, $x \leqslant y \cdot x$ and X is a TPUP-algebra, then there exist three open sets U_1, G and H containing x, $y * x$ and $y \cdot x$, respectively such that $U_1 \cdot G \subseteq W$ and $U_1 * H \subseteq W$. Again, since X is a TPUP-algebra, then there exist two open sets V and U_2 containing y and x, respectively such that $V \cdot U_2 \subseteq H$ and $V * U_2 \subseteq G$. Let $U = U_1 \cap U_2$, then U is an open set containing x. Therefore, $U \cdot (V * U) \subseteq U \cdot G \subseteq W$ and $U * (V \cdot U) \subseteq U * H \subseteq W$. □

Proposition 3.11. *In a TPUP-algebra* X*,* $\{0\}$ *is an open set if and only if* X *is a discrete topology.*

Proof. Let $\{0\}$ be an open set in X. Since $x \cdot x = 0 \in \{0\}$, $x * x = 0 \in \{0\}$ and X is a TPUP-algebra, then by Proposition 3.10, for each $x \in X$ there exists an open set U containing x such that $U \cdot U \subseteq \{0\}$ and $U * U \subseteq \{0\}$. Now, if $x \neq y$ and $y \in U$ then $x \leqslant y$ and $y \leqslant x$ which is a contradiction. Hence, $U = \{x\}$ implies that $\{x\}$ is open for each $x \in X$.

Conversely, Let X be a discrete topology. Then $\{0\}$ is an open set in X. □

Note that in Example 3.9, $\{0\}$ is not an open set and (X, τ) is not a discrete topology.

Corollary 3.12. *In a TPUP-algebra* X*, if* $\{0\}$ *is an open set, then* (X, τ) *is a disconnected space.*

The converse of Corollary 3.12 may not be true in general. In Example 3.9, every element of τ is a clopen set. Therefore, (X, τ) is a disconnected space and $\{0\}$ is not an open set.

Proposition 3.13. *In a TPUP-algebra* X*,* $\{0\}$ *is a closed set if and only if* X *is a* T_2*.*

Proof. Suppose that $\{0\}$ is a closed set in X and let $x, y \in X$ such that $x \neq y$. Hence, we have $x \nleqslant y$ or $y \nleqslant x$ if we assume that $x \nleqslant y$. Then $\{0\}^c$ is an open set containing $x \cdot y$ and $x * y$. Since X is a TPUP-algebra, then there exist two open sets U and V containing x and y, respectively such that $U \cdot V \subseteq \{0\}^c$ and $U * V \subseteq \{0\}^c$. We claim that $U \cap V = \phi$. If $U \cap V \neq \phi$, then there is $z \in U \cap V$, so $0 = z \cdot z \in U \cdot V \subseteq \{0\}^c$ and $0 = z * z \in U * V \subseteq \{0\}^c$ which is a contradiction. Hence (X, τ) is a T_2.

The converse is obvious. □

Proposition 3.14. *In a TPUP-algebra* $((X, \leqslant), \cdot, *, 0, \tau)$*, the following statements are equivalent:*

1. X *is a* T_0*;*
2. X *is a* T_1*;*
3. X *is a* T_2*.*

Proof.

(1) $\Longrightarrow$ (2): Suppose that X is a T_0 and $x, y \in X$ such that $x \neq y$. Thus, we have $x \nleqslant y$ or $y \nleqslant x$ without loss of generality. Assume that $x \nleqslant y$, then we have two cases.

Case 1: There exists an open set W containing either $x \cdot y$ or $x * y$ but not $\{0\}$. Since X is a TPUP-algebra, then there exist two open sets U and V containing x and y, respectively such that $U \cdot V \subseteq W$ or $U * V \subseteq W$. But $0 \notin W$, then $0 \notin U \cdot V$ and $0 \notin U * V$. If $U \cap V \neq \phi$, then there is $z \in U \cap V$. Hence, $0 = z \cdot z \in U \cdot V \subseteq W$ and $0 = z * z \in U * V \subseteq W$ which is a contradiction. Therefore, $y \notin U$.

Case 2: There exists an open W containing 0 but not $x \cdot y$ and $x * y$. Since $x \leqslant x$ and X is a TPUP-algebra, then there exist two open sets U and V containing x such that $U \cdot V \subseteq W$ and $U * V \subseteq W$. But $x \cdot y, x * y \notin W$, then $x \cdot y \notin U \cdot V$ and $x * y \notin U * V$. Therefore, $y \notin V$.

(2) $\Longrightarrow$ (3): Suppose that X is a T_1, then $\{0\}$ is a closed set. Therefore, by Proposition 3.13, X is a T_2. □

Proposition 3.15. *Every open pseudo-UP subalgebra S of a TPUP-algebra X is also a TPUP-algebra.*

Proof. Let $x, y \in S$, and let O, W be any two open sets in S containing $x \cdot y$ and $x * y$, respectively. Since S is an open set in X, then O and W are two open sets in X also. Since X is a TPUP-algebra, then there exist two open sets U and V in X containing x and y, respectively such that $U \cdot V \subseteq O$ and $U * V \subseteq O$. Hence, $U \cap S = G$ and $V \cap S = H$ are open sets in S containing x and y, respectively such that $G \cdot H \subseteq O$ and $G * H \subseteq W$. Therefore, S is a TPUP-algebra. □

Proposition 3.16. *Let S be a pseudo-UP subalgebra of a TPUP-algebra X, then $\mathrm{cl}(S)$ is a pseudo-UP subalgebra.*

Proof. Let $x, y \in \mathrm{cl}(S)$ and W be an open set containing $x \cdot y$. Since X is a TPUP-algebra, then there exist two open sets U and V containing x and y, respectively such that $U \cdot V \subseteq W$. Since $x, y \in \mathrm{cl}(S)$, then there are points $a \in U \cap S \neq \phi$ and $b \in V \cap S \neq \phi$. Since $a, b \in S$ and S is a pseudo-UP subalgebra of X, then $a \cdot b \in W \cap S \neq \phi$. Since W be any open set containing $x \cdot y$, then $x \cdot y \in \mathrm{cl}(S)$. By similar statements we can prove that $x * y \in \mathrm{cl}(S)$. This implies that $\mathrm{cl}(S)$ is a pseudo-UP subalgebra. □

Proposition 3.17. *Let $((X, \leqslant), \cdot, *, 0, \tau)$ be a TPUP-algebra and M_0 be the minimal open set containing 0. If $x \in M_0$, then M_0 is the minimal open set containing x.*

Proof. Suppose that $x \in M_0$ and W is any open set containing x. Since $0 \cdot x = x$, $0 * x = x$ and X is a TPUP-algebra, then there exist two open sets U and V containing 0 and x, respectively such that $U \cdot V \subseteq W$ and $U * V \subseteq W$. Since U is an open set containing 0, it follows from assumption that $0 = x \cdot x \in M_0 \cdot V \subseteq U \cdot V \subseteq W$ and $0 = x * x \in M_0 * V \subseteq U * V \subseteq W$. Therefore, W is an open set containing 0. Then by assumption $M_0 \subseteq W$. Hence, M_0 is the minimal open set containing x. □

Lemma 3.18. *Let $((X, \leqslant), \cdot, *, 0, \tau)$ be a TPUP-algebra and let $\tau^* = \tau \setminus \{\phi\}$. If $0 \in \bigcap_{U \in \tau^*} U$, then $V \subseteq V \cdot V$ and $V \subseteq V * V$ for all $V \in \tau^*$.*

Proof. If $x \in V$, then $0 \in V$ and we have $x = 0 \cdot x \in V \cdot V$ and $x = 0 * x \in V * V$. Hence the proof. □

Proposition 3.19. *Let $((X, \leqslant), \cdot, *, 0, \tau)$ be a TPUP-algebra. If $0 \in \bigcap_{U \in \tau^*} U$, then $B \subseteq X$ is an open set if and only if 0 is an interior point of B.*

Proof. If B is an open set, clearly 0 is an interior point of B. Conversely, let 0 be an interior point of B. Since $x \cdot x = 0$ and $x * x = 0$, then there exists an open set W containing 0 such that $x \cdot x = 0 \in W \subseteq B$ and $x * x = 0 \in W \subseteq B$. Since X is a TPUP-algebra, then there exists an open set V containing x such that $V \cdot V \subseteq W$ and $V * V \subseteq W$. By assumption, $0 \in V$, and so by Lemma 3.18, $x \in V \subseteq V \cdot V \subseteq W \subseteq B$ and $x \in V \subseteq V * V \subseteq W \subseteq B$. This shows that x is an interior point of B. □

Proposition 3.20. *Let $((X, \leqslant), \cdot, *, 0, \tau)$ be a TPUP-algebra and $\mathcal{F}$ be a pseudo-UP filter of a pseudo-UP algebra X, then the following statements hold.*

1. *0 is an interior point of $\mathcal{F}$ if and only if $\mathcal{F}$ is an open set in X.*
2. *If $\mathcal{F}$ is an open set in X, then $\mathcal{F}$ is a closed set in X.*
3. *If M_0 is a minimal open set containing 0 and $\mathcal{F}$ is a closed set in X, then $\mathcal{F}$ is an open set in X.*

Proof.

1. Suppose that 0 is an interior point of $\mathcal{F}$, then there exists $W \in \tau$ such that $0 \in W \subseteq \mathcal{F}$. Let $x \in \mathcal{F}$. Since $x \cdot x = 0 \in W$, $x * x = 0 \in W$ and X is a TPUP-algebra, then there exist two open sets U and V containing x such that $U \cdot V \subseteq W \subseteq \mathcal{F}$ and $U * V \subseteq W \subseteq \mathcal{F}$. To prove that $V \subseteq \mathcal{F}$, let $y \in V$ then $x \cdot y \in U \cdot V \subseteq W \subseteq \mathcal{F}$ and $x * y \in U * V \subseteq W \subseteq \mathcal{F}$. Since $x \in \mathcal{F}$ and $\mathcal{F}$ is a pseudo-UP filter of X, then $y \in \mathcal{F}$ and so $V \subseteq \mathcal{F}$. Hence, $\mathcal{F}$ is an open set in X.

2. Suppose that $\mathcal{F}$ is an open set in X and let $x \in \mathcal{F}^c$. Since $x \cdot x = 0$, $x * x = 0$ and X is a TPUP-algebra, then there exist two open sets U and V containing x such that $U \cdot V \subseteq \mathcal{F}$ and $U * V \subseteq \mathcal{F}$. If $U \nsubseteq \mathcal{F}^c$, then $s \in U$ for some $s \in \mathcal{F}$. Therefore, $s \cdot y \in \mathcal{F}$ and $s * y \in \mathcal{F}$ for all $y \in V$. Since $s \in \mathcal{F}$ and $\mathcal{F}$ is a pseudo-UP filter of X, then $y \in \mathcal{F}$ and so $V \subseteq \mathcal{F}$. Thus, $x \in \mathcal{F}$ which is a contradiction. Hence, $U \subseteq \mathcal{F}^c$ and so $\mathcal{F}^c$ is an open set in X. Therefore, $\mathcal{F}$ is a closed set in X.

3. Suppose that M_0 is a minimal open set containing 0 and $\mathcal{F}$ is a closed set in X. Hence, $\mathcal{F}^c$ is an open set in X. Assume that $\mathcal{F}$ is not an open set in X, then by (1) we have 0 is not an interior point of $\mathcal{F}$. Thus, $U \nsubseteq \mathcal{F}$ for all open sets U containing 0. Therefore, $M_0 \nsubseteq \mathcal{F}$ and so $M_0 \cap \mathcal{F}^c \neq \phi$, then there exists $x \in M_0 \cap \mathcal{F}^c$. By Proposition 3.17, $M_0 \subseteq \mathcal{F}^c$ and so $0 \in \mathcal{F}^c$ which is a contradiction. Hence, $\mathcal{F}$ is an open set in X. □

Proposition 3.21. *Let $((X, \leqslant), \cdot, *, 0, \tau)$ be a TPUP-algebra and M_0 be the minimal open set containing 0, then M_0 is a pseudo-UP filter of X.*

Proof. Let $x, x \cdot y, x * y \in M_0$. By Proposition 3.17, M_0 is the minimal open set containing x. Since $x \cdot y, x * y \in M_0$ and X is a TPUP-algebra, then there exist two open sets U and V containing x and y, respectively such that $U \cdot V \subseteq M_0$ and $U * V \subseteq M_0$. Thus, $y = 0 \cdot y \in M_0 \cdot V \subseteq U \cdot V \subseteq M_0$ and $y = 0 * y \in M_0 * V \subseteq U * V \subseteq M_0$. Hence, M_0 is a pseudo-UP filter of X. □

Proposition 3.22. *Let $((X, \leqslant), \cdot, *, 0, \tau)$ be a TPUP-algebra and M_x, M_y be two minimal open sets containing x, y, respectively. If $x \cdot y, x * y \notin M_0$, then $y \notin M_x$ and $x \notin M_y$ where $x \neq 0$ and $y \neq 0$.*

Proof. Suppose that $y \in M_x$, then $\{x, y\} \subseteq M_x$. Since $x \cdot y \in M_{x \cdot y}$, $x * y \in M_{x*y}$ and X is a TPUP-algebra, then there exist two open sets U_1 and U_2 containing x and y, respectively such that $U_1 \cdot U_2 \subseteq M_{x \cdot y}$ and $U_1 * U_2 \subseteq M_{x*y}$. Hence, we have $y \in M_x \subseteq U_1$, $y \in M_y \subseteq U_2$ and thus $0 = y \cdot y \in M_x \cdot M_y \subseteq M_{x \cdot y}$ and $0 = y * y \in M_x * M_y \subseteq M_{x*y}$. Pick $z = x \cdot y$ and $z = x * y$. Since $z \cdot z = 0 \in M_0$ and $z * z = 0 \in M_0$, then there exist two open sets V_1 and V_2 containing z such that $V_1 \cdot V_2 \subseteq M_0$ and $V_1 * V_2 \subseteq M_0$. Therefore, $0 \cdot z \in M_z \cdot M_z \subseteq V_1 \cdot V_2 \subseteq M_0$ and $0 * z \in M_z * M_z \subseteq V_1 * V_2 \subseteq M_0$. Hence, $x \cdot y = z \in M_0$ and $x * y = z \in M_0$ which is a contradiction. Similarly, $x \notin M_y$. □

Definition 3.23. Let B be a non-empty subset of a pseudo-UP algebra X and $a \in X$. The subsets ${}_aB$ and B_a are defined as follows: ${}_aB = \{x \in X : a \cdot x \in B \text{ and } a * x \in B\}$ and $B_a = \{x \in X : x \cdot a \in B \text{ and } x * a \in B\}$. If $A \subseteq X$, then

$$ {}_AB = \bigcup_{a \in A} {}_aB \quad \text{and} \quad B_A = \bigcup_{a \in A} B_a. $$

Proposition 3.24. *Let X be any pseudo-UP algebra and A, B, C, F be non-empty subsets of X, then the following statements hold.*

1. *If $B \subseteq C$, then ${}_AB \subseteq {}_AC$ and $B_A \subseteq C_A$.*
2. *If $F \subseteq X$, then $({}_aF)^c = {}_a(F^c)$ and $(F_a)^c = (F^c)_a$ for all $a \in X$.*

Proposition 3.25. *Let B and F be two non-empty subsets of a TPUP-algebra X, then the following statements hold for all $a \in X$.*

1. *If B is an open set, then B_a and $_aB$ are open sets.*
2. *If F is a closed set, then F_a and $_aF$ are closed sets.*

Proof.

1. Let B be an open set, and for any $a \in X$ let $x \in B_a$. Then, $x \cdot a \in B$ and $x * a \in B$. Since X is a TPUP-algebra, then there exist two open sets U and V containing x and a, respectively such that $U \cdot V \subseteq B$ and $U * V \subseteq B$, which implies that $x \cdot a \in U_a \subseteq B$ and $x * a \in U_a \subseteq B$, and so $U \cdot a \subseteq B$ and $U * a \subseteq B$. Thus, $x \in U \subseteq B_a$. Therefore, B_a is an open set. By similar statements we can prove $_aB$ is an open set.

2. The proof follows from Proposition 3.24 and part (1). □

Definition 3.26. A TPUP-algebra X is called a transitive open TPUP-algebra, if the right maps are both continuous and open.

For a fixed element s of a TPUP-algebra X, define the right maps $R_s : X \to X$ by $R_s(x) = x \cdot s$ and $r_s : X \to X$ by $r_s(x) = x * s$ for all $x \in X$.

Proposition 3.27. *In a TPUP-algebra X, the right maps are continuous.*

Proof. Let $s \in X$, define the right maps $R_s : X \to X$ by $R_s(x) = x \cdot s$ and $r_s : X \to X$ by $r_s(x) = x * s$ for all $x \in X$. Let W be an open set containing $R_s(x)$ and $r_s(x)$. Since X is a TPUP-algebra, then there exist two open sets U and V containing x and s, respectively such that $U \cdot V \subseteq W$ and $U * V \subseteq W$. Clearly, $U \cdot s \subseteq W$ and $U * s \subseteq W$. Hence, $R_s(U) \subseteq W$ and $r_s(U) \subseteq W$. Therefore, R_s and r_s are continuous. □

Proposition 3.28. *Let U be an open set in a transitive open TPUP-algebra X, then the following statements hold.*

1. *$R_s(U) = U \cdot s$ and $r_s(U) = U * s$ are open sets in X.*
2. *$R_s^{-1}(U) = \{x \in X | x \cdot s = R_s(x)\}$ and $r_s^{-1}(U) = \{x \in X | x * s = r_s(x)\}$ are open sets in X.*
3. *$U \cdot V$ and $U * V$ are open sets in X.*

Proof.

1. Let $s \in X$. Since the right maps R_s and r_s are open and U is an open set, then $R_s(U)$ and $r_s(U)$ are open sets in X.

2. Let $s \in X$. Since the right maps R_s and r_s are continuous, then $R_s^{-1}(U)$ and $r_s^{-1}(U)$ are open sets in X.

3. Since $U \cdot V = \bigcup_{s \in V}(U \cdot s)$ and $U * V = \bigcup_{s \in V}(U * s)$, then by (1) we get $U \cdot V$ and $U * V$ are open sets in X. □

Proposition 3.29. *Let F and P be two disjoint subsets of a TPUP-algebra X. If F is a compact set, P is a closed set and the right maps are open from X to X, then there exists an open set U containing 0 such that $(U \cdot F) \cap P = \phi$ and $(U * F) \cap P = \phi$.*

Proof. Let $x \in F \subseteq X \setminus P$. Since $0 * (0 \cdot x) = x \in X \setminus P$, $0 \cdot (0 * x) = x \in X \setminus P$ and X is a TPUP-algebra, then there exists an open set U_0 containing 0 and an open set V containing $0 \cdot x$ and $0 * x$ such that $U_0 * V \subseteq X \setminus P$ and $U_0 \cdot V \subseteq X \setminus P$. Also, there exists an open set U_1 containing 0 such that $U_1 \cdot x \subseteq V$ and $U_1 * x \subseteq V$. If $U_x = U_0 \cap U_1$, then U_x is an open set containing 0 and $U_x * (U_x \cdot x) \subseteq U_0 * V \subseteq X \setminus P$, $U_x \cdot (U_x * x) \subseteq U_0 \cdot V \subseteq X \setminus P$. Since the right maps are open, then $C = \{U_x \cdot x : x \in F\}$ and $C = \{U_x * x : x \in F\}$ are open cover of the compact set F. Therefore, there exist $U_{x_1} \cdot x_1, \ldots, U_{x_n} \cdot x_n \in C$ and $U_{x_1} * x_1, \ldots, U_{x_n} * x_n \in C$ such that

$$F \subseteq \bigcup_{i=1}^{n}(U_{x_i} \cdot x_i) \quad \text{and} \quad F \subseteq \bigcup_{i=1}^{n}(U_{x_i} * x_i).$$

Suppose that $U = \bigcap_{i=1}^{n}(U_{x_i})$, then U is an open set containing 0 such that for all $y \in F$, $y \in U_{x_i} \cdot x_i$, $y \in U_{x_i} * x_i$ for some x_i and

$$U * y \subseteq U * (U_{x_i} \cdot x_i) \subseteq U_0 * V \subseteq X \setminus P,$$

and

$$U \cdot y \subseteq U \cdot (U_{x_i} * x_i) \subseteq U_0 \cdot V \subseteq X \setminus P.$$

Hence, we obtain that $(U \cdot F) \cap P = \phi$ and $(U * F) \cap P = \phi$. □

Proposition 3.30. *Let $((X, \leqslant), \cdot, *, 0_X, \tau_X)$, and $((Y, \leqslant_Y), \cdot_Y, *_Y, 0_Y, \tau_Y)$ be two transitive open TPUP-algebras and $f : X \to Y$ be a pseudo-UP homomorphism. If f is a continuous map at 0_X, then f is a continuous on X.*

Proof. Let $x \in X$ and W be an open set containing $y = f(x)$. Since the right maps on Y are continuous, then there exists an open set V containing 0_Y such that $R_y(V) = V \cdot_Y y \subseteq W$ and $r_y(V) = V *_Y y \subseteq W$. Since f is a continuous at 0_X, then there exists an open set U containing 0_X such that $f(U) \subseteq V$. Since the right maps on X are open, then $0 \cdot x = x \in U \cdot x$ and $0 * x = x \in U * x$ are open sets containing x. Now, we have

$$f(U \cdot x) = f(U) \cdot_Y f(x) = f(U) \cdot_Y y \subseteq V \cdot_Y y \subseteq W,$$

and

$$f(U * x) = f(U) *_Y f(x) = f(U) *_Y y \subseteq V *_Y y \subseteq W.$$

This proves that f is a continuous at x. Since x is any arbitrary element in X, then f is a continuous on X. □

Proposition 3.31. *Suppose that X, Y, and Z are tansitive open TPUP-algebras and $\psi : X \to Y$, $\xi : X \to Z$ are pseudo-UP homomorphisms such that $\xi(X) = Z$ and $\mathrm{Ker}\xi \subseteq \mathrm{Ker}\psi$, then there exists a pseudo-UP homomorphism $f : Z \to Y$ such that $\psi = f \circ \xi$. Also, for each open set U containing 0_Y, there exists an open set V containing 0_Z such that $\xi^{-1}(V) \subseteq \psi^{-1}(U)$, then f is a continuous.*

Proof. Suppose that U is an open set containing 0_Y. By assumption, there exists an open set V containing 0_Z such that

$$\xi^{-1}(V) \subseteq \psi^{-1}(U).$$

Therefore,

$$\psi(\xi^{-1}(V)) \subseteq \psi(\psi^{-1}(U)),$$

and thus

$$f(V) \subseteq U.$$

Hence, f is a continuous map at 0_Z. By Proposition 3.30, we get f is a continuous. □

Definition 3.32. Let$((X, \leqslant), \cdot, *, 0_X, \tau_X)$, and $((Y, \leqslant_Y), \cdot_Y, *_Y, 0_Y, \tau_Y)$ be two TPUP-algebras. A map $f : X \to Y$ is called a topological pseudo-UP homomorphism if:

1. f is a pseudo-UP homomorphism;
2. f is a contininuous.

Proposition 3.33. *Let $((X, \leqslant), \cdot, *, 0_X, \tau_X)$ and $((Y, \leqslant_Y), \cdot_Y, *_Y, 0_Y, \tau_Y)$ be two transitive open TPUP-algebras, and $f : X \to Y$ be a pseudo-UP homomorphism. Then the following statements hold.*

1. *for every open set H containing 0_Y, there exists an open set G containing 0_X such that $f(G) \subseteq H$. Then f is a continuous and hence f is a topological pseudo-UP homomorphism.*
2. *for every open set G containing 0_X, there exists an open set H containing 0_Y such that $H \subseteq f(G)$. Then, f is an open map.*

Proof.

1. Suppose that V is an open set in Y. If $V \cap \mathrm{Im}(f) = \phi$, then $f^{-1}(V) = \phi$ is an open set in X. Let $V \cap \mathrm{Im}(f) \neq \phi$ and $x \in f^{-1}(V)$, then $y := f(x) \in V \cap \mathrm{Im}(f)$. By Proposition 3.28, $R_y^{-1}(V) = \{b \in Y | b \cdot_Y y = R_y(b) \in V\}$ and $r_y^{-1}(V) = \{b \in Y | b *_Y y = r_y(b) \in V\}$ are open sets in Y. Let $v \in H := R_y^{-1}(V) \cap r_y^{-1}(V)$. Therefore, $0_Y \cdot_Y y = y \in V$ and $0_Y *_Y y = y \in V$ and thus $0_Y \in H$. By assumption, there exists an open set G containing 0_X such that $f(G) \subseteq H$. Since the right maps are open, then $G \cdot x$ and $G * x$ are open sets in X. Thus, $x = 0_X \cdot x \in G \cdot x$ and $x = 0_X * x \in G * x$. Since $v \cdot_Y y \in H \cdot_Y y$ and $v *_Y y \in H *_Y y$, then $v \cdot_Y y \in V$ and $v *_Y y \in V$ and hence $H \cdot_Y y \subseteq V$ and $H *_Y y \subseteq V$. Now, $f(G \cdot x) = f(G) \cdot_Y f(x) = f(G) \cdot_Y y \subseteq H \cdot_Y y \subseteq V$ and $f(G * x) = f(G) *_Y f(x) = f(G) *_Y y \subseteq H *_Y y \subseteq V$. Thus, $x \in G \cdot x \subseteq f^{-1}(f(G \cdot x)) \subseteq f^{-1}(V)$ and $x \in G * x \subseteq f^{-1}(f(G * x)) \subseteq f^{-1}(V)$. This implies that $f^{-1}(V)$ is an open set in X. Hence, f is a continuous and hence f is a topological pseudo-UP homomorphism.

2. Suppose that U is an open set in X and let $y \in f(U)$. Then, $y = f(x)$ for some $x \in U$. Since the right maps are continuous, then $R_x^{-1}(U) = \{a \in X | a \cdot x = R_x(a) \in U\}$ and $r_x^{-1}(U) = \{a \in X | a * x = r_x(a) \in U\}$ are open sets in X. Let $u \in G := R_x^{-1}(U) \cap r_x^{-1}(U)$. Therefore, $0_X \cdot x = x \in U$, $0_X * x = x \in U$ and so $0_X \in G$. By assumption, there exists an open set H containing 0_Y such that $H \subseteq f(G)$. By Proposition 3.28, $H \cdot_Y y$ and $H *_Y y$ are open sets in Y. Thus, $y = 0_Y \cdot_Y y \in H \cdot_Y y$ and $y = 0_Y *_Y y \in H *_Y y$. Since $u \cdot x \in G \cdot x$ and $u * x \in G * x$, then $u \cdot x \in U$ and $u * x \in U$. Hence, $G \cdot x \subseteq U$ and $G * x \subseteq U$. Therefore, $f(G \cdot x) \subseteq f(U)$ and $f(G * x) \subseteq f(U)$. Now, $H \cdot_Y y = H \cdot_Y f(x) \subseteq f(G) \cdot_Y f(x) = f(G \cdot x) \subseteq f(U)$ and $H *_Y y = H *_Y f(x) \subseteq f(G) *_Y f(x) = f(G * x) \subseteq f(U)$. Thus, $y \in H \cdot_Y y \subseteq f(U)$ and $y \in H *_Y y \subseteq f(U)$. This implies that $f(U)$ is an open set in Y. Hence, f is an open map. □

4. Conclusion

In this article, the concept of topological pseudo-UP algebra is introduced. Minimal open sets and some separation axioms (T_i-spaces $i = 0, 1, 2$) are discussed in such spaces. Several topological properties and relations among pseudo-UP algebras are obtained by using pseudo-UP homomorphisms. Moreover, this work can be extended into supra (infra) topological pseudo-UP algebras.

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