J. Math. Computer Sci., 26 (2022), 101–112

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

New structure of Fibonacci numbers using concept of Δ–operator

Dowlath Fathima[a], Mashael M. AlBaidani[b], Abdul Hamid Ganie[c,*], Afroza Akhter[d]

[a] *Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University-Jeddah Female, Kingdom of Saudi Arabia.*

[b] *Department of Mathematics, College of Science and Humanities Studies, Prince Sattam Bin Abdulaziz University, Al Kharj 11942, Kingdom of Saudi Arabia.*

[c] *Basic Science Department, College of Science and Theoretical Studies, Saudi Electronic University-Abha Male, Kingdom of Saudi Arabia.*

[d] *Department of Applied Science and Humanities, SSM College of Engineering Parihaspora–Pattan, Jammu and Kashmir, India.*

Abstract

The theory of sequence spaces is the fundamental of summability and applications to various sequences like Fibonacci sequences were deeply studied. In [A. H. Ganie, In: Matrix Theory-Applications and Theorems, **2018** (2018), 75–86], the author has analyzed the Fibonacci sequences and studied its various properties. By utilizing this concept, the notion of this paper is to introduce new scenario of spaces using Fibonacci numbers. By using Kizmaz operator, we shall introduce the difference sequence spaces $c_0^J(\widetilde{\mho_g})$, $c^J(\widetilde{\mho_g})$ and $\ell_\infty^J(\widetilde{\mho_g})$ by involving Fibonacci sequence and the idea of ideal convergence. We will prove certain basic inclusion relations and study these for some topological properties.

Keywords: Fibonacci numbers, ideal convergence, BK-spaces.

2020 MSC: 40A05, 46A45, 40C05.

1. Introduction

The theory of sequence spaces is the fundamental of summability. Summability is a wide field of mathematics, mainly in analysis and functional analysis, and has many applications, for instance, in numerical analysis to speed up the rate of convergence, in operator theory, the theory of orthogonal series, and approximation theory. This subsection serves as a motivation of what follows. Let us represent $\mathbf{C}$ to denote the set of complex numbers and $\mathbb{N}$ to represent the set of whole numbers. Then, we symbolize by $\Psi = \mathbf{C}^{\mathbb{N}} = \left\{ \rho = (\rho_i) : \rho : \mathbf{N} \to \mathbf{C}, j \to \rho_j \right\}$, and it represents the set of all real or complex valued sequences $\rho = (\rho_j)_{j=0}^{\infty}$. Any linear subspace of Ψ is called as a sequence space. Denote by ℓ_∞, c and c_0 to be the set of all bounded sequences, convergent sequences, and the sequences converging to zero, respectively.

*Corresponding author

Email address: a.ganie@seu.edu.sa (Abdul Hamid Ganie)

doi: 10.22436/jmcs.026.02.01

Received: 2021-09-09 *Revised:* 2021-10-13 *Accepted:* 2021-10-16

We let the j-section of any sequence $\rho = (\rho_r)$ be denoted by $\rho^{[j]} = \sum_{i=0}^{j} \rho_i e_i$, where e_i has value as 1 in i-th place and 0 elsewhere and we represent e to be $e = (1, 1, 1, \cdots)$.

A coordinate space (or K-space) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space X with a linear topology is called a K-space provided each of the maps $\pi_j : X \to \mathbf{C}$ defined by $\pi_j(x) = x_j$ is continuous for all $j \in \mathbf{N}$. A BK-space is a K-space, which is also a Banach space with continuous coordinate functionals $f_j(x) = x_j$, with $j = 1, 2, \cdots$. A K-space is called an FK-space provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. A BK-space $\mathcal{X} \supset \varphi$, the set of all finite sequences that terminate in zeros, is said to have AK if every sequence $\rho = (\rho_m) \in \mathcal{X}$ can be expressed in one and only one way as $\rho = \sum_{m=0}^{\infty} \rho_m e_m$.

As in [1, 13, 35–38], let $\mathcal{B} = (b_{ij})$ be an infinite matrix of complex numbers b_{ij} and $\mathcal{U}$, $\mathcal{V}$ two subsets of Ψ. We say that the matrix $\mathcal{B}$ defines a matrix transformation from $\mathcal{U}$ into $\mathcal{V}$ if for every sequence $\rho = (\rho_j) \in \mathcal{U}$ the sequence $\mathcal{B}(s) = (\mathcal{B}_j(\rho)) \in \mathcal{V}$, where $\mathcal{B}_j(\rho) = \sum_i b_{ji}\rho_i$ converges for each j. We denote the class of matrix transformations from $\mathcal{U}$ into $\mathcal{V}$ by $(\mathcal{U}, \mathcal{V})$.

The matrix domain of an infinite matrix $\mathcal{B}$ in a sequence space Λ is defined by

$$\Lambda_{\mathcal{B}} = \{\rho = (\rho_j) \in \Psi : \mathcal{B}\rho \in \Lambda\}, \tag{1.1}$$

which is a sequence space. If $\mathcal{B} = \triangle$, where Δ is the backward difference matrix defined by

$$\triangle = \triangle_{rk} = \begin{cases} (-1)^{r-k}, & r-1 \leqslant k \leqslant r, \\ 0, & 0 \leqslant k < r-1 \text{ or } k > r, \end{cases}$$

for all $r, k \in \mathbb{N}$, then $\Lambda_{\triangle}$ is called the difference sequence space defined by the domain of a triangle matrix $\mathcal{B}$ whenever λ is a normed linear space or paranormed sequence space as can be seen in [15–18, 21, 22, 29], and many others.

The notion of difference sequence spaces was introduced by Kizmaz [26] and can be seen in [10, 12] as follows:

$$\Lambda(\triangle) := \{\rho = (\rho_m) \in \Psi : (\rho_m - \rho_{m+1}) \in \Lambda\},$$

for $\lambda \in \{\ell_\infty, c, c_0\}$. In recent years, some researchers have addressed the approach to constructing a new sequence space by means of the matrix domain of a particular limitation method; see, for instance, [3, 4, 7, 31] and the references therein. In [23], the author has studied the following difference sequence space

$$\ell_\infty(\widetilde{\mho}) = \left\{\rho = (\rho_j) \in \Psi : \sup_{j \in \mathbb{N}} \left| \frac{\mathfrak{f}_j}{\mathfrak{f}_{j+1}} \rho_j - \frac{\mathfrak{f}_{n+1}}{\mathfrak{f}_j} \rho_{j-1} \right| < \infty \right\},$$

which is derived by the Fibonacci difference matrix $\widetilde{\mho} = (\widetilde{\mho}_{rk})$ defined as follows:

$$\widetilde{\mho}_{rk} = \begin{cases} -\frac{\mathfrak{f}_{r+1}}{\mathfrak{f}_r}, & \text{if } k = r-1, \\ \\ \frac{\mathfrak{f}_r}{\mathfrak{f}_{r+1}}, & \text{if } 0 \leqslant k < r-1 \ \text{ or } \ k > r, \end{cases} \tag{1.2}$$

for all $r, k \in \mathbb{N}$, and as in [2, 6, 14, 27, 44], $\{\mathfrak{f}_r\}_{r=0}^{\infty}$ represents the sequence of Fibonacci numbers defined by the linear recurrence equalities $\mathfrak{f}_0 = \mathfrak{f}_1 = 1$ and $\mathfrak{f}_r = \mathfrak{f}_{r-1} + \mathfrak{f}_{r-2}$, $r \geqslant 2$, with the following fundamental properties:

$$\begin{aligned}
&\lim_{r\to\infty}\frac{\mathfrak{f}_{r+1}}{\mathfrak{f}_r}=\frac{1+\sqrt{5}}{2}=\alpha \quad \text{(Golden Ratio)},\\
&\sum_{k=0}^{r}\mathfrak{f}_k=\mathfrak{f}_{r+2}-1 \quad (r\in\mathbb{N}),\\
&\sum_{k}\frac{1}{\mathfrak{f}_k} \text{ converges},\\
&\mathfrak{f}_{r-1}\mathfrak{f}_{r+1}-\mathfrak{f}_r^2=(-1)^{r+1},\quad r\geqslant 1 \quad \text{(Cassini's formula)},
\end{aligned} \tag{1.3}$$

which yields $\mathfrak{f}_{r-1}^2+\mathfrak{f}_r\mathfrak{f}_{r-1}-\mathfrak{f}_r^2=(-1)^{r+1}$ by substituting for $\mathfrak{f}_{r+1}$ in Cassini's formula.

Also, let Ψ denotes any of the spaces l_∞, c , c_0 and l_p $(1\leqslant p<\infty)$. Then, as in [11], the Fibonacci sequence space $\Psi(\mho)$ is defined by

$$\Psi(\mho)=\{\rho=(\rho_k)\in\Psi:\zeta=(\zeta_k)\in\Psi\},$$

where the sequence $\zeta=(\zeta_k)$ is the $\mho$-transform of the sequence $\rho=(\rho_k)$ and is given by

$$\zeta_k=\mho_k(\rho)=\frac{1}{\mathfrak{f}_k\mathfrak{f}_{k+1}}\sum_{i=0}^{k}\mathfrak{f}_i^2 g_k\rho_i,\quad \forall\, k\in\mathbb{N}.$$

For a more detailed information about Fibonacci sequence spaces, we refer to [5, 20, 25, 40]. By using the same infinite Fibonacci matrix $\widetilde{\mho_g}$ and the same technique, Başarir et al. [2] have introduced the Fibonacci difference sequence spaces $c_0(\widetilde{\mho_g})$ and $c(\widetilde{\mho})$ as the sets of all sequences whose $\widetilde{\mho_g}$-transforms are in the spaces c_0 and c, respectively, that is,

$$c_0(\widetilde{\mho_g}):=\left\{\rho=(\rho_n)\in\Psi:\lim_{n\to\infty}\widetilde{\mho_g}_n(\rho)=0\right\},$$

and

$$c(\widetilde{\mho_g}):=\left\{\rho=(\rho_n)\in\Psi:\exists\ell\in\mathbb{R}\ni\lim_{n\to\infty}\widetilde{\mho_g}_n(\rho)=\ell\right\},$$

where the sequence $\widetilde{\mho_g}_n(\rho)$ is the $\widetilde{\mho_g}$-transform of a sequence $\rho=(\rho_n)\in\Psi$, defined as follows:

$$\widetilde{\mho_g}_n(\rho)=\begin{cases}\frac{\mathfrak{f}_0}{\mathfrak{f}_1}g_k\rho_0=g_0\rho_0, & n=0,\\ \frac{\mathfrak{f}_n}{\mathfrak{f}_{n+1}}g_k\rho_n-\frac{\mathfrak{f}_{n+1}}{\mathfrak{f}_n}g_k\rho_{n-1}, & n\geqslant 1.\end{cases} \tag{1.4}$$

As in [30, 32, 43], by an ideal we mean a family of sets $I\subset P(X)$ (where $P(X)$ is the power set of X) such that

(i) $\emptyset\in I$;

(ii) $A\cup B\in I$ for all $A,B\in I$; and

(iii) for each $A\in I$ and $B\subset A$, we have $B\in I$;

I is called admissible in X if it contains all singletons, that is, if $I\supset\{\{x\}:x\in X\}$.

A filter on X is a nonempty family of sets $\mathcal{F}\subset P(X)$ satisfying

(i) $\emptyset\notin\mathcal{F}$;

(ii) $A,B\in\mathcal{F}$ implies that $A\cap B\in\mathcal{F}$; and

(iii) for any $A\in\mathcal{F}$ and $B\supset A$, we have $B\in\mathcal{F}$.

For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I (a filter associated with ideal I), that is, $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$, where $K^c = X \setminus K$. In 1999, Kostyrko et al. [28] defined the notion of I-convergence, which depends on the structure of ideals of subsets of $\mathbb{N}$ as a generalization of statistical convergence introduced by Fast [9] and Steinhaus [39] in 1951. Later on, the notion of I-convergence was further investigated from the sequence space point of view and linked with the summability theory by Šalát et al. [33], Tripathy and Hazarika [41, 42], Das et al. [8], and many other authors. Šalát et al. [34] extended the notion of summability fields of an infinite matrix of operators A with the help of the notion of I-convergence, that is, the notion of I-summability and introduced new sequence spaces c_A^I and m_A^I, the I-convergence field and bounded I-convergence field of an infinite matrix A, respectively. For further details on ideal convergence, we refer to [1, 19, 24], etc.

Throughout the paper, c_0^I, c^I and ℓ_∞^I denote the I-null, I-convergent, and I-bounded sequence spaces, respectively. In this paper, by combining the definitions of Fibonacci difference matrix $\widetilde{\mho_g}$ and ideal convergence we introduce the sequence spaces $c_0^I(\widetilde{\mho_g}), c^I(\widetilde{\mho_g})$ and $\ell_\infty^I(\widetilde{\mho_g})$. Further, we study some topological and algebraic properties of these spaces. Also, we study some inclusion relations concerning these spaces.

Following [9, 28, 33, 39], we recall some definitions and lemmas, which will be used throughout the paper.

Definition 1.1. A sequence $\rho = (\rho_n) \in \Psi$ is said to be statistically convergent to a number $\ell \in \mathbb{R}$ if, for every $\epsilon > 0$, the natural density of the set $\{n \in \mathbb{N} : |\rho_n - \ell| \geqslant \epsilon\}$ equals zero, and we write $st\text{-}\lim \rho_n = \ell$. If $\ell = 0$, then $\rho = (\rho_n) \in \Psi$ is said to be st-null.

Definition 1.2. A sequence $\rho = (\rho_n) \in \Psi$ is said to be I-Cauchy if, for every $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that the set $\{n \in \mathbb{N} : |\rho_n - \rho_N| \geqslant \epsilon\} \in I$.

Definition 1.3. A sequence $\rho = (\rho_n) \in \Psi$ is said to be I-convergent to a number $\ell \in \mathbb{R}$ if, for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |\rho_n - \ell| \geqslant \epsilon\} \in I$, and we write $I\text{-}\lim \rho_n = \ell$. If $\ell = 0$, then $(\rho_n) \in \Psi$ is said to be I-null.

Definition 1.4. A sequence $\rho = (\rho_n) \in \Psi$ is said to be I-bounded if there exists $K > 0$ such that the set $\{n \in \mathbb{N} : |\rho_n| \geqslant K\} \in I$.

Definition 1.5. Let $\rho = (\rho_n)$ and $z = (z_n)$ be two sequences. We say that $\rho_n = z_n$ for almost all n relative to I (in short, a.a.n.r.I) if the set $\{n \in \mathbb{N} : \rho_n \neq z_n\} \in I$.

Definition 1.6. A sequence space E is said to be solid or normal if $(\alpha_n \rho_n) \in E$ for any sequence $(\rho_n) \in E$ and any sequence of scalars $(\alpha_n) \in \Psi$ with $|\alpha_n| < 1$ for all $n \in \mathbb{N}$.

Lemma 1.7. *Every solid space is monotone.*

Definition 1.8. Let $K = \{n_i \in \mathbb{N} : n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$, and let E be a sequence space. The K-step space of E is the sequence space

$$\lambda_K^E = \big\{(\rho_{n_i}) \in \Psi : (\rho_n) \in E\big\}.$$

A canonical preimage of a sequence $(\rho_{n_i}) \in \lambda_K^E$ is the sequence $(\mu_n) \in \Psi$ defined as

$$\mu_n = \begin{cases} \rho_n, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of the step space λ_K^E is the set of canonical preimages of all elements in λ_K^E, that is, μ is in the canonical preimage of λ_K^E iff y is the canonical preimage of some element $\rho \in \lambda_K^E$.

Definition 1.9. A sequence space E is said to be monotone if it contains the canonical preimages of its step space (i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(\rho_n) \in E$, the sequence $(\alpha_n \rho_n)$ with $\alpha_n = 1$ for $n \in K$ and $\alpha_n = 0$ otherwise belongs to E).

Definition 1.10. A map h defined on a domain $D \subset \mathcal{X}$(i.e., $h : D \subset \mathcal{X} \longrightarrow \mathbb{R}$) is said to satisfy the Lipschitz condition if $|h(\rho) - h(\mu)| \leqslant K|\rho - \mu|$, where K is called the Lipschitz constant.

Remark 1.11. As in [33], the convergence field of I-convergence is the set

$$\mathcal{F}(I) = \left\{ \rho = (\rho_k) \in \ell_\infty : \text{ there exists } I - \lim \rho \in \mathbf{R} \right\}.$$

Definition 1.12. The convergence field $\mathcal{F}(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $\mathcal{F}(I) = \ell_\infty \cap c^J$.

Lemma 1.13. *As in [34], let* $K \in \mathcal{F}(I)$ *and* $M \subseteq \mathbb{N}$. *If* $M \notin I$, *then* $M \cap K \notin I$.

Definition 1.14. The function $h : D \subset X \longrightarrow \mathbb{R}$ defined by $h(\rho) = I\text{-}\lim \rho$ for all $\rho \in \mathcal{F}(I)$ is a Lipschitz function.

2. I-Convergence Fibonacci difference sequence spaces

This portion of the paper deals with introducing the sequence spaces as the sets of sequences whose $\widetilde{\mho_g}$-transforms are in the spaces c_0^J, c^J and ℓ_∞^J, where $g = (g_k)$ is fixed non-zero sequence of real numbers. We give certain inclusion results and basic topological structures will be studied. Throughout the manuscript, choose a sequence $\rho = (\rho_n) \in \Psi$ and $\widetilde{\mho_{g_n}}(\rho)$ having connected by relation (1.4) and I is an admissible ideal of subset of $\mathbb{N}$. We define

$$c_0^J(\widetilde{\mho_g}) = \left\{ \rho = (\rho_n) \in \Psi : \left\{ n \in \mathbb{N} : \left| \widetilde{\mho_{g_n}}(x) \right| \geqslant \epsilon \right\} \in I \right\}, \tag{2.1}$$

$$c^J(\widetilde{\mho_g}) = \left\{ \rho = (\rho_n) \in \Psi : \left\{ n \in \mathbb{N} : \left| \widetilde{\mho_{g_n}}(\rho) - L \right| \geqslant \epsilon, \text{ for some } L \in \mathbb{R} \right\} \in I \right\}, \tag{2.2}$$

$$\ell_\infty^J(\widetilde{\mho_g}) = \left\{ \rho = (\rho_n) \in \Psi : \exists K > 0 \text{ s.t. } \left\{ n \in \mathbb{N} : \left| \widetilde{\mho_{g_n}}(\rho) \right| \geqslant K \right\} \in I \right\}. \tag{2.3}$$

We write

$$m_0^J(\widetilde{\mho_g}) = c_0^J(\widetilde{\mho_g}) \cap \ell_\infty(\widetilde{\mho_g}), \tag{2.4}$$

and

$$m^J(\widetilde{\mho_g}) := c^J(\widetilde{\mho_g}) \cap \ell_\infty(\widetilde{\mho_g}). \tag{2.5}$$

With notation (1.1), the spaces $c_0^J(\widetilde{\mho_g}), c^J(\widetilde{\mho_g}), \ell_\infty^J(\widetilde{\mho_g}), m^J(\widetilde{\mho_g})$ and $m_0^J(\widetilde{\mho_g})$ can be redefined as follows:

$$\begin{aligned} &c_0^J(\widetilde{\mho_g}) = (c_0^J)_{\widetilde{\mho_g}}, \quad c^J(\widetilde{\mho_g}) = (c^J)_{\widetilde{H}}, \quad \ell_\infty^J(\widetilde{\mho_g}) = (\ell_\infty^J)_{\widetilde{\mho_g}}, \\ &m^J(\widetilde{\mho_g}) = (m^J)_{\widetilde{\mho_g}}, \quad m_0^J(\widetilde{H}) = (m_0^J)_{\widetilde{\mho_g}}. \end{aligned}$$

Definition 2.1. Let I be an admissible ideal of subsets of $\mathbb{N}$. A sequence $\rho = (\rho_n) \in \Psi$ is called Fibonacci I-Cauchy if for each $\epsilon > 0$, there exists a number $N = N(\epsilon) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : |\widetilde{\mho_{g_n}}(\rho) - \widetilde{\mho_{g_N}}(\rho)| \geqslant \epsilon\} \in I.$$

Example 2.2. Define $I_f = \{A \subseteq \mathbf{N} : \text{ A is finite}\}$. Then I_f is an admissible ideal in $\mathbf{N}$ and $c^{I_f}(\widetilde{\mho_g}) = c(\widetilde{\mho_g})$.

Example 2.3. Define the nontrivial ideal $I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the natural density of a set A. In this case, $c^{I_d}(\widetilde{\mho_g}) = S(\widetilde{\mho_g})$, where $S(\widetilde{\mho_g})$ is the space of Fibonacci difference statistically convergent sequence defined as

$$S(\widetilde{\mho_g}) := \left\{ \rho = (\rho_n) \in \Psi : d\left(\left\{ n \in \mathbb{N} : \left| \widetilde{\mho_{g_n}}(\rho) - L \right| \geqslant \epsilon \right\} \right) = 0, \text{ for some } L \in \mathbb{R} \right\}. \tag{2.6}$$

Theorem 2.4. *The sequence spaces* $c^J(\widetilde{\mho_g}), c_0^J(\widetilde{\mho_g}), \ell_\infty^J(\widetilde{\mho_g}), m_0^J(\widetilde{\mho_g})$*, and* $m^J(\widetilde{\mho_g})$ *are linear over* $\mathbb{R}$.

Proof. Let $x = (x_n)$ and $y = (y_n)$ be two arbitrary elements of the space $c^J(\widetilde{\mho_g})$, and let α, β are scalars. Then, for given $\epsilon > 0$, there exist $L_1, L_2 \in \mathbb{R}$ such that

$$\left\{n \in \mathbb{N} : \left|\widetilde{\mho_g}_n(x) - L_1\right| \geqslant \frac{\epsilon}{2}\right\} \in I,$$

and

$$\left\{n \in \mathbb{N} : \left|\widetilde{\mho_g}_n(y) - L_2\right| \geqslant \frac{\epsilon}{2}\right\} \in I.$$

Now, let

$$\mathfrak{A}_1 = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_g}_n(x) - L_1\right| < \frac{\epsilon}{2|\alpha|}\right\} \in \mathcal{F}(I),$$

and

$$\mathfrak{A}_2 = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_g}_n(y) - L_2\right| < \frac{\epsilon}{2|\beta|}\right\} \in \mathcal{F}(I),$$

be such that $\mathfrak{A}_1^c, \mathfrak{A}_2^c \in I$. Then

$$\begin{aligned} \mathfrak{A}_3 =& \left\{n \in \mathbb{N} : \left|\alpha\widetilde{\mho_g}_n(x) + \beta\widetilde{\mho_g}_n(y) - (\alpha L_1 + \beta L_2)\right| < \epsilon\right\} \\ \supseteq& \left\{\left\{n \in \mathbb{N} : \left|\widetilde{\mho_g}_n(x) - L_1\right| < \frac{\epsilon}{2|\alpha|}\right\} \cap \left\{n \in \mathbb{N} : \left|\widetilde{\mho_g}_n(y) - L_2\right| < \frac{\epsilon}{2|\beta|}\right\}\right\}. \end{aligned} \tag{2.7}$$

Thus, the sets on the right-hand side of (2.7) belong to $\mathcal{F}(I)$. By the definition of the filter associated with an ideal the complement of the set on the left-hand side of (2.7) belongs to I. This implies that $(\alpha x + \beta y) \in c^J(\widetilde{\mho_g})$. Hence $c^J(\widetilde{\mho_g})$ is a linear space. The proof of the remaining results is similar. □

Theorem 2.5. *The spaces* $X(\widetilde{\mho_g})$ *are normed spaces with the norm*

$$\|x\|_{X(\widetilde{\mho_g})} = \sup_n \left|\widetilde{\mho_g}_n(x)\right|, \quad \text{where } X \in \{m^J, m_0^J\}. \tag{2.8}$$

Proof. The proof of the result is easy by existing techniques and hence is omitted. □

Theorem 2.6. *Let* $I \subseteq 2^{\mathbb{N}}$ *be a nontrivial ideal. Then the inclusion* $c(\widetilde{\mho_g}) \subset c^J(\widetilde{\mho_g})$ *is strict.*

Proof. We know that $c \subseteq c^J$ and, for any X and Y spaces, $X \subseteq Y$ implies $X(\widetilde{\mho_g}) \subseteq Y(\widetilde{\mho_g})$ (see [25]- Lemma 2.1). Hence it is easy to see that $c(\widetilde{H}) \subset c^J(\widetilde{\mho_g})$. □

The following example shows the strictness of the inclusion.

Example 2.7. Define the sequence $\rho = (\rho_n) \in \Psi$ such that

$$\widetilde{\mho_g}_n(\rho) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\rho \in c^{I_d}(\widetilde{\mho_g})$, but $\rho \notin c(\widetilde{\mho_g})$.

Example 2.8. Define the ideal I such that $\mathfrak{A} \in I \iff \mathfrak{A}$ eventually contains only even natural numbers. Then I is a nontrivial ideal in $\mathbb{N}$. When

$$\widetilde{\mho_g}_n(\rho) = (1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 0, 1, 0, \cdots),$$

by taking $g = 1$, then we have

$$\mathfrak{A}_\epsilon = \left\{n \in \mathbb{N} : \widetilde{\mho_{g_n}}(\rho) \neq 0\right\} = \{1, 2, 3, 5, 6, 7, 10, 12, 14, 16, 18, \cdots\},$$

and $(\rho_n) \in c_0^J(\widetilde{\mho_g})$. Hence $\mathfrak{A}_\epsilon \in I$ and $\widetilde{\mho_{g_n}}(\rho) \in c^J$. Now let us look at the statistical convergence of the sequence:

$$\lim_{n\to\infty} \frac{1}{n}|\mathfrak{A}_\epsilon| = \lim_{n\to\infty} \frac{1}{n}\left|\mathfrak{B} + \frac{n}{2}\right| = \frac{1}{2},$$

where $\mathfrak{B}$ is a finite number, and $|\mathfrak{A}_\epsilon|$ is the cardinality of $\mathfrak{A}_\epsilon$. Hence $\widetilde{\mho_{g_n}}(\rho) \notin S$.

Theorem 2.9. *A sequence $\rho = (\rho_n) \in \Psi$ Fibonacci I-converges if and only if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that*

$$\left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho) - \widetilde{\mho_{g_N}}(\rho)\right| < \epsilon\right\} \in \mathcal{F}(I). \tag{2.9}$$

Proof. Suppose that a sequence $\rho = (\rho_n) \in \Psi$ is Fibonacci I-convergent to some number $L \in \mathbb{R}$. Then, for given $\epsilon > 0$, the set

$$\mathfrak{B}_\epsilon = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho) - L\right| < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I).$$

Fix an integer $N = N(\epsilon) \in \mathfrak{B}_\epsilon$. Then we have

$$\left|\widetilde{\mho_{g_n}}(\rho) - \widetilde{\mho_{g_N}}(\rho)\right| \leqslant \left|\widetilde{\mho_{g_n}}(\rho) - L\right| + \left|L - \widetilde{\mho_{g_N}}(\rho)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $n \in \mathfrak{B}_\epsilon$. Hence (2.9) holds.

Conversely, suppose that (2.9) holds for all $\epsilon > 0$. Then

$$\mathfrak{C}_\epsilon = \left\{n \in \mathbb{N} : \widetilde{\mho_{g_n}}(\rho) \in \left[\widetilde{\mho_{g_n}}(\rho) - \epsilon, \widetilde{\mho_{g_n}}(\rho) + \epsilon\right]\right\} \in \mathcal{F}(I), \quad \forall \epsilon > 0.$$

Let $\mathfrak{J}_\epsilon = [\widetilde{\mho_{g_n}}(\rho) - \epsilon, \widetilde{\mho_{g_n}}(\rho) + \epsilon]$. Fixing $\epsilon > 0$, we have $\mathfrak{C}_\epsilon \in \mathcal{F}(I)$ and $\mathfrak{C}_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Hence $\mathfrak{C}_\epsilon \cap \mathfrak{C}_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$\mathfrak{J} = \mathfrak{J}_\epsilon \cap \mathfrak{J}_{\frac{\epsilon}{2}} \neq \emptyset,$$

that is,

$$\left\{n \in \mathbb{N} : \widetilde{\mho_{g_n}}(\rho) \in \mathfrak{J}\right\} \in \mathcal{F}(I),$$

and thus

$$\operatorname{diam}(\mathfrak{J}) \leqslant \frac{1}{2}\operatorname{diam}(\mathfrak{J}_\epsilon),$$

where $\operatorname{diam}(\mathfrak{J})$ denotes the length of an interval $\mathfrak{J}$. Proceeding in this way, by induction we get a sequence of closed intervals

$$\mathfrak{J}_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots,$$

such that

$$\operatorname{diam}(I_n) \leqslant \frac{1}{2}\operatorname{diam}(I_{n-1}), \quad \text{for } n = 2, 3, \cdots,$$

and

$$\left\{n \in \mathbb{N} : \widetilde{\mho_{g_n}}(\rho) \in I_n\right\} \in \mathcal{F}(I).$$

Then there exists a number $L \in \bigcap_{n\in\mathbb{N}} I_n$, and it is a routine work to verify that $\mathfrak{L} = I\text{-}\lim \widetilde{\mho_{g_n}}(\rho)$, showing that $\rho = (\rho_n) \in \Psi$ Fibonacci I-converges. □

Theorem 2.10. *The inclusions $c_0^J(\widetilde{\mho_g}) \subset c^J(\widetilde{\mho_g}) \subset \ell_\infty^J(\widetilde{\mho_g})$ are strict.*

Proof. The inclusion $c_0^J(\widetilde{\mho_g}) \subset c^J(\widetilde{\mho_g})$ is obvious. Now, to show its strictness, consider the sequence $\rho = (\rho_n) \in \Psi$ such that $\widetilde{\mho_{g_n}}(\rho) = 1$. It easy to see that $\widetilde{\mho_{g_n}}(\rho) \in c^J$ but $\widetilde{\mho_{g_n}}(\rho) \notin c_0^J$, that is, $\rho \in c^J(\widetilde{\mho_g}) \setminus c_0^J(\widetilde{\mho_g})$. Now, let $\rho = (\rho_n) \in c^J(\widetilde{\mho_g})$. Then there exists $L \in \mathbb{R}$ such that $I\text{-}\lim |\widetilde{\mho_{g_n}}(x) - \mathfrak{L}| = 0$, that is,

$$\{n \in \mathbb{N} : |\widetilde{\mho_{g_n}}(\rho) - \mathfrak{L}| \geqslant \epsilon\} \in I.$$

We have

$$|\widetilde{\mho_{g_n}}(\rho)| = |\widetilde{\mho_{g_n}}(\rho) - \mathfrak{L} + \mathfrak{L}| \leqslant |\widetilde{\mho_{g_n}}(\rho) - \mathfrak{L}| + |\mathfrak{L}|.$$

From this, it easily follows that the sequence (ρ_n) must belong to $\ell_\infty^J(\widetilde{\mho_g})$. Further, we show the strictness of the inclusion $c^J(\widetilde{\mho_g}) \subset \ell_\infty^J(\widetilde{\mho_g})$ by constructing the following example.

Example 2.11. For $g = 1$, define the sequence $\rho = (\rho_n) \in \Psi$ such that

$$\widetilde{\mho_{g_n}}(\rho) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is a square,} \\ 1, & \text{if } n \text{ is odd nonsquare,} \\ 0, & \text{if } n \text{ is even nonsquare.} \end{cases}$$

Then $\widetilde{\mho_{g_n}}(\rho) \in \ell_\infty^J$, but $\widetilde{\mho_{g_n}}(\rho) \notin c^J$, which implies that $\rho \in \ell_\infty^J(\widetilde{\mho_g}) \setminus c^J(\widetilde{\mho_g})$.

Thus the inclusion $c_0^J(\widetilde{\mho_g}) \subset c^J(\widetilde{\mho_g}) \subset \ell_\infty^J(\widetilde{\mho_g})$ is strict. □

Remark 2.12. A Fibonacci bounded sequence is obviously Fibonacci I-bounded as the empty set belongs to the ideal I. However, the converse is not true. For example, consider the sequence

$$\widetilde{\mho_{g_n}}(\rho) = \begin{cases} n, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\widetilde{\mho_{g_n}}(\rho)$ is not a bounded sequence. However, $\{n \in \mathbb{N} : |\widetilde{\mho_{g_n}}(\rho)| \geqslant \frac{1}{2}\} \in I$. Hence $\rho = (\rho_n)$ is Fibonacci I-bounded.

Theorem 2.13. *The spaces $m^J(\widetilde{\mho_g})$ and $m_0^J(\widetilde{\mho_g})$ are Banach spaces normed by* (2.8).

Proof. Let $(\rho_n^{(i)})$ be a Cauchy sequence in $m^J(\widetilde{\mho_g}) \subset \ell_\infty(\widetilde{\mho_g})$. Then $(\rho_n^{(i)})$ converges in $\ell_\infty(\widetilde{\mho_g})$, and $\lim_{i\to\infty} \widetilde{H}_n^{(i)}(\rho) = \widetilde{\mho_{g_n}}(\rho)$. Let $I\text{-}\lim \widetilde{\mho_{g_n}}^{(j)}(x) = \mathfrak{L}_i$ for $i \in \mathbb{N}$. Then we have to show that

(i) $(\mathfrak{L}_i)$ is convergent say to $\mathfrak{L}$;

(ii) $I\text{-}\lim \widetilde{\mho_{g_n}}(\rho) = \mathfrak{L}$.

(i) Since $(\rho_n^{(i)})$ is a Cauchy sequence, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|\widetilde{\mho_{g_n}}^{(i)}(\rho) - \widetilde{\mho_{g_n}}^{(j)}(\rho)| < \frac{\epsilon}{3}, \quad \forall i, j \geqslant n_0. \tag{2.10}$$

Now let $\mathfrak{E}_i$ and $\mathfrak{E}_j$ be the following sets in I

$$\mathfrak{E}_i = \left\{n \in \mathbb{N} : |\widetilde{\mho_{g_n}}^{(i)}(\rho) - \mathfrak{L}_i| \geqslant \frac{\epsilon}{3}\right\}, \tag{2.11}$$

and

$$\mathfrak{E}_j = \left\{n \in \mathbb{N} : |\widetilde{\mho_{g_n}}^{(j)}(\rho) - \mathfrak{L}_j| \geqslant \frac{\epsilon}{3}\right\}. \tag{2.12}$$

Consider $i, j \geqslant n_0$ and $n \notin \mathfrak{E}_i \cap \mathfrak{E}_j$. Then we have by (2.10), (2.11) and (2.12)

$$|\mathfrak{L}_i - \mathfrak{L}_j| \leqslant \left|\widetilde{\mho_{g_n}}^{(i)}(\rho) - \mathfrak{L}_i\right| + \left|\widetilde{\mho_{g_n}}^{(j)}(\rho) - \mathfrak{L}_j\right| + \left|\widetilde{H}_n^{(i)}(\rho) - \widetilde{\mho_{g_n}}^{(j)}(\rho)\right| < \epsilon.$$

Thus $(\mathfrak{L}_i)$ is a Cauchy sequence in $\mathbb{R}$ and thus converges, say to $\mathfrak{L}$, that is, $\lim\limits_{i\to\infty} \mathfrak{L}_i = \mathfrak{L}$.

(ii) Let $\delta > 0$ be given. Then we can find m_0 such that

$$|\mathfrak{L}_i - \mathfrak{L}| < \frac{\delta}{3} \quad \text{for each } i > m_0. \tag{2.13}$$

We have $(\rho_n^{(i)}) \to \rho_n$ as $i \to \infty$. Thus

$$\left|\widetilde{\mho_{g_n}}^{(i)}(\rho) - \widetilde{\mho_{g_n}}(\rho)\right| < \frac{\delta}{3} \quad \text{for each } i > m_0. \tag{2.14}$$

Since $(\widetilde{\mho_{g_n}}^{(j)})$ is I-converges to $\mathfrak{L}_j$, there exists $\mathfrak{D} \in I$ such that, for each $n \notin \mathfrak{D}$, we have

$$\left|\widetilde{\mho_{g_n}}^{(j)}(\rho) - \mathfrak{L}_j\right| < \frac{\delta}{3}. \tag{2.15}$$

Assume without loss of generality that $j > m_0$. Then, for all $n \notin \mathfrak{D}$, we have by (2.13), (2.14), and (2.15) that

$$\left|\widetilde{\mho_{g_n}}(\rho) - \mathfrak{L}\right| \leqslant \left|\widetilde{\mho_{g_n}}(\rho) - \widetilde{H}_n^{(j)}(\rho)\right| + \left|\widetilde{\mho_{g_n}}^{(j)}(\rho) - \mathfrak{L}_j\right| + |\mathfrak{L}_j - \mathfrak{L}| < \delta.$$

Hence (ρ_n) is Fibonacci I-convergent to $\mathfrak{L}$. Thus $m^J(\widetilde{\mho_g})$ is a Banach space. The other cases can be similarly established. □

The following results are consequences of Theorem 2.13.

Theorem 2.14. *The spaces $m^J(\widetilde{\mho_g})$ and $m_0^J(\widetilde{\mho_g})$ are K-spaces.*

Theorem 2.15. *The set $m^J(\widetilde{\mho_g})$ is a closed subspace of $\ell_\infty(\widetilde{\mho_g})$.*

As the inclusions $m^J(\widetilde{\mho_g}) \subset \ell_\infty(\widetilde{\mho_g})$ and $m_0^J(\widetilde{\mho_g}) \subset \ell_\infty(\widetilde{\mho_g})$ are strict, in view of Theorem 2.15, we have the following result.

Theorem 2.16. *The spaces $m^J(\widetilde{\mho_g})$ and $m_0^J(\widetilde{\mho_g})$ are nowhere dense subsets of $\ell_\infty(\widetilde{\mho_g})$.*

Theorem 2.17. *The spaces $c_0^J(\widetilde{\mho_g})$ and $m_0^J(\widetilde{\mho_g})$ are solid and monotone.*

Proof. We will prove the result for $c_0^J(\widetilde{\mho_g})$; for $m_0^J(\widetilde{\mho_g})$, the result can be established similarly.

Let $\rho = (\rho_n) \in c_0^J(\widetilde{\mho_g})$. For $\epsilon > 0$, the set

$$\left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho)\right| \geqslant \epsilon\right\} \in I. \tag{2.16}$$

Let $\alpha = (\alpha_n)$ be a sequence of scalars with $|\alpha| \leqslant 1$ for all $n \in \mathbb{N}$. Then

$$\left|\widetilde{\mho_{g_n}}(\alpha\rho)\right| = \left|\alpha\widetilde{\mho_{g_n}}(\rho)\right| \leqslant |\alpha|\left|\widetilde{\mho_{g_n}}(\rho)\right| \leqslant \left|\widetilde{\mho_{g_n}}(\rho)\right|, \quad \forall n \in \mathbb{N}.$$

From this inequality and from (2.16) we have that

$$\left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\alpha\rho)\right| \geqslant \epsilon\right\} \subseteq \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho)\right| \geqslant \epsilon\right\} \in I,$$

implies

$$\left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\alpha\rho)\right| \geqslant \epsilon\right\} \in I.$$

Therefore $(\alpha\rho_n) \in c_0^J(\widetilde{\mho_g})$. Hence the space $c_0^J(\widetilde{\mho_g})$ is solid, and hence by Lemma 1.7 the space $c_0^J(\widetilde{\mho_g})$ is monotone. □

Theorem 2.18. *The spaces* $c_0^J(\widetilde{\mho_g})$ *and* $c^J(\widetilde{\mho_g})$ *are sequence algebras.*

Proof. Let $\rho = (\rho_n),\ y = (y_n) \in c_0^J(\widetilde{\mho_g})$. Then

$$\text{I-}\lim_{n\to\infty}\left|\widetilde{\mho_{g_n}}(\rho)\right| = 0 \quad \text{and} \quad \text{I-}\lim_{n\to\infty}\left|\widetilde{\mho_{g_n}}(\mu)\right| = 0. \tag{2.17}$$

Therefore, from (2.17) we have I-lim $|\widetilde{H}_n(\rho\cdot\mu)| = 0$. This implies that $\{n \in \mathbb{N} : |\widetilde{H}_n(\rho\cdot\mu)| \geqslant \epsilon\} \in I$. Thus, $(\rho\cdot\mu) \in c_0^J(\widetilde{\mho_g})$. Hence $c_0^J(\widetilde{\mho_g})$ is sequence algebra. Similarly, we can prove that $c^J(\widetilde{\mho_g})$, is a sequence algebra. □

Theorem 2.19. *The function* $h : m^J(\widetilde{\mho_g}) \to \mathbb{R}$ *defined by* $\mathfrak{H}(x) = |I - \lim \widetilde{\mho_{g_n}}(\rho)|$, *where*

$$m^J(\widetilde{\mho_g}) = \ell_\infty(\widetilde{\mho_g}) \cap c^J(\widetilde{\mho_g}),$$

is a Lipschitz function and hence uniformly continuous.

Proof. To prove the result, we first prove that the function is well defined. For this, let $\rho, \sigma \in m^J(\widetilde{\mho_g})$ be such that

$$\begin{aligned} \rho = \sigma \quad &\Rightarrow \quad \text{I-}\lim \widetilde{\mho_{g_n}}(\rho) = \text{I-}\lim \widetilde{\mho_{g_n}}(\sigma) \\ &\Rightarrow \quad \left|\text{I-}\lim \widetilde{\mho_{g_n}}(\rho)\right| = \left|\text{I-}\lim \widetilde{\mho_{g_n}}(\sigma)\right| \quad \Rightarrow \quad \mathfrak{H}(\rho) = \mathfrak{H}(\sigma). \end{aligned}$$

Thus $\mathfrak{H}$ is well defined. Now let $\rho = (\rho_n), \sigma = (\sigma_n) \in m^J(\widetilde{H}), \rho \neq \sigma$. Then

$$\mathfrak{A}_\rho = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho) - \mathfrak{H}(\rho)\right| \geqslant |\rho - \sigma|_*\right\} \in I,$$

and

$$\mathfrak{A}_\sigma = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\sigma) - \mathfrak{H}(\sigma)\right| \geqslant |\rho - \sigma|_*\right\} \in I,$$

where $|\rho - \sigma|_* = \sup_n |\widetilde{\mho_{g_n}}(\rho) - \widetilde{\mho_{g_n}}(\sigma)|$. Thus

$$\mathfrak{B}_\rho = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho) - \mathfrak{H}(\rho)\right| < |\rho - \sigma|_*\right\} \in \mathcal{F}(I),$$

and

$$\mathfrak{B}_\sigma = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\sigma) - \mathfrak{H}(\sigma)\right| < |\rho - \sigma|_*\right\} \in \mathcal{F}(I).$$

Hence $\mathfrak{B} = \mathfrak{B}_\rho \cap \mathfrak{B}_\sigma \in \mathcal{F}(I)$, so that $\mathfrak{B}$ is a nonempty set. Therefore, choosing $n \in \mathfrak{B}$, we have

$$\begin{aligned} \left|\mathfrak{H}(\rho) - \mathfrak{H}(\sigma)\right| &\leqslant \left|\mathfrak{H}(\rho) - \widetilde{\mho_{g_n}}(\rho)\right| + \left|\widetilde{\mho_{g_n}}(\rho) - \widetilde{\mho_{g_n}}(\sigma)\right| + \left|\widetilde{\mho_{g_n}}(\sigma) - \mathfrak{H}(\sigma)\right| \\ &\leqslant 3|\rho - \sigma|_*. \end{aligned}$$

Thus, $\mathfrak{H}$ is a Lipschitz function and hence uniformly continuous.

□

Theorem 2.20. *If* $\rho = (\rho_n),\ \sigma = (\sigma_n) \in m^J(\widetilde{\mho_g})$ *with* $\widetilde{\mho_{g_n}}(\rho\cdot\sigma) = \widetilde{\mho_{g_n}}(\rho)\cdot\widetilde{\mho_{g_n}}(\sigma)$, *then* $(\rho\cdot\sigma) \in m^J(\widetilde{\mho_g})$ *and* $\mathfrak{H}(\rho\cdot\sigma) = \mathfrak{H}(\rho)\cdot\mathfrak{H}(\sigma)$, *where* $\mathfrak{H} : m^J(\widetilde{\mho_g}) \to \mathbb{R}$ *is defined by* $\mathfrak{H}(\rho) = |\text{I-}\lim \widetilde{\mho_{g_n}}(\rho)|$.

Proof. For $\epsilon > 0$,

$$\mathfrak{B}_\rho = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\rho) - \mathfrak{H}(\rho)\right| < \epsilon\right\} \in \mathcal{F}(I), \tag{2.18}$$

and

$$\mathfrak{B}_\sigma = \left\{n \in \mathbb{N} : \left|\widetilde{\mho_{g_n}}(\sigma) - \mathfrak{H}(\sigma)\right| < \epsilon\right\} \in \mathcal{F}(I), \tag{2.19}$$

where $\epsilon = |\rho - \sigma|_* = \sup_n |\widetilde{\mho_{g_n}}(\rho) - \widetilde{\mho_{g_n}}(\sigma)|$. Now, we have

$$\begin{aligned}\left|\widetilde{\mho_{g_n}}(\rho\cdot\sigma) - \mathfrak{H}(\rho)\mathfrak{H}(\sigma)\right| &= \left|\widetilde{\mho_{g_n}}(\rho)\widetilde{\mho_{g_n}}(\sigma) - \widetilde{\mho_{g_n}}(\rho)\mathfrak{H}(\sigma) + \widetilde{\mho_{g_n}}(\rho)\mathfrak{H}(\sigma) - \mathfrak{H}(\rho)\mathfrak{H}(\sigma)\right| \\ &\leqslant \left|\widetilde{\mho_{g_n}}(\rho)\right|\left|\widetilde{\mho_{g_n}}(\sigma) - \mathfrak{H}(\sigma)\right| + \left|\mathfrak{H}(\sigma)\right|\left|\widetilde{\mho_{g_n}}(\rho) - \mathfrak{H}(\rho)\right|.\end{aligned} \tag{2.20}$$

As $m^J(\widetilde{\mho_g}) \subseteq \ell_\infty(\widetilde{\mho_g})$, there exists $\mathfrak{M} \in \mathbb{R}$ such that $|\widetilde{\mho_{g_n}}(\rho)| < \mathfrak{M}$. Therefore, from (2.18), (2.19) and (2.20) we have

$$\begin{aligned}\left|\widetilde{\mho_{g_n}}(\rho\sigma) - \mathfrak{H}(\rho)\mathfrak{H}(\sigma)\right| &= \left|\widetilde{\mho_{g_n}}(\rho)\cdot\widetilde{\mho_{g_n}}(\sigma) - \mathfrak{H}(\rho)\mathfrak{H}(\sigma)\right| \\ &\leqslant M\epsilon + \left|\mathfrak{H}(\sigma)\right|\epsilon = \epsilon_1 \quad \text{(say)}\end{aligned}$$

for all $n \in \mathfrak{B}_\rho \cap \mathfrak{B}_\sigma \in \mathcal{F}(I)$. Hence $(\rho\cdot\sigma) \in m^J(\widetilde{\mho_g})$ and $\mathfrak{H}(\rho\cdot\sigma) = \mathfrak{H}(\rho)\cdot\mathfrak{H}(\sigma)$. □

Acknowledgment

We are thankful to the reviewers for the careful reading and suggestions that improved the presentation of the paper.

References

[1] V. Baláž, T. Visnyai, I-*Convergence of Arithmetical Functions*, In: Number Theory and Its Applications, InTechOpen, **2020** (2020), 125–134. 1, 1

[2] M. Başarir, F. Başar, E. E. Kara, *On the spaces of Fibonacci difference absolutely* p*-summable, null and convergent sequences*, Sarajevo J. Math., **12** (2016), 167–182. 1, 1

[3] M. Candan, E. E. Kara, *A study on topological and geometrical characteristics of new Banach sequence spaces*, Gulf J. Math., **3** (2015), 67–84. 1

[4] M. Candan, K. Kayaduman, *Almost convergent sequence space derived by generalized Fibonacci matrix and Fibonacci core*, J. Adv. Math. Comput. Sci., **7** (2015), 150–167. 1

[5] A. Das, B. Hazarika, *Some new Fibonacci difference spaces of non-absolute type and compact operators*, Linear Multilinear Algebra, **65** (2017), 2551–2573. 1

[6] A. Das, B. Hazarika, *Matrix transformation of Fibonacci band matrix on generalized* bv*-space and its dual spaces*, Bol. Soc. Parana. Mat. (3), **36** (2018), 41–52. 1

[7] A. Das, B. Hazarika, *Some properties of generalized Fibonacci difference bounded and* p*-absolutely convergent sequences*, Bol. Soc. Parana. Mat. (3), **36** (2018), 37–50. 1

[8] P. Das, P. Kostyrko, W. Wilczynski, P. Malik, *I and* I^**-convergence of double sequences*, Math. Slovaca, **58** (2008), 605–620. 1

[9] H. Fast, *Sur la convergence statistique*, (French) Colloq. Math., **2** (1951), 241–244. 1

[10] D. Fathema, A. H. Ganie, *On some new scenario of* Δ*-spaces*, J. Nonlinear Sci. Appl., **14** (2021), 163–167. 1

[11] A. H. Ganie, *Nature of Phylllotaxy and topology of H-matrix*, In: Matrix Theory-Applications and Theorems, **2018** (2018), 75–86. 1

[12] A. H. Ganie, *Some new approach of spaces of non-integral order*, J. Nonlinear Sci. Appl., **14** (2021), 89–96. 1

[13] A. H. Ganie, *New spaces over modulus function*, Bol. da Soc. Parana. de Mat., (accepted). 1

[14] A. H. Ganie, M. M. AlBaidani, *Matrix Structure of Jacobsthal numbers*, J. Funct. Space, **2021** (2021), 5 pages. 1

[15] A. H. Ganie, A. Antesar, *Certain sequence spaces using* Δ*-operator*, Adv. Stud. Contemp. Math. (Kyungshang), **30** (2020), 17–27. 1

[16] A. H. Ganie, D. Fathima, *Almost convergence property of generalized Riesz spaces*, J. Appl. Math. Comput., **4** (2020), 249–253.

[17] A. H. Ganie, S. A. Lone, *Generalised Cesaro difference sequence space of non- absolute type*, J. Sci. Data Anal., **20** (2020), 147–153.

[18] A. H. Ganie, N. A. Sheikh, *Infinite matrices and almost convergence*, Filomat, **29** (2015), 1183–1188. 1

[19] D. N. Georgiou, S. D. Iliadis, A. C. Megaritis, G. A. Prinos, *Ideal-convergence classes*, Topology Appl., **222** (2017), 217–226. 1

[20] F. GökçeK, M. A. Sarigöl, *Some matrix and compact operators of the absolute Fibonacci spaces*, Kragujevac J. Math., **44** (2020), 273–286. 1

[21] T. Jalal, *Some ideal convergent multiplier sequence spaces using de la Vallee Poussin mean and Zweier operator*, Proyecciones, **39** (2020), 91–105. 1

[22] T. Jalal, S. A. Gupkari, A. H. Ganie, *Infinite matrices and* σ*-convergent sequences*, Southeast Asian Bull. Math., **36** (2012), 825–830. 1

[23] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl., **2013** (2013), 15 pages. 1

[24] E. E. Kara, M. Ilkhan, *Lacunary* I-*convergent and lacunary* I-*bounded sequence spaces defined by an Orlicz function*, Electron. J. Math. Anal. Appl., **4** (2016), 150–159. 1

[25] M. Kirişçi, A. Karaisa, *Fibonacci numbers, statistical convergence and applications*, arXiv, **2016** (2016), 16 pages. 1, 2

[26] H. Kizmaz, *On certain sequences spaces*, Canad. Math. Bull., **24** (1981), 169–176. 1

[27] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, (2001). 1

[28] P. Kostyrko, M. Macaj, T. Šalát, *Statistical convergence and* I-*convergence*, Real Analysis Exchange, New York, (1999). 1

[29] E. Malkowsky, V. Rakocevic, *Summability Methods and Applications*, Mathematical Institute of the Serbian Academy of Sciences and Arts, Serbia, (2020). 1

[30] S. A. Mohiuddine, B. Hazarika, *Some Classes of Ideal Convergent Sequences and Generalized Difference Matrix Operator*, Filomat, **31** (2017), 1827–1834. 1

[31] M. Mursaleen, A. H. Ganie, N. A. Sheikh, *New type of difference sequenence spaces and matrix transformation*, Filomat, **28** (2014), 1381–1392. 1

[32] M. Mursaleen, S. K. Sharma, *Spaces of ideal convergent sequences*, Sci. World J., **2014** (2014), 6 pages. 1

[33] T. Šalát, B. C. Tripathy, M. Ziman, *On some properties of I-convergence*, Tatra Mt. Math. Publ., **28** (2004), 279–286. 1, 1.11

[34] T. Šalát, B. C. Tripathy, M. Ziman, *On* I-*convergence field*, Ital. J. Pure Appl. Math., **17** (2005), 45–54. 1, 1.13

[35] N. A. Sheikh, A. H. Ganie, *A new paranormed sequence space and some matrix transformation*, Acta Math. Acad. Paed. Nyir., **28** (2012), 47–58. 1

[36] N. A. Sheikh, A. H. Ganie, *Some matrix transformations and almost convergence*, Kathmandu Uni. J. Sci. Eng. Tech., **8** (2012), 89–92.

[37] N. A. Sheikh, A. H. Ganie, *A new type of sequence space of non-absolute type and matrix transformation*, WSEAS Trans. J. Math., **8** (2013), 852–859.

[38] N. A. Sheikh, T. Jalal, A. H. Ganie, *New type of sequence spaces of non-absolute type and some matrix transformations*, Acta Math. Acade. Paedag. Nyireg., **29** (2013), 51–66. 1

[39] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74. 1

[40] T. A. Tarray, P. A. Naik, R. A. Najar, *Matrix representation of an all-inclusive Fibonacci sequence*, Asian J. Math. Stat., **11** (2018), 18–26. 1

[41] B. C. Tripathy, B. Hazarika, I-*convergent sequence spaces associated with multiplier sequences*, Math. Inequal. Appl., **11** (2008), 543–548. 1

[42] B. C. Tripathy, B. Hazarika, *Some* I-*convergent sequence spaces defined by Orlicz functions*, Acta Math. Appl. Sin. Engl. Ser., **27** (2011), 149–154. 1

[43] O. Tuğ, *On some Generalized Nörlund Ideal Convergent Sequence Spaces*, ZANCO J. Pure Appl. Sci., **28** (2016), 97–103. 1

[44] F. Yaşar, K. Kayaduman, *On the domain of the Fibonacci difference matrix*, Mathematics, **7** (2019), 10 pages. 1