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Null boundary controllability of nonlinear integrodifferential systems with Rosenblatt process

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Abstract

By using stochastic analysis, compact semigroups and Schauder fixed-point theorem, we discuss the null boundary controllability of nonlinear integrodifferential system with Rosenblatt process. In addition, the null boundary controllability of Sobolev-type neutral integro-differential system with Rosenblatt is studied. Finally, an example is given to illustrate the obtained results. The null controllability results for stochastic differential systems with Rosenblatt process and control on the boundary have not yet been considered in the literature, and this fact motivates this work.

Keywords: Rosenblatt process, stochastic analysis, Sobolev-type integro-differential system, null boundary controllability.

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1. Introduction

In recent years fractional differential equations have attracted the attentions of many researchers and are becoming increasingly popular due to their practical applications in various fields of science and engineering (see [1, 9, 17, 26, 30, 37]). The deterministic system often fluctuates due to environmental noise. So, it is important and necessary for us to deal with stochastic differential equations (SDEs). The noises arise in mathematical finance, physics, telecommunication networks, hydrology and medicine etc., can be modeled by fractional Brownian motions (fBm). The fBm can be expressed as a Wiener integral with respect to the standard Wiener process, i.e. the integral of a deterministic kernal with respect to a standard Brownian motion, the Hermite process of order 1 is fBm and of order 2 is the Rosenblatt process. Firstly, Tudor investigated the Rosenblatt process which is a self-similar process with stationary increments and it appears as limit of long-range dependent stationary series in the Non-Central Limit theorem, for more details see [2, 7, 29, 35]. Stochastic differential equations driven by fractional Brownian motion have been considered greatly by research community in various aspects due to its salient features for real world problems (see [8, 15, 16, 20, 31, 34]). On the other hand, controllability problems for

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different kinds of dynamical systems have been studied by several authors (see [3, 5, 10, 12, 18, 19, 21, 32, 36], and references therein). There are many applications for boundary control systems (see [22, 24]). Many authors studied the controllability for linear and nonlinear system when the control on the boundary, for example, Barbu studied the boundary control problems with convex cost criterion (see [14]). Balachandran and Anandhi discussed the boundary controllability for ntegrodifferential systems and delay integrodifferential systems in Banach spaces (see [11, 13]). Ahmed obtained the boundary controllability of nonlinear fractional integrodifferential systems (see [4]). Lizzy and Balachandran studied the boundary controllability of nonlinear stochastic fractional system in Hilbert space (see [25]). Ahmed studied the boundary controllability of impulsive nonlinear fractional delay integrodifferential system (see [6]). Wang established sufficient condition for approximate boundary controllability for Semilinear delay differential equations (see [36]). Palanisamy and Chinnathambi investigated the approximate boundary controllability of Sobolev-type stochastic differential systems in Hilbert spaces (see [27]). However, the null controllability results for stochastic differential systems with Rosenblatt process and control on the boundary have not yet been considered in the literature, and this fact motivates this work.

The contributions of this paper exist in the following aspects.

- Stochastic differential systems with Rosenblatt process and control on the boundary are introduced.
- Sufficient conditions for null boundary controllability of nonlinear integrodifferential system with Rosenblatt process are established.
- Sufficient conditions for null boundary controllability of Sobolev type neutral stochastic integrodifferential system with Rosenblatt process are established.

This paper is prepared as follows. In Section 2, we present some basic definitions and lemmas which are useful to prove the main results. In Section 3, we give sufficient conditions to prove the null boundary controllability of nonlinear integrodifferential system with Rosenblatt process. In Section 4, we study the null boundary controllability of Sobolev-type neutral integrodifferential system with Rosenblatt process. In Section 5, we consider an example to verify the theoretical results.

2. Preliminaries

Let (Ω, F, P) be a complete probability space equipped with a normal filtration $F_t, t \in [0, b]$ where F_t is the σ-algebra generated by random variables $\{\omega(s), Z_H(s), s \in [0, b]\}$ and all P-null sets.

Fix a time interval $[0, T]$ and let $\{Z_H(t), t \in [0, T]\}$ represents one-dimensional Rosenblatt process with parameter $H \in (1/2, 1)$. The Rosenblatt process with parameter $H > \frac{1}{2}$ can be written as [28]

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial \nu}(\nu, y_1) \frac{\partial K^{H'}}{\partial \nu}(\nu, y_2) \right] dB(y_1) dB(y_2),$$

where $K^H(t, s)$ is given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (\nu - s)^{H-\frac{3}{2}} \nu^{H-\frac{1}{2}} d\nu, \quad \text{for } s < t,$$

with $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(\cdot, \cdot)$ denotes the Beta function, $K_H(t, s) = 0$ when $t \leqslant s$, $\{B(t), t \in [0, T]\}$ is a Brownian motion, $H' = \frac{H+1}{2}$ and $d(H) = \frac{1}{H+1}\sqrt{\frac{H}{2H-1}}$ is a normalizing constant. The covariance of the Rosenblatt process $Z_H(t), t \in [0, T]$ satisfies $E(Z_H(t)Z_H(s)) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})$.

Let X and Y be two real, separable Hilbert spaces and let $L(Y, X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $L(Y, X)$. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ =$

$\sum_{n=1}^{\infty}\lambda_n < \infty$ where $\lambda_n \geqslant 0$ $(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in Y. We define the infinite dimensional Q-Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}e_n z_n(t),$$

where $(z_n)_{n\geqslant 0}$ is a family of real, independent Rosenblatt process.

In order to define Wiener integrals with respect to the Q-Rosenblatt process, we introduce the space $L_2^0 := L_2^0(Y, X)$ of all Q-Hilbert Schmidt operators $\psi : Y \to X$. We recall that $\psi \in L(Y, X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{L_2^0} := \sum_{n=1}^{\infty}\|\sqrt{\lambda}_n\psi e_n\|^2 < \infty$$

and that the space L_2^0 equipped with the inner product $\langle\vartheta, \psi\rangle_{L_2^0} = \sum_{n=1}^{\infty}\langle\vartheta e_n, \psi e_n\rangle$ is a separable Hilbert space. Let $\phi(s);\ s \in [0, T]$ be a function with values in $L_2^0(Y, X)$, the Wiener integral of ϕ with respect to Z_Q is defined by

$$\int_0^t \phi(s)dZ_Q(s) = \sum_{n=1}^{\infty}\int_0^t \sqrt{\lambda}K_H^*(\phi e_n)(y_1, y_2)dB(y_1)dB(y_2), \tag{2.1}$$

where

$$K_H^*(\vartheta)(y_1, y_2) = \int_{y_1\vee y_2}^{T}\vartheta(r)\frac{\partial K}{\partial r}(r, y_1, y_2)dr.$$

For more details (see [33]).

Lemma 2.1 ([23]). *If $\psi : [0, T] \to L_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|^2_{L_2^0} < \infty$, then the above sum in (2.1) is well defined as X-valued random variable and we have*

$$E\left\|\int_0^t \psi(s)dZ_H(s)\right\|^2 \leqslant 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{L_2^0}ds.$$

The collection of all strongly-measurable, square-integrable, X-valued random variables, denoted by $L_2(\Omega, X)$, is a Banach space equipped with norm

$$\| x(\cdot) \|_{L_2(\Omega,X)} = (E \| x(., \omega) \|^2)^{\frac{1}{2}},$$

where the expectation, E is defined by $E(x) = \int_\Omega x(\omega)dP$. Let $C(J, L_2(\Omega, X))$ be the Banach space of all continuous maps from J into $L_2(\Omega, X)$ satisfying the condition $\sup_{t\in J} E \| x(t) \|^2 < \infty$. Define $\bar{C} = \{x : x(t) \in C(J, L_2(\Omega, X))\}$, with norm $\|\cdot\|_{\bar{C}}$ defined by

$$\|\cdot\|_{\bar{C}} = \left(\sup_{t\in J} E\|x(t)\|^2\right)^{\frac{1}{2}}.$$

Obviously, $\bar{C}$ is a Banach space. Also, let us introduce the set $B_r = \{v \in \bar{C} : \|v\|^2_{\bar{C}} \leqslant r\}$, where $r > 0$.

3. Nonlinear integrodifferential system

In this section, we study the null boundary controllability of nonlinear integrodifferential system with Rosenblatt process in the following form

$$\begin{cases} \frac{dx(t)}{dt} = \sigma x(t) + f_1(t, x(t)) + \int_0^t f_2(s, x(s))d\omega(s) + g(t, x(t))\frac{dZ_H(t)}{dt}, & t \in J = [0, b], \\ \tau x(t) = B_1 u(t), & t \in J = [0, b], \\ x(0) = x_0, \end{cases} \tag{3.1}$$

where the state $x(\cdot)$ takes values in the separable Hilbert space X with inner product $\langle .,. \rangle$ and norm $\| \cdot \|$ and the control function $u(\cdot)$ is given in $L_2(J,U)$, the Hilbert space of admissible control functions with U a Hilbert space. Here, σ, B_1 are bounded linear operators and τ is a linear operator with domain in X and range in separable Hilbert space Y. Let $A : X \to X$ be the linear operator defined by $D(A) = \{x \in D(\sigma);\ \tau x = 0\}$, $Ax = \{\sigma x, \text{ for } x \in D(A)\}$. Suppose $\{\omega(t)\}_{t\geqslant 0}$ is a Wiener process defined on $(\Omega, F, \{F_t\}_{t\geqslant 0}, P)$ with values in Hilbert space K and $\{Z_H(t)\}_{t\geqslant 0}$ is a Rosenblatt process with parameter $H \in (\frac{1}{2}, 1)$ defined on $(\Omega, F, \{F_t\}_{t\geqslant 0}, P)$ with values in Hilbert space Y. The nonlinear operators $f_1 : J \times X \to X$, $f_2 : J \times X \to L(K,X)$ and $g : J \times X \to L_2^0(Y,X)$ are given. To establish the result, we need the following hypotheses.

(H1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

(H2) There exists a linear continuous operator $B : U \to X$ such that $\sigma B \in L(U,X)$; $\tau(Bu) = B_1 u$ for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|Bu\| \leqslant C\|B_1 u\|$ for all $u \in U$, where C is some positive constant.

(H3) The operator A is the infinitesimal generator of a compact semigroup $T(t)$ on X and there exists a constant $M > 0$ such that $\|T(t)\| \leqslant M$.

(H4) For all $t \in (0, b]$ and $u \in U$, $T(t)Bu \in D(A)$. Moreover, there exists a positive constant $M_1 > 0$ such that $\|AT(t)\| \leqslant M_1$.

(H5) The function $f_1 : J \times X \to X$ satisfies the following two conditions:
  (i) the function $f_1 : J \times X \to X$ is continuous;
  (ii) for each positive number $r \in N$, there is a positive function $\rho_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \leqslant r} E\|f_1(t, x(t))\|^2 \leqslant \rho_r(t),$$

the function $s \to \rho_r(s) \in L^1([0,t], R^+)$, and there exists a $\delta > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t \rho_r(s)ds}{r} = \delta < \infty,\ t \in J.$$

(H6) The function $f_2 : J \times X \to L(K,X)$ satisfies the following two conditions:
  (i) the function $f_2 : J \times X \to L(K,X)$ is continuous;
  (ii) for each positive number $r \in N$, there is a positive function $h_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \leqslant r} \int_0^t E\|f_2(s, x(s))\|_Q^2\, ds \leqslant h_r(t),$$

the function $s \to h_r(s) \in L^1([0,t], R^+)$, and there exists a $\delta_1 > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t h_r(s)ds}{r} = \delta_1 < \infty,\ t \in J.$$

(H7) The function $g : J \times X \to L_2^0(Y,X)$ satisfies the following two conditions:
  (i) the function $g : J \times X \to L_2^0(Y,X)$ is continuous;
  (ii) for each positive number $r \in N$, there is a positive function $k_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \leqslant r} E\|g(t, x(t))\|_{L_2^0}^2 \leqslant k_r(t),$$

the function $s \to k_r(s) \in L^1([0,t], R^+)$, and there exists a $\delta_2 > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t k_r(s)ds}{r} = \delta_2 < \infty,\ t \in J.$$

Let $x(t)$ be the solution of the system (3.1). Then we can define a function $z(t) = x(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(A)$. Hence (3.1) can be written in terms of A and B as

$$\begin{cases} \frac{dx(t)}{dt} = Az(t) + \sigma Bu(t) + f_1(t, x(t)) + \int_0^t f_2(s, x(s))d\omega(s) + g(t, x(t))\frac{dZ_H(t)}{dt}, \ t \in J, \\ x(t) = z(t) + Bu(t), \\ x(0) = x_0. \end{cases}$$

If u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to the Cauchy problem

$$\begin{cases} \frac{dz(t)}{dt} = Az(t) + \sigma Bu(t) - B\frac{du(t)}{dt} + f_1(t, x(t)) + \int_0^t f_2(s, x(s))d\omega(s) + g(t, x(t))\frac{dZ_H(t)}{dt}, \ t \in J, \\ z(0) = x(0) - Bu(0). \end{cases} \tag{3.2}$$

Then the mild solution of (3.2) is given by

$$\begin{aligned} z(t) = T(t)z(0) + \int_0^t T(t-s)[\sigma Bu(s) - B\frac{du(s)}{ds}]ds + \int_0^t T(t-s)f_1(s, x(s))ds \\ + \int_0^t \int_0^s T(t-s)f_2(\eta, x(\eta))d\omega(\eta)ds + \int_0^t T(t-s)g(s, x(s))dZ_H(s). \end{aligned} \tag{3.3}$$

Since, $x(t) = z(t) + Bu(t)$, then the solution of (3.1) is given by

$$\begin{aligned} x(t) = T(t)[x_0 - Bu(0)] + Bu(t) + \int_0^t T(t-s)[\sigma Bu(s) - B\frac{du(s)}{ds}]ds \\ + \int_0^t T(t-s)f_1(s, x(s))ds + \int_0^t \int_0^s T(t-s)f_2(\eta, x(\eta))d\omega(\eta)ds + \int_0^t T(t-s)g(s, x(s))dZ_H(s). \end{aligned} \tag{3.4}$$

By using integration by parts, we get

$$\int_0^t [T(t-s)B\frac{du(s)}{ds}]ds = Bu(t) - T(t)Bu(0) + \int_0^t AT(t-s)Bu(s)ds.$$

By substitution in (3.3) we obtain

$$\begin{aligned} x(t) = T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds + \int_0^t T(t-s)f_1(s, x(s))ds \\ + \int_0^t \int_0^s T(t-s)f_2(\eta, x(\eta))d\omega(\eta)ds + \int_0^t T(t-s)g(s, x(s))dZ_H(s). \end{aligned}$$

which it is a mild solution of (3.1). To study the exact null controllability of (3.1), we consider the linear boundary control system

$$\begin{cases} \frac{d\mu(t)}{dt} = \sigma\mu(t) + F(t), & t \in J = [0, b], \\ \tau\mu(t) = B_1u(t), & t \in J = [0, b], \\ \mu(0) = \mu_0, \end{cases} \tag{3.5}$$

associated with the system (3.1). Consider

$$L_0^b u = \int_0^b [T(b-s)\sigma B - AT(b-s)B]u(s)ds \ : L_2(J, U) \to X,$$

where $L_0^b u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(J, U)/\ker(L_0^b)$, and

$$N_0^b(\mu, F) = T(b)\mu + \int_0^b T(b-s)F(s)ds : X \times L_2(J, X) \to X.$$

Definition 3.1 ([21]). The system (3.5) is said to be exactly null controllable on J if $\mathrm{Im}L_0^b \supset \mathrm{Im}N_0^b$ or there exists a $\gamma > 0$ such that $\|(L_0^b)^*\mu\|^2 \geqslant \gamma\|(N_0^b)^*\mu\|^2$ for all $\mu \in X$.

Lemma 3.2 ([19]). *Suppose that the linear system (3.5) is exactly null controllable on J. Then the linear operator $(L_0)^{-1}N_0^b : X \times L_2(J,X) \to L_2(J,U)$ is bounded and the control*

$$u(t) = -(L_0)^{-1}\left[T(b)\mu_0 + \int_0^b T(b-s)F(s)ds\right](t)$$

transferees the system (3.5) from μ_0 to 0, where L_0 is the restriction of L_0^b to $[\ker L_0^b]^\perp$ and $F \in L_2(J,X)$.

Definition 3.3. We say $x \in \bar{C}$ is a mild solution to (3.1) if it satisfies that

$$\begin{aligned} x(t) = T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds + \int_0^t T(t-s)f_1(s,x(s))ds \\ + \int_0^t \int_0^s T(t-s)f_2(\eta,x(\eta))d\omega(\eta)ds + \int_0^t T(t-s)g(s,x(s))dZ_H(s), t \in J. \end{aligned}$$

Definition 3.4. The system (3.1) is said to be exact null controllable on the interval J if there exists a stochastic control $u \in L_2(J,U)$ such that the solution $x(t)$ of the system (3.1) satisfies $x(b) = 0$.

To prove the main result in this section, we need addition hypothesis:

(H8) The linear system (3.5) is exactly null controllable on J.

Theorem 3.5. *If the hypotheses* (H1)-(H8) *are satisfied, then the boundary control system* (3.1) *is exactly null controllable on J provided that*

$$25M^2\left[\delta + \mathrm{Tr}(Q)\delta_1 + 2Hb^{2H-1}\delta_2\right]\left[1 + [M^2\|\sigma\|^2 + M_1^2]\|B\|^2\|(L_0)^{-1}\|^2\right] < 1. \tag{3.6}$$

Proof. For an arbitrary $x(\cdot)$ define the operator Φ on $\bar{C}$ as follows

$$\begin{aligned} (\Phi x)(t) = T(t)x_0 + \int_0^t [T(t-s)\sigma - AT(t-s)]Bu(s)ds + \int_0^t T(t-s)f_1(s,x(s))ds \\ + \int_0^t \int_0^s T(t-s)f_2(\eta,x(\eta))d\omega(\eta)ds + \int_0^t T(t-s)g(s,x(s))dZ_H(s), t \in J, \end{aligned}$$

where

$$\begin{aligned} u(t) = -(L_0)^{-1}\{T(b)x_0 + \int_0^b T(b-s)f_1(s,x(s))ds + \int_0^b \int_0^s T(b-s)f_2(\eta,x(\eta))d\omega(\eta)ds \\ + \int_0^b T(b-s)g(s,x(s))dZ_H(s)\}. \end{aligned}$$

It will be shown that the operator Φ from $\bar{C}$ into itself has a fixed point. We claim that there exists a positive number r such that $\Phi(B_r) \subseteq B_r$. If it is not true, then for each positive number r, there is a function $x_r(\cdot) \in B_r$, but $\Phi(x_r) \notin B_r$, that is $\|\Phi x_r\|_{\bar{C}}^2 > r$ for some $t = t(r) \in J$, where $t(r)$ denotes that t is dependent of r.

From our hypotheses, we obtain

$$\begin{aligned} r \leqslant \|\Phi x_r\|_{\bar{C}}^2 &= \sup_{t\in J} E\|\Phi(x_r)(t)\|^2 \\ &\leqslant 25M^2E\|x_0\|^2 + 25M^2b[M^2\|\sigma\|^2 + M_1^2]\|B\|^2\|(L_0)^{-1}\|^2 \\ &\quad \times [\|x_0\|_{\bar{C}}^2 + \int_0^b \rho_r(s)ds + \mathrm{Tr}(Q)\int_0^b h_r(s)ds + 2Hb^{2H-1}\int_0^b k_r(s)ds] \\ &\quad + 25M^2\int_0^t \rho_r(s)ds + 25\mathrm{Tr}(Q)M^2\int_0^t h_r(s)ds + 50Hb^{2H-1}M^2\int_0^t k_r(s)ds. \end{aligned} \tag{3.7}$$

Dividing both sides of (3.7) by r and taking the lower limit $r \to +\infty$, we get

$$25M^2 \left[\delta + Tr(Q)\delta_1 + 2Hb^{2H-1}\delta_2\right] \left[1 + [M^2\|\sigma\|^2 + M_1^2]\|B\|^2\|(L_0)^{-1}\|^2\right] \geqslant 1.$$

This contradicts (3.6). Hence, for positive r, $\Phi(B_r) \subseteq B_r$ for positive number r. In fact, the operator Φ maps B_r into a compact subset of B_r. To prove this, we first show that the set $V_r(t) = \{(\Phi x)(t) : x \in B_r\}$ is a precompact in X, for every fixed $t \in J$. This is trivial for $t = 0$, since $V_r(0) = \{x_0\}$. Let t, $0 < t \leqslant b$, be fixed. For $0 < \epsilon < t$, take

$$\begin{aligned}
(\Phi^\epsilon x)(t) &= T(t)x_0 + \int_0^{t-\epsilon} [T(t-s)\sigma - AT(t-s)]Bu(s)ds + \int_0^{t-\epsilon} T(t-s)f_1(s,x(s))ds \\
&\quad + \int_0^{t-\epsilon}\int_0^s T(t-s)f_2(\eta,x(\eta))d\omega(\eta)ds + \int_0^{t-\epsilon} T(t-s)g(s,x(s))dZ_H(s) \\
&= T(t)x_0 + T(\epsilon)\int_0^{t-\epsilon} [T(t-\epsilon-s)\sigma - AT(t-\epsilon-s)]Bu(s)ds + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)f_1(s,x(s))ds \\
&\quad + T(\epsilon)\int_0^{t-\epsilon}\int_0^s T(t-s)f_2(\eta,x(\eta))d\omega(\eta)ds + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)g(s,x(s))dZ_H(s).
\end{aligned}$$

Since $u(s)$ is bounded and $T(\epsilon)(\epsilon > 0)$ is a compact operator then the set $V_\epsilon(t) = \{(\Phi^\epsilon x)(t) : x \in B_r\}$ is precompact in X for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $x \in B_r$, we have

$$\begin{aligned}
\|\Phi x - \Phi^\epsilon x\|_{\tilde{C}}^2 &= \sup_{t\in J} E\|(\Phi x)(t) - (\Phi^\epsilon x)(t)\|^2 \\
&\leqslant 25M^2 \int_{t-\epsilon}^t [M^2\|\sigma\|^2 + M_1^2]\|B\|^2\|(L_0)^{-1}\|^2 \\
&\quad \times [E\|x_0\|^2 + \int_0^b \rho_r(s)ds + Tr(Q)\int_0^b h_r(s)ds + 2Hb^{2H-1}\int_0^b k_r(s)ds](s)ds \\
&\quad + 25M^2\int_{t-\epsilon}^t \rho_r(s)ds + 25Tr(Q)M^2\int_{t-\epsilon}^t h_r(s)ds + 50Hb^{2H-1}M^2\int_{t-\epsilon}^t k_r(s)ds.
\end{aligned}$$

We see that for each $x \in B_r$, $\|\Phi x - \Phi^\epsilon x\|_{\tilde{C}}^2 \to 0$ as $\epsilon \to 0^+$. Therefore, there are precompact sets arbitrary close to the set $V_r(t)$ and so $V_r(t)$ is precompact in X.

Next we prove that the family $\{\Phi x : x \in B_r\}$ is an equicontinuous family of functions. Let $x \in B_r$ and $t_1, t_2 \in J$ such that $0 < t_1 < t_2$, then

$$\begin{aligned}
&\|(\Phi x)(t_2) - (\Phi x)(t_1)\|_{\tilde{C}}^2 \\
&\leqslant 25\|T(t_2) - T(t_1)\|^2\|x_0\|_{\tilde{C}}^2 + 25\| \int_0^{t_1} [T(t_2-s)\sigma - AT(t_2-s) - T(t_1-s)\sigma + AT(t_1-s)] \\
&\quad \times -(L_0)^{-1}\{T(b)x_0 + \int_0^b T(b-s)f_1(s,x(s))ds + \int_0^b\int_0^\eta T(b-s)f_2(\eta,x(\eta))d\omega(\eta)ds \\
&\quad + \int_0^b T(b-s)g(s,x(s))dZ_H(s)\}(s)ds\|_{\tilde{C}}^2 + 25\| \int_{t_1}^{t_2} [T(t_2-s)\sigma - AT(t_2-s)] \\
&\quad \times -(L_0)^{-1}\{T(b)x_0 + \int_0^b T(b-s)f_1(s,x(s))ds + \int_0^b\int_0^\eta T(b-s)f_2(\eta,x(\eta))d\omega(\eta)ds \\
&\quad + \int_0^b T(b-s)g(s,x(s))dZ_H(s)\}(s)ds\|_{\tilde{C}}^2 \\
&\quad + 25\| \int_0^{t_1} [T(t_2-s) - T(t_1-s)] \int_0^s f_2(\eta,x(\eta))d\omega(\eta)ds\|_{\tilde{C}}^2
\end{aligned}$$

$$+25\|\int_{t_1}^{t_2} T(t_2-s)\int_0^s f_2(\eta,x(\eta))d\omega(\eta)ds\|_{\tilde{C}}^2$$
$$+25\|\int_0^{t_1} [T(t_2-s)-T(t_1-s)]g(s,x(s))dZ_H(s)\|_{\tilde{C}}^2$$
$$+25\|\int_{t_1}^{t_2} T(t_2-s)g(s,x(s))dZ_H(s)\|_{\tilde{C}}^2.$$

From the above fact, we see that $\|(\Phi x)(t_2)-(\Phi x)(t_1)\|_{\tilde{C}}^2$ tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Since the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, $\Phi(B_r)$ is both equicontinuous and bounded. By the Arzela-Ascoli theorem $\Phi(B_r)$ is precompact in X. Hence Φ is a completely continuous operator on X. From the Schauder fixed point theorem, Φ has a fixed point in B_r. Any fixed point of Φ is a mild solution of (3.1) on J. Therefore the system (3.1) is exact null controllable on J. □

4. Sobolev-type neutral integrodifferential system

In this section, we investigate the exact null controllability for Sobolev type neutral stochastic integrodifferential system with Rosenblatt process in the following form

$$\begin{cases} \frac{d}{dt}[Gx(t)+f_1(t,x(t))] = \sigma x(t)+\int_0^t f_2(s,x(s))d\omega(s)+g(t,x(t))\frac{dZ_H(t)}{dt}, & t\in J=[0,b],\\ \tau x(t)=B_1u(t), & t\in J=[0,b],\\ x(0)=x_0, \end{cases} \tag{4.1}$$

where $G : D(G) \subset X \to X$ is a linear operator. Let $y(t) = Gx(t)$ for $x \in X$, then the equation (4.1) can be written as

$$\begin{cases} \frac{d}{dt}[y(t)+f_1(t,G^{-1}y(t))] = \sigma G^{-1}y(t)+\int_0^t f_2(s,G^{-1}y(s))d\omega(s)\\ \quad +g(t,G^{-1}y(t))\frac{dZ_H(t)}{dt}, & t\in J=[0,b],\\ \tau G^{-1}y(t)=B_1u(t), & t\in J=[0,b],\\ y(0)=y_0, \end{cases} \tag{4.2}$$

where $\tau G^{-1} : X \to X$ is a linear operator. Let $AG^{-1} : X \to X$ be a linear operator defined by

$$D(AG^{-1})=\{\theta\in D(\sigma G^{-1}) : \tau G^{-1}\theta = 0\},\quad AG^{-1}\theta = \sigma G^{-1}\theta, \text{ for } \theta\in D(AG^{-1}).$$

The operators $A : D(A) \subset X \to X$ and $G : D(G) \subset X \to X$ satisfy the following hypotheses:

(H9) A and G are closed linear operators.
(H10) $D(G) \subset D(A)$ and G is bijective.
(H11) $G^{-1} : X \to D(G)$ is continuous.

Here, (H9) and (H10) together with the closed graph theorem imply the boundedness of the linear operator $AG^{-1} : X \to X$.

(H12) For each $t \in J$ and for $\lambda \in (\rho(AG^{-1}))$, the resolvent of AG^{-1}, the resolvent $R(\lambda, AG^{-1})$ is compact operator.

Lemma 4.1 ([28]). *Let $T(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.*

From the above fact, AG^{-1} generates a compact semigroup $\{S(t), t > 0\}$ in X, which means that there exists $M > 1$ such that $\sup_{t\in J} \|S(t)\| \leqslant M$. We suppose that $0 \in \rho(AG^{-1})$, the resolvent set of AZ^{-1}, for every $t > 0$. We define the fractional power $(AG^{-1})^{-\gamma}$ by

$$(AG^{-1})^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} S(t)dt, \quad \gamma > 0.$$

For $\gamma \in (0, 1]$, $(AG^{-1})^{\gamma}$ is a closed linear operator on its domain $D((AG^{-1})^{\gamma})$. Furthermore, the subspace $D((AG^{-1})^{\gamma})$ is dense in X. We will introduce the following basic properties of $(AG^{-1})^{\gamma}$.

Theorem 4.2 ([31]).

(1) *Let* $0 < \gamma \leqslant 1$, *then* $X_\gamma := D((AG^{-1})^{\gamma})$ *is a Banach space with the norm* $\| x \|_\gamma = \| (AG^{-1})^{\gamma} x \|$, $x \in X_\gamma$.
(2) *If* $0 < \beta < \gamma \leqslant 1$, *then* $D((AG^{-1})^{\gamma}) \hookrightarrow D((AG^{-1})^{\beta})$ *and the embedding is compact whenever the resolvent operator of* (AG^{-1}) *is compact.*
(3) *For every* $0 < \gamma \leqslant 1$, *there exists a positive constant* C_γ *such that*

$$\| (AG^{-1})^{\gamma} S(t) \| \leqslant \frac{C_\gamma}{t^{\gamma}}, \ 0 < t \leqslant b.$$

Let $y(t)$ be the solution of the system (4.2). Then we can define a function $z(t) = y(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(AG^{-1})$. Hence (4.2) can be written in terms of A and B as

$$\begin{cases} \frac{d}{dt}[y(t) + f_1(t, G^{-1}y(t))] = AG^{-1}z(t) + \sigma G^{-1}Bu(t) + \int_0^t f_2(s, G^{-1}y(s))d\omega(s) \\ \quad +g(t, G^{-1}y(t))\frac{dZ_H(t)}{dt}, \ t \in J, \\ y(t) = z(t) + Bu(t), \\ y(0) = y_0. \end{cases}$$

If u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to the Cauchy problem

$$\begin{cases} \frac{d}{dt}[z(t) + f_1(t, G^{-1}y(t))] = AG^{-1}z(t) + \sigma G^{-1}Bu(t) - B\frac{du}{dt} + \int_0^t f_2(s, G^{-1}y(s))d\omega(s) \\ \quad +g(t, G^{-1}y(t))\frac{dZ_H(t)}{dt}, \ t \in J, \\ z(0) = y(0) - Bu(0). \end{cases}$$

Then the mild solution of (4.2) is given by

$$\begin{aligned} y(t) = S(t)[y(0) + f_1(0, G^{-1}y(0)) - Bu(0)] - f_1(t, G^{-1}y(t)) + \int_0^t AG^{-1}S(t-s)f_1(s, G^{-1}y(s)) \\ + Bu(t) + \int_0^t S(t-s)[\sigma G^{-1}Bu(s) - B\frac{du(s)}{ds}]ds \\ + \int_0^t \int_0^s S(t-s)f_2(\eta, G^{-1}y(\eta))d\omega(\eta)ds + \int_0^t S(t-s)g(s, G^{-1}y(s))dZ_H(s). \end{aligned}$$

By using integration by parts, we obtain

$$\begin{aligned} y(t) = S(t)[y(0) + f_1(0, G^{-1}y(0))] - f_1(t, G^{-1}y(t)) + \int_0^t AG^{-1}S(t-s)f_1(s, G^{-1}y(s)) \\ + \int_0^t [S(t-s)\sigma G^{-1}B - AG^{-1}S(t-s)B]u(s)ds \\ + \int_0^t \int_0^s S(t-s)f_2(\eta, G^{-1}y(\eta))d\omega(\eta)ds + \int_0^t S(t-s)g(s, G^{-1}y(s))dZ_H(s). \end{aligned}$$

Hence the mild solution of system (4.1) is given by

$$\begin{aligned} x(t) &= G^{-1}S(t)[Gx(0) + f_1(0, x(0))] - f_1(t, x(t)) + \int_0^t G^{-1}AG^{-1}S(t-s)f_1(s, x(s)) \\ &+ \int_0^t G^{-1}[S(t-s)\sigma G^{-1}B - AG^{-1}S(t-s)B]u(s)ds \\ &+ \int_0^t \int_0^s S(t-s)G^{-1}f_2(\eta, x(\eta))d\omega(\eta)ds + \int_0^t G^{-1}S(t-s)g(s, x(s))dZ_H(s). \end{aligned}$$

To study the exact null controllability of (4.1), we consider the Soblev-type linear boundary control system

$$\begin{cases} \frac{d}{dt}[G\mu(t)] = \sigma\mu(t) + F(t), & t \in J = [0, b], \\ \tau\mu(t) = B_1 u(t), & t \in J = [0, b], \\ \mu(0) = \mu_0, \end{cases} \tag{4.3}$$

associated with the system (4.1). Consider

$$L_0^b u = \int_0^b G^{-1}[T(b-s)\sigma G^{-1}B - AG^{-1}T(b-s)B]u(s)ds \ : L_2(J, U) \to X,$$

where $L_0^b u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(J, U)/\ker(L_0^b)$, and

$$N_0^b(\mu, F) = G^{-1}T(b)G\mu + \int_0^b G^{-1}T(b-s)F(s)ds : X \times L_2(J, X) \to X.$$

Definition 4.3. The system (4.3) is said to be exactly null controllable on J if $\mathrm{Im}L_0^b \supset \mathrm{Im}N_0^b$ or there exists a $\gamma > 0$ such that $\|(L_0^b)^*\mu\|^2 \geqslant \gamma\|(N_0^b)^*\mu\|^2$ for all $\mu \in X$.

Lemma 4.4. *Suppose that the linear system* (4.3) *is exactly null controllable on* J. *Then the linear operator* $(L_0)^{-1}N_0^b : X \times L_2(J, X) \to L_2(J, U)$ *is bounded and the control*

$$u(t) = -(L_0)^{-1}\left[G^{-1}T(b)G\mu_0 + \int_0^b G^{-1}T(b-s)F(s)ds\right](t)$$

transferees the system (4.3) *from* μ_0 *to* 0, *where* L_0 *is the restriction of* L_0^b *to* $[\ker L_0^b]^\perp$ *and* $F \in L_2(J, X)$.

Definition 4.5. We say $x \in \bar{C}$ is a mild solution to (4.1) if it satisfies that

$$\begin{aligned} x(t) &= G^{-1}S(t)[Gx(0) + f_1(0, x(0))] - G^{-1}f_1(t, x(t)) + \int_0^t G^{-1}AG^{-1}S(t-s)f_1(s, x(s)) \\ &+ \int_0^t G^{-1}[S(t-s)\sigma G^{-1}B - AG^{-1}S(t-s)B]u(s)ds \\ &+ \int_0^t \int_0^s S(t-s)G^{-1}f_2(\eta, x(\eta))d\omega(\eta)ds + \int_0^t G^{-1}S(t-s)g(s, x(s))dZ_H(s). \end{aligned}$$

Definition 4.6. The system (4.1) is said to be exact null controllable on the interval J if there exists a stochastic control $u \in L_2(J, U)$ such that the solution $x(t)$ of the system (4.1) satisfies $x(b) = 0$.

To prove the main result in this section, we need addition hypotheses.

(H13) For all $t \in (0, b]$ and $u \in U$, $S(t)Bu \in D(AG^{-1})$. Moreover, there exists a positive constant $C > 0$ such that $\|AG^{-1})S(t)\| \leqslant C$.

(H14) The linear system (4.3) is exactly null controllable on J.

(H15) $f_1 : J \times X \to X$ is a continuous function, and there exists a constant $\frac{1}{2} < \beta < 1$ and $M_1,\ M_2\ > 0$ such that for all $t \in J$ and $x, y \in X$ the following inequalities are satisfied:

$$E \parallel (AG^{-1})^{\beta} f_1(t,x) - (AG^{-1})^{\beta} f_1(t,y) \parallel^2 \leqslant M_1 E \parallel x - y \parallel^2,$$
$$E \parallel (AG^{-1})^{\beta} f_1(t,x) \parallel^2 \leqslant M_2(1 + E \parallel x \parallel^2).$$

Theorem 4.7. *If the hypotheses* (H1), (H2), (H6), (H7), *and* (H9)-(H15) *are satisfied, then the boundary control system* (4.1) *is exactly null controllable on* J *provided that*

$$\begin{aligned} &\left[M_2[M_0^2 + \frac{C_{1-\beta}^2 b^{2\beta-1}\|G^{-1}\|^2}{2\beta-1}] + Tr(Q)M^2\|G^{-1}\|^2\delta_1 + 2HM^2\|G^{-1}\|^2 b^{2H-1}\delta_2\right] \\ &\quad \times 36\left[1 + \|G^{-1}\|^2 b[M^2\|\sigma\|^2\|G^{-1}\|^2 + C^2]\|B\|^2\|(L_0)^{-1}\|^2\right] \leqslant 1, \\ &M_0 = \|(AG^{-1})^{-\beta}\|. \end{aligned} \tag{4.4}$$

Proof. For an arbitrary $x(\cdot)$ define the operator Ψ on $\bar{C}$ as follows

$$\begin{aligned} (\Psi x)(t) &= G^{-1}S(t)[Gx(0) + f_1(0,x(0))] - f_1(t,x(t)) + \int_0^t G^{-1}AG^{-1}S(t-s)f_1(s,x(s))ds \\ &\quad + \int_0^t G^{-1}[S(t-s)\sigma G^{-1}B - AG^{-1}S(t-s)B]u(s)ds \\ &\quad + \int_0^t \int_0^s S(t-s)G^{-1}f_2(\eta,x(\eta))d\omega(\eta)ds + \int_0^t G^{-1}S(t-s)g(s,x(s))dZ_H(s),\ t \in J, \end{aligned}$$

where

$$\begin{aligned} u(t) &= -(L_0)^{-1}\{G^{-1}S(b)[Gx(0) + f_1(0,x(0))] - f_1(b,x(b)) + \int_0^b G^{-1}AG^{-1}S(b-s)f_1(s,x(s))ds \\ &\quad + \int_0^b \int_0^s S(b-s)G^{-1}f_2(\eta,x(\eta))d\omega(\eta)ds + \int_0^b G^{-1}S(b-s)g(s,x(s))dZ_H(s)\}. \end{aligned}$$

It will be shown that the operator Ψ from $\bar{C}$ into itself has a fixed point. We claim that there exists a positive number r such that $\Psi(B_r) \subseteq B_r$. If it is not true, then for each positive number r, there is a function $x_r(\cdot) \in B_r$, but $\Psi(x_r) \notin B_r$, that is $\|\Psi x_r\|_{\bar{C}}^2 > r$ for some $t = t(r) \in J$, where $t(r)$ denotes that t is dependent of r. From our hypotheses, we obtain

$$\begin{aligned} r &\leqslant \|\Psi x_r\|_{\bar{C}}^2 \\ &= \sup_{t\in J} E\|\Psi(x_r)(t)\|^2 \\ &\leqslant 36\|G^{-1}\|^2[M^2\|G\|^2 E\|x_0\|^2 + M_2 C_{-\beta}^2 b^{2\beta}(1 + E\|x_0\|^2)] + [M_0^2 + \frac{C_{1-\beta}^2 b^{2\beta-1}\|G^{-1}\|^2}{2\beta-1}]36M_2(1+r) \\ &\quad + 36\|G^{-1}\|^2 b[M^2\|\sigma\|^2\|G^{-1}\|^2 + C^2]\|B\|^2\|(L_0)^{-1}\|^2[\|G^{-1}\|^2[M^2\|G\|^2 E\|x_0\|^2 + M_2 C_{-\beta}^2 b^{2\beta}(1 + E\|x_0\|^2)] \\ &\quad + [M_0^2 + \frac{C_{1-\beta}^2 b^{2\beta-1}\|G^{-1}\|^2}{2\beta-1}]M_2(1+r) + Tr(Q)M^2\|G^{-1}\|^2\int_0^b h_r(s)ds + 2HM^2\|G^{-1}\|^2 b^{2H-1}\int_0^b k_r(s)ds] \\ &\quad + 36M^2\|G^{-1}\|^2 Tr(Q)\int_0^t h_r(s)ds + 72HM^2\|G^{-1}\|^2 b^{2H-1}\int_0^t k_r(s)ds. \end{aligned}$$

Dividing both sides of above equation by r and taking the lower limit $r \to +\infty$, we get

$$\left[M_2[M_0^2 + \frac{C_{1-\beta}^2 b^{2\beta-1}\|G^{-1}\|^2}{2\beta-1}] + Tr(Q)M^2\|G^{-1}\|^2\delta_1 + 2HM^2\|G^{-1}\|^2 b^{2H-1}\delta_2\right]$$

$$\times 36\left[1+\|G^{-1}\|^2 b[M^2\|\sigma\|^2\|G^{-1}\|^2+C^2]\|B\|^2\|(L_0)^{-1}\|^2\right]\geqslant 1.$$

This contradicts (4.4). Hence, for positive r, $\Psi(B_r)\subseteq B_r$ for positive number r. In fact, the operator Ψ maps B_r into a compact subset of B_r. To prove this, we first show that the set $V_r(t)=\{(\Psi x)(t):x\in B_r\}$ is a precompact in X, for every fixed $t\in J$. This is trivial for $t=0$, since $V_r(0)=\{x_0\}$. Let t, $0<t\leqslant b$, be fixed. For $0<\epsilon<t$, take

$$\begin{aligned}(\Psi^\epsilon x)(t)&=G^{-1}S(t)[Gx(0)+f_1(0,x(0))]-f_1(t,x(t))+\int_0^{t-\epsilon}G^{-1}AG^{-1}S(t-s)f_1(s,x(s))ds\\&+\int_0^{t-\epsilon}G^{-1}[S(t-s)\sigma G^{-1}B-AG^{-1}S(t-s)B]u(s)ds\\&+\int_0^{t-\epsilon}\int_0^{s}S(t-s)G^{-1}f_2(\eta,x(\eta))d\omega(\eta)ds+\int_0^{t-\epsilon}G^{-1}S(t-s)g(s,x(s))dZ_H(s),\ t\in J\\&=G^{-1}S(t)[Gx(0)+f_1(0,x(0))]-f_1(t,x(t))+S(\epsilon)\int_0^{t-\epsilon}G^{-1}AG^{-1}S(t-\epsilon-s)f_1(s,x(s))ds\\&+S(\epsilon)\int_0^{t-\epsilon}G^{-1}[S(t-\epsilon-s)\sigma G^{-1}B-AG^{-1}S(t-\epsilon-s)B]u(s)ds\\&+S(\epsilon)\int_0^{t-\epsilon}\int_0^{s}S(t-\epsilon-s)G^{-1}f_2(\eta,x(\eta))d\omega(\eta)ds\\&+S(\epsilon)\int_0^{t-\epsilon}G^{-1}S(t-\epsilon-s)g(s,x(s))dZ_H(s),\ t\in J.\end{aligned}$$

Since $u(s)$ is bounded and $S(\epsilon)(\epsilon>0)$ is a compact operator, then the set $V_\epsilon(t)=\{(\Psi^\epsilon x)(t):x\in B_r\}$ is precompact in X for every $\epsilon, 0<\epsilon<t$. Moreover, for every $x\in B_r$, we have

$$\begin{aligned}\|\Psi x-\Psi^\epsilon x\|_{\tilde{C}}^2&=\sup_{t\in J}E\|(\Psi x)(t)-(\Psi^\epsilon x)(t)\|^2\\&\leqslant 36\int_{t-\epsilon}^{t}G^{-1}AG^{-1}S(t-s)f_1(s,x(s))ds+36\int_{t-\epsilon}^{t}\|G^{-1}\|^2[M^2\|\sigma\|^2\|G^{-1}\|^2+C^2]\|B\|^2\|(L_0)^{-1}\|^2\\&\times\{\|G^{-1}\|^2[M^2\|G\|^2E\|x_0\|^2+M_2C_{-\beta}^2b^{2\beta}(1+E\|x_0\|^2)]+Tr(Q)\int_0^b h_r(s)ds\\&+2Hb^{2H-1}\int_0^b k_r(s)ds\}(s)ds+36Tr(Q)M^2\int_{t-\epsilon}^{t}h_r(s)ds+72Hb^{2H-1}M^2\int_{t-\epsilon}^{t}k_r(s)ds.\end{aligned}$$

We see that for each $x\in B_r$, $\|\Psi x-\Psi^\epsilon x\|_{\tilde{C}}^2\to 0$ as $\epsilon\to 0^+$. Therefore, there are precompact sets arbitrary close to the set $V_r(t)$ and so $V_r(t)$ is precompact in X.

Next we prove that the family $\{\Psi x:x\in B_r\}$ is an equicontinuous family of functions. Let $x\in B_r$ and $t_1,t_2\in J$ such that $0<t_1<t_2$, then

$$\begin{aligned}&\|(\Psi x)(t_2)-(\Psi x)(t_1)\|_{\tilde{C}}^2\\&\leqslant 36\|G^{-1}\|^2\|S(t_2)-S(t_1)\|^2\|Gx(0)+f_1(0,x(0))\|_{\tilde{C}}^2+36\|f(t_2,x(t_2))-f(t_1,x(t_1))\|_{\tilde{C}}^2\\&+36\|\int_0^{t_1}G^{-1}[S(t_2-s)\sigma G^{-1}-AG^{-1}S(t_2-s)-S(t_1-s)\sigma G^{-1}+AG^{-1}S(t_1-s)]\\&\times-(L_0)^{-1}\{T(b)x_0+\int_0^b T(b-s)f_1(s,x(s))ds+\int_0^b\int_0^\eta T(b-s)f_2(\eta,x(\eta))d\omega(\eta)ds\\&+\int_0^b T(b-s)g(s,x(s))dZ_H(s)\}(s)ds\|_{\tilde{C}}^2+25\|\int_{t_1}^{t_2}[T(t_2-s)\sigma-AT(t_2-s)]\\&\times-(L_0)^{-1}\{G^{-1}S(b)[Gx(0)+f_1(0,x(0))]-f_1(b,x(b))+\int_0^b G^{-1}AG^{-1}S(b-s)f_1(s,x(s))ds\end{aligned}$$

$$+\int_0^b\int_0^\eta S(b-s)G^{-1}f_2(\eta,x(\eta))d\omega(\eta)ds+\int_0^b S(b-s)G^{-1}g(s,x(s))dZ_H(s)\}(s)ds\|_{\bar{C}}^2$$
$$+36\|\int_0^{t_1}[S(t_2-s)-S(t_1-s)]G^{-1}\int_0^s f_2(\eta,x(\eta))d\omega(\eta)ds\|_{\bar{C}}^2$$
$$+36\|\int_{t_1}^{t_2}S(t_2-s)G^{-1}\int_0^s f_2(\eta,x(\eta))d\omega(\eta)ds\|_{\bar{C}}^2$$
$$+36\|\int_0^{t_1}[S(t_2-s)-S(t_1-s)]G^{-1}g(s,x(s))dZ_H(s)\|_{\bar{C}}^2+36\|\int_{t_1}^{t_2}S(t_2-s)G^{-1}g(s,x(s))dZ_H(s)\|_{\bar{C}}^2.$$

From the above fact, we see that $\|(\Psi x)(t_2)-(\Psi x)(t_1)\|_{\bar{C}}^2$ tends to zero independently of $x\in B_r$ as $t_2\to t_1$. Since the compactness of $S(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus, $\Psi(B_r)$ is both equicontinuous and bounded. By the Arzela-Ascoli theorem $\Psi(B_r)$ is precompact in X. Hence Ψ is a completely continuous operator on X. From the Schauder fixed point theorem, Ψ has a fixed point in B_r. Any fixed point of Φ is a mild solution of (4.1) on J. Therefore the system (4.1) is exact null controllable on J. □

5. Application

Consider the following boundary control stochastic integro-partial differential system with Rosenblatt process of the form,

$$\begin{cases}\frac{\partial x(t,y)}{\partial t}=\Delta x(t,y)+\cos(x(t,y))+\int_0^t\tan^{-1}(x(s,y))d\omega(s)+e^{-x(t,y)}\frac{dZ_H(t)}{dt}, & \text{in } \bar{Q}=(0,b)\times\Pi,\\ x(t,0)=u(t,0), & \text{on } \Sigma=(0,b)\times\Gamma,\ t\in J,\\ x(t,y)=0,\ \ x(0,y)=x_0(y), & \text{for } y\in\Pi,\end{cases}\tag{5.1}$$

where Π is a bounded and open subset of R^n with sufficiently smooth boundary Γ. Let $x_0\in L_2(\Pi)$, $\omega(t)$ is Wiener process, Z_H is a Rosenblatt process with parameter $H\in(\frac{1}{2},1)$ and $u\in L_2(\Sigma)$. The functions are $x(t)(y)=x(t,y)$, $f_1(t,x(t))(y)=\cos(x(t,y))$, $f_2(s,x(s))(y)=\tan^{-1}(x(s,y))$, and $g(t,x(t))(y)=e^{-x(t,y)}$. Let $X=Y=K=L_2(\Pi)$, $U=L_2(\Gamma)$ $B_1=I$, the identity operator and $\sigma x=\Delta x$ with domain $D(\sigma)=\{x\in L_2(\Pi):\Delta x\in L_2(\Pi)\}$.

The operator θ is the trace operator such that $\theta x=x|_\Gamma$ is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $x\in D(\sigma)$.

Define the operator $A:D(A)\subset X\to X$ is given by $Ax=\Delta x$ with domain $D(A)=H_0^1(\Omega)\cup H^2(\Omega)$, where $H^k(\Omega)$, $H^\beta(\Gamma)$ are usual Sobolev space on Ω, Γ. Then A can be written as

$$Ax=\sum_{n=1}^\infty n^2(x,x_n)x_n,\ \ z\in D(A),$$

where $x_n(y)=\sqrt{2}\sin ny$, $n=1,\ 2,\ 3,\ldots$, is the orthogonal set of eigenvectors of A. Furthermore, for $x\in X$

$$T(t)x=\sum_{n=1}^\infty e^{\frac{n^2}{1+n^2}t}(x,x_n)x_n.$$

We define the linear operator $B:L_2(\Gamma)\to L_2(\Omega)$ by $Bu=v_u$, where v_u is the unique solution to the Dirichlet boundary value problem,

$$\Delta v_u=0,\quad \text{in } \Omega,$$
$$v_u=u,\quad \text{in } \Gamma.$$

Choose b and other constants such that all the hypotheses of Theorem 3.5 are satisfied and

$$25M^2\left[\delta+Tr(Q)\delta_1+2Hb^{2H-1}\delta_2\right]\left[1+[M^2\|\sigma\|^2+M_1^2]\|B\|^2\|(L_0)^{-1}\|^2\right]<1,$$

so system (5.1) is exactly null controllable on J.

6. Conclusion

Stochastic differential systems with Rosenblatt process and control on the boundary are introduced. By using fractional calculus and stochastic analysis, the sufficient conditions for null boundary controllability of nonlinear integrodifferential system with Rosenblatt process and Sobolev type neutral stochastic integrodifferential system with Rosenblatt process are established. Moreover, we provided an example to illustrate our results.

For future work, we can present the null controllability of noninstantaneous impulsive fractional stochastic evolution inclusions.

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