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$\mathcal{SUP}$-Hesitant fuzzy ideals of Γ-semigroups

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Abstract

As a generalization of the concepts of interval-valued fuzzy ideals and hesitant fuzzy ideals of Γ-semigroups, the concept of $\mathcal{SUP}$-hesitant fuzzy ideals is introduced. Characterizations of $\mathcal{SUP}$-hesitant fuzzy ideals are discussed in terms of sets, fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, and hesitant fuzzy sets. Further, $\mathcal{SUP}$-hesitant fuzzy translations of $\mathcal{SUP}$-hesitant fuzzy ideals of Γ-semigroups are introduced and their related properties are investigated.

Keywords: Γ-semigroup, $\mathcal{SUP}$-hesitant fuzzy ideal, hesitant fuzzy ideal, interval-valued fuzzy ideal, $\mathcal{SUP}$-hesitant fuzzy translation.

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1. Introduction

The theory of fuzzy sets (FSs), introduced by Zadeh [20], has provided an important and useful mathematical tool for describing the behavior of the systems that are illdefined or too complex to admit precise mathematical analysis by classical methods and tools. However, it presents limitations to deal with imprecise and vague information when different sources of vagueness appear simultaneously. In order to overcome such limitations, Torra and Narukawa [17, 18] proposed a extension of FSs so-called a hesitant fuzzy set (HFS) which is a function from a reference set to a power set of the unit interval and a generalization of intuitionistic fuzzy sets (IFSs) and interval-valued fuzzy sets (IvFSs). The HFS theories developed by Torra and Narukawa, and others have found many applications in the domain of mathematics and elsewhere.

After introducing the concept of HFSs, several pieces of research were actualized on the generalizations of the concept of HFSs and application to many algebraic structures, such as in 2015, Jun et al. [7] characterized hesitant fuzzy left (right, generalized bi-, bi-, two-sided) ideals of semigroups. In UP-algebras, Senapati et al. [15] introduced the concept of interval-valued intuitionistic fuzzy UP-subalgebras (UP-ideals) in 2017. In ternary semigroups, Talee et al. [16] introduced hesitant fuzzy left (right, lateral, two-sided) ideals and characterized regular ternary semigroups by HFSs in 2018. In the same year, in

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Γ-semigroups, Abbasi et al. [1] introduced hesitant fuzzy left (right, two-sided, bi-, interior) ideals and characterized simple Γ-semigroups by HFSs. Later Mosrijai et al. [9] introduced a new concept derived from HFSs in UP-algebras, namely SUP-hesitant fuzzy UP-subalgebras (UP-filters, UP-ideals, strong UP-ideals). In 2019, Muhiuddin and Jun [11] introduced SUP-hesitant fuzzy subalgebras and their translations and extensions. In Γ-hemirings, Mandal [8] introduced and studied hesitant fuzzy h-ideals (h-bi-ideals, h-quasi-ideals) in 2020. In the same year, in BCK/BCI-algebras, Muhiuddin et al. [10] introduced SUP-hesitant fuzzy ideals. Harizavi and Jun [5] introduced SUP-hesitant fuzzy quasi-associative ideal in BCI-algebras. Later Dey et al. [4] developed the concept of hesitant multi-fuzzy sets by combining the hesitant fuzzy set with the multi-fuzzy set. In 2021, Jittburus and Julatha [6] proposed SUP-hesitant fuzzy ideals of semigroups and gave its characterizations in terms of sets, FSs, HFSs and IvFSs.

As previously stated, It motivated us to establish the concept of $\mathcal{SUP}$-hesitant fuzzy ideals of Γ-semigroups, which is the general concept of interval-valued fuzzy ideals and hesitant fuzzy ideals. Characterizations of $\mathcal{SUP}$-hesitant fuzzy ideals are investigated in terms of sets, FSs, IFSs, IvFSs, and HFSs. Further, we discuss the relation between ideals and generalizations of the characteristic interval-valued fuzzy sets and the characteristic hesitant fuzzy sets. Finally, $\mathcal{SUP}$-hesitant fuzzy translations of $\mathcal{SUP}$-hesitant fuzzy ideals of Γ-semigroups are introduced and their relations are investigated.

2. Preliminaries

In this section, the basic definitions and necessary results to be used in this paper are provided. First, we recall the definition of Γ-semigroups which is defined by Sen and Saha [13].

By a Γ-semigroup we mean a nonempty set M with a nonempty set Γ and a function $M \times \Gamma \times M \to M$, written as $(x, \gamma, y) \mapsto x\gamma y$ satisfying the identity

$$(x\alpha y)\beta z = x\alpha(y\beta z), \quad \forall x, y, z \in M, \ \forall \alpha, \beta \in \Gamma.$$

From now on throughout this paper, M is represented as a Γ-semigroup and X a nonempty set unless otherwise specified.

For nonempty subsets X, Y of M, let

$$X\Gamma Y = \{x\alpha y \mid x \in X, y \in Y, \gamma \in \Gamma\}.$$

If $s \in M$ and $\gamma \in \Gamma$, we let $X\Gamma s := X\Gamma\{s\}$, $s\Gamma X := \{s\}\Gamma X$, and $X\gamma Y := X\{\gamma\}Y$.

By a left (right) ideal of M we mean a nonempty subset I of M such that $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$). A nonempty subset of M is called an ideal of M if it is both a left and a right ideal of M. Then a nonempty subset I of M is an ideal of M if and only if

$$x\gamma a, a\gamma x \in I, \quad \forall x \in M, \ \forall a \in I, \ \forall \gamma \in \Gamma.$$

A fuzzy set (FS) f [20] in X (or a fuzzy subset of X) is an arbitrary function from X into the unit segment of the real line [0, 1]. A FS f in M is called a fuzzy ideal (FI) of M if

$$\max\{f(x), f(y)\} \leqslant f(x\gamma y), \ \ \forall x, y \in M, \ \forall \gamma \in \Gamma.$$

An intuitionistic fuzzy set (IFS) A [3] in X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and

$$\nu_A : X \to [0, 1],$$

define the degree of membership and the degree of nonmembership of an element $x \in X$ to the set A, which is a subset of X, respectively, and $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1 \ \forall x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for an IFS $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$. An IFS

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in M\} \in M,$$

can be identified to an ordered pair (μ_A, ν_A) in $[0,1]^M \times [0,1]^M$. An IFS $A = (\mu_A, \nu_A)$ in M is called an intuitionistic fuzzy ideal (IFI) [19] of M if (IFI1) $\mu_A(x\gamma y) \geqslant \max\{\mu_A(x), \mu_A(y)\}$, $\forall x, y \in M, \forall \gamma \in \Gamma$ and (IFI2) $\nu_A(x\gamma y) \leqslant \min\{\nu_A(x), \nu_A(y)\}$, $\forall x, y \in M, \forall \gamma \in \Gamma$.

By an interval number $\bar{a}$ we mean an interval $[a^-, a^+]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Especially, we denoted $\bar{1} := [1,1]$ and $\bar{0} := [0,0]$. The set of all interval numbers is denoted by $\mathcal{D}[0,1]$. For any

$$\bar{a} = [a^-, a^+], \bar{b} = [b^-,\ b^+] \in \mathcal{D}[0,1],$$

define the relations $\preceq, =, \prec$ and the operation rmax on $\mathcal{D}[0,1]$ as follows:

(1) $\bar{a} \preceq \bar{b} \Leftrightarrow a^- \leqslant b^-$ and $a^+ \leqslant b^+$;

(2) $\bar{a} = \bar{b} \Leftrightarrow a^- = b^-$ and $a^+ = b^+$;

(3) $\bar{a} \prec \bar{b} \Leftrightarrow \bar{a} \preceq \bar{b}$ and $\bar{a} \neq \bar{b}$;

(4) $\mathrm{rmax}\{\bar{a}, \bar{b}\} = [\max\{a^-, b^-\},\ \max\{a^+, b^+\}]$.

A function $\tilde{A} : X \to \mathcal{D}[0,1]$ is called an interval-valued fuzzy set (IvFS) [21] on X, where

$$\tilde{A}(x) = [A^-(x), A^+(x)], \quad \forall x \in X,$$

and A^- and A^+ are FSs in X such that $A^-(x) \leqslant A^+(x)$, $\forall x \in X$. An IvFS $\tilde{A}$ on M is called an interval-valued fuzzy ideal (IvFI) of M if $\mathrm{rmax}\{\tilde{A}(x), \tilde{A}(y)\} \preceq \tilde{A}(x\gamma y)$, $\forall x, y \in M, \forall \gamma \in \Gamma$. Then $\tilde{A}$ is an IvFI of M if and only if $\tilde{A}(y) \preceq \tilde{A}(x\gamma y)$ and $\tilde{A}(y) \preceq \tilde{A}(y\gamma x)$ $\forall x, y \in M, \forall \gamma \in \Gamma$. Torra [17, 18] introduced a hesitant fuzzy set (HFS) on X in terms of a function h that when applied to X returns a subset of $[0,1]$, that is, $h : X \to \mathcal{P}[0,1]$, where $\mathcal{P}[0,1]$ denote the set of all subset of $[0,1]$. It is well known that the concept of a HFS on X is a generalization of an IvFS on X. A HFS h on M is called a hesitant fuzzy ideal (HFI) [1] of M if $h(x) \cup h(y) \subseteq h(x\gamma y)$, $\forall x, y \in M, \forall \gamma \in \Gamma$. Then h is a HFI of M if and only if $h(y) \subseteq h(x\gamma y) \cap h(y\gamma x)$, $\forall x, y \in M, \forall \gamma \in \Gamma$.

3. $\mathcal{SUP}$-hesitant fuzzy ideals

In this section, the concepts of $\mathcal{SUP}$-hesitant fuzzy ideals of M and characterize $\mathcal{SUP}$-hesitant fuzzy ideals of M by sets, FSs, IFSs, IvFSs and HFSs are introduced and their related properties are studied.

For any HFS h on X and $\Theta \in \mathcal{P}[0,1]$, define $\mathcal{SUP}\,\Theta$ and $\mathcal{S}[h;\Theta]$ [6] by

$$\mathcal{SUP}\,\Theta = \begin{cases} \sup\Theta & \text{if } \Theta \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{S}[h;\Theta] = \{x \in X \mid \mathcal{SUP}\,h(x) \geqslant \mathcal{SUP}\,\Theta\}.$$

Then the following assertions are true:

(1) For every IvFS $\tilde{A}$ on X, $\mathcal{SUP}\,\tilde{A}(x) = \sup\tilde{A}(x) = A^+(x)$, $\forall x \in X$;

(2) If $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\Theta \subseteq \Psi$, then $\mathcal{SUP}\,\Theta \leqslant \mathcal{SUP}\,\Psi$ and $\mathcal{S}[h;\Psi] \subseteq \mathcal{S}[h;\Theta]$.

Definition 3.1. A HFS h on M is called a $\mathcal{SUP}$-hesitant fuzzy ideal of M related to Θ (Θ-$\mathcal{SUP}$-HFI) if the set $\mathcal{S}[h;\Theta]$ is an ideal of M. We say that h is a $\mathcal{SUP}$-hesitant fuzzy ideal ($\mathcal{SUP}$-HFI) of M if h is a Θ-$\mathcal{SUP}$-HFI of M, $\forall\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[h;\Theta] \neq \emptyset$.

Proposition 3.2. *If $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\,\Theta = \mathcal{SUP}\,\Psi$ and h is a Θ-$\mathcal{SUP}$-HFI of M, then h is a Ψ-$\mathcal{SUP}$-HFI of M.*

Proof. Straightforward. □

Lemma 3.3. *Every IvFI of* M *is a* $\mathcal{SUP}$*-HFI of* M.

Proof. Assume that $\tilde{A}$ is an IvFI of M and let $\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[\tilde{A};\Theta] \neq \emptyset$. Let $x \in M, \gamma \in \Gamma$, and $y \in \mathcal{S}[\tilde{A};\Theta]$. Then $\sup \tilde{A}(y) \geqslant \mathcal{SUP}\,\Theta$. By assumption, we have $\tilde{A}(y) \preceq \tilde{A}(x\gamma y)$ and $\tilde{A}(y) \preceq \tilde{A}(y\gamma x)$. Thus $\mathcal{SUP}\,\Theta \leqslant \sup \tilde{A}(y) = A^+(y) \leqslant A^+(x\gamma y) = \sup \tilde{A}(x\gamma y)$, which implies that $x\gamma y \in \mathcal{S}[\tilde{A};\Theta]$. Similarly, we have $y\gamma x \in \mathcal{S}[\tilde{A};\Theta]$. Hence, $\mathcal{S}[\tilde{A};\Theta]$ is an ideal of M and so $\tilde{A}$ is a Θ-$\mathcal{SUP}$-HFI of M. Therefore, $\tilde{A}$ is a $\mathcal{SUP}$-HFI of M. □

Lemma 3.4. *Every HFI of* M *is a* $\mathcal{SUP}$*-HFI of* M.

Proof. Assume that h is a HFI of M and let $\Theta \in \mathcal{P}[0,1]$ with $\mathcal{S}[h;\Theta] \neq \emptyset$. Let $x \in M, \gamma \in \Gamma$, and $y \in \mathcal{S}[h;\Theta]$. Then $\mathcal{SUP}\,h(y) \geqslant \mathcal{SUP}\,\Theta$. By assumption, we have $h(y) \subseteq h(x\gamma y)$ and $h(y) \subseteq h(y\gamma x)$. Thus

$$\mathcal{SUP}\,h(y) \leqslant \mathcal{SUP}\,h(x\gamma y),$$

and $\mathcal{SUP}\,h(y) \leqslant \mathcal{SUP}\,h(y\gamma x)$, which imply that $x\gamma y, y\gamma x \in \mathcal{S}[h;\Theta]$. Hence, $\mathcal{S}[h;\Theta]$ is an ideal of M and then h is a Θ-$\mathcal{SUP}$-HFI of M. Therefore, h is a $\mathcal{SUP}$-HFI of M. □

The following example is shown that the converses of Lemma 3.3 and Lemma 3.4 do not hold in general.

Example 3.5. Let $\mathbb{Z}^-$ be the set of all negative integers, $M = \mathbb{Z}^- \cup \{0\}$ and $\Gamma = 2M$. Then M is a Γ-semigroup with respect to usual multiplication.

(1) Define a HFS h on M by $\forall x \in M$,

$$h(x) = \begin{cases} [l]\{0,1\} & \text{if } x = 0, \\ \{0.1, 0.2, 0.3\} & \text{if } x \in \{-1,-2\}, \\ [0, 0.3] & \text{otherwise.} \end{cases}$$

Then h is a $\mathcal{SUP}$-HFI of M but not a HFI of M because

$$h(-1) \cup h(-4) = [0, 0.3] \not\subset \{0,1\} = h(0) = h((-1)(0)(-4)).$$

(2) Define an IvFS $\tilde{A}$ on M by $\forall x \in M$,

$$\tilde{A}(x) = \begin{cases} [0,1] & \text{if } x = 0, \\ \bar{1} & \text{if } x \in 2\mathbb{Z}^-, \\ \bar{0} & \text{otherwise.} \end{cases}$$

Then $\tilde{A}$ is a $\mathcal{SUP}$-HFI of M but not an IvFI of M because

$$\tilde{A}((-1)(0)(-2)) = \tilde{A}(0) = [0,1] \prec \bar{1} = \mathrm{rmax}\{\bar{0}, \bar{1}\} = \mathrm{rmax}\{\tilde{A}(-1), \tilde{A}(-2)\}.$$

By Lemmas 3.3 and 3.4, and Example 3.5, we have that a $\mathcal{SUP}$-HFI of M is a generalized concept of a hesitant and an IvFI of M.

For every HFS h on X and $\Theta \in \mathcal{P}[0,1]$, we define the HFS $\mathcal{H}(h;\Theta)$ [6] on X by $\forall x \in X$,

$$\mathcal{H}(h;\Theta)(x) = \{t \in \Theta \mid \mathcal{SUP}\,h(x) \geqslant t\}.$$

We denote $\mathcal{H}(h; \bigcup_{x \in X} h(x))$ by $\mathcal{H}_h$ and denote $\mathcal{H}(h;[0,1])$ by $\mathcal{I}_h$. Then the following assertions are true:

(1) $\mathcal{I}_h$ is an IvFS on X;

(2) $h(x) \subseteq \mathcal{H}_h(x) \subseteq \mathcal{I}_h(x), \forall x \in X$;

(3) $\mathcal{SUP}\, h(x) = \mathcal{SUP}\, \mathcal{H}_h(x) = \sup \mathcal{I}_h(x), \forall x \in X$;

(4) $\mathcal{H}(h; \Theta)(x) \subseteq \Theta, \forall x \in X$;

(5) For all $x \in X$, we have $\mathcal{H}(h; \Theta)(x) = \Theta$ if and only if $x \in \mathcal{S}[h; \Theta]$.

Next, we study a $\mathcal{SUP}$-HFI h of a Γ-semigroup via the HFS $\mathcal{H}(h; \Theta)$.

Lemma 3.6. *A HFS h on M is a $\mathcal{SUP}$-HFI of M if and only if $\mathcal{H}(h; \Theta)$ is a HFI of M, $\forall \Theta \in \mathcal{P}[0, 1]$.*

Proof. Let $\Theta \in \mathcal{P}[0, 1], \gamma \in \Gamma$, and $x, y \in M$. Suppose that $t \in \mathcal{H}(h; \Theta)(x) \cup \mathcal{H}(h; \Theta)(y)$. Then $t \in \mathcal{H}(h; \Theta)(x)$ or $t \in \mathcal{H}(h; \Theta)(y)$, which implies that

$$\mathcal{SUP}\,(h(x) \cup h(y)) = \max\{\mathcal{SUP}\, h(x), \mathcal{SUP}\, h(y)\} \geqslant t \in \Theta.$$

Thus $x \in \mathcal{S}[h; h(x) \cup h(y)]$ or $y \in \mathcal{S}[h; h(x) \cup h(y)]$. Since h is a $\mathcal{SUP}$-HFI of M, we have

$$x\gamma y \in \mathcal{S}[h; h(x) \cup h(y)].$$

This implies that

$$\mathcal{SUP}\, h(x\gamma y) \geqslant \mathcal{SUP}\,(h(x) \cup h(y)) \geqslant t \in \Theta.$$

Thus $t \in \mathcal{H}(h; \Theta)(x\gamma y)$. Therefore, $\mathcal{H}(h; \Theta)(x) \cup \mathcal{H}(h; \Theta)(y) \subseteq \mathcal{H}(h; \Theta)(x\gamma y)$. Consequently, $\mathcal{H}(h; \Theta)$ is a hesitant fuzzy ideal of M.

Conversely, let $\Theta \in \mathcal{P}[0, 1], x \in M, a \in \mathcal{S}[h; \Theta]$, and $\gamma \in \Gamma$. Then $\mathcal{H}(h; \Theta)(a) = \Theta$ and by assumption, we get

$$\Theta = \mathcal{H}(h; \Theta)(a) \subseteq \mathcal{H}(h; \Theta)(x) \cup \mathcal{H}(h; \Theta)(a) \subseteq \mathcal{H}(h; \Theta)(x\gamma a),$$

and so $\Theta \subseteq \mathcal{H}(h; \Theta)(x\gamma a)$. Similarly, we have $\Theta \subseteq \mathcal{H}(h; \Theta)(a\gamma x)$. Hence, $\mathcal{SUP}\, h(x\gamma a) \geqslant \mathcal{SUP}\, \Theta$ and $\mathcal{SUP}\, h(a\gamma x) \geqslant \mathcal{SUP}\, \Theta$, which imply that $x\gamma a, a\gamma x \in \mathcal{S}[h; \Theta]$. Therefore, $\mathcal{S}[h; \Theta]$ is an ideal of M, that is, h is a Θ-$\mathcal{SUP}$-HFI of M. Consequently, h is a $\mathcal{SUP}$-HFI of M. □

The following theorem, some characterizations of $\mathcal{SUP}$-HFIs of Γ-semigroups are investigated in terms of IvFSs and HFSs.

Theorem 3.7. *For any HFS h on M, the following assertions are equivalent.*

(1) *h is a $\mathcal{SUP}$-HFI of M.*

(2) *$\mathcal{H}_h$ is a HFI of M.*

(3) *$\mathcal{H}_h$ is a $\mathcal{SUP}$-HFI of M.*

(4) *$\mathcal{I}_h$ is an IvFI of M.*

(5) *$\mathcal{I}_h$ is a $\mathcal{SUP}$-HFI of M.*

(6) *$\mathcal{I}_h$ is a HFI of M.*

Proof.
(1) ⇒ (2) and (1) ⇒ (6). The proof is given by Lemma 3.6.
(2) ⇒ (3) and (6) ⇒ (5). The proof is given by Lemma 3.4.
(4) ⇒ (5). The proof is given by Lemma 3.3.
(3) ⇒ (1). Let $\Theta \in \mathcal{P}[0, 1], x \in M, a \in \mathcal{S}[h; \Theta]$, and $\gamma \in \Gamma$. Then $\mathcal{SUP}\, \mathcal{H}_h(a) = \mathcal{SUP}\, h(a) \geqslant \mathcal{SUP}\, \Theta$, that is, $a \in \mathcal{S}[\mathcal{H}_h; \Theta]$. By assumption (3), we have $\mathcal{S}[\mathcal{H}_h; \Theta]$ is an ideal of M and then $x\gamma a, a\gamma x \in \mathcal{S}[\mathcal{H}_h; \Theta]$.

Thus $\mathcal{SUP}\, h(x\gamma a) = \mathcal{SUP}\, \mathcal{H}_h(x\gamma a) \geqslant \mathcal{SUP}\, \Theta$ and $\mathcal{SUP}\, h(a\gamma x) = \mathcal{SUP}\, \mathcal{H}_h(a\gamma x) \geqslant \mathcal{SUP}\, \Theta$, which implies that $x\gamma a, a\gamma x \in \mathcal{S}[h;\Theta]$. Hence, $\mathcal{S}[h;\Theta]$ is an ideal of M. Therefore, h is a $\mathcal{SUP}$-HFI of M.
(1) ⇒ (4). Let $x, y \in M$ and $\gamma \in \Gamma$. Then $y \in \mathcal{S}[h; h(y)]$ and by assumption (1), we have

$$x\gamma y, y\gamma x \in \mathcal{S}[h; h(y)].$$

Thus $\mathcal{SUP}\, h(y) \leqslant \mathcal{SUP}\, h(x\gamma y)$ and $\mathcal{SUP}\, h(y) \leqslant \mathcal{SUP}\, h(y\gamma x)$. Hence,

$$\mathcal{I}_h(y) = [0, \mathcal{SUP}\, h(y)] \preceq [0, \mathcal{SUP}\, h(x\gamma y)] = \mathcal{I}_h(x\gamma y),$$

and so $\mathcal{I}_h(y) \preceq \mathcal{I}_h(x\gamma y)$. In a similar way, we can prove that $\mathcal{I}_h(y) \preceq \mathcal{I}_h(y\gamma x)$. Hence, $\mathcal{I}_h$ is an IvFI of M.
(5) ⇒ (1). It is same as proving that (3) implies (1). □

For every HFS h on X, define the FS $\mathcal{F}_h$ [9] in X by $\mathcal{F}_h(x) = \mathcal{SUP}\, h(x), \forall x \in X$. Now, the following lemma, a characterization of a $\mathcal{SUP}$-HFI h of a Γ-semigroup is discussed by the FS $\mathcal{F}_h$.

Lemma 3.8. *A HFS h on M is a $\mathcal{SUP}$-HFI of M if and only if $\mathcal{F}_h$ is a fuzzy ideal of M.*

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Then $h(x) \cup h(y) = \Theta$ for some $\Theta \in \mathcal{P}[0,1]$. Thus $x \in \mathcal{S}[h;\Theta]$ or $y \in \mathcal{S}[h;\Theta]$. By assumption, we have $x\gamma y \in \mathcal{S}[h;\Theta]$. Hence,

$$\mathcal{F}_h(x\gamma y) = \mathcal{SUP}\, h(x\gamma y) \geqslant \mathcal{SUP}\, \Theta = \mathcal{SUP}\, (h(x) \cup h(y)) = \max\{\mathcal{SUP}\, h(x), \mathcal{SUP}\, h(y)\} = \max\{\mathcal{F}_h(x), \mathcal{F}_h(y)\}.$$

Therefore, $\mathcal{F}_h$ is a fuzzy ideal of M.
Conversely, let $\Theta \in \mathcal{P}[0,1], a \in \mathcal{S}[h;\Theta], x \in M$, and $\gamma \in \Gamma$. Then

$$\mathcal{SUP}\, h(x\gamma a) = \mathcal{F}_h(x\gamma a) \geqslant \mathcal{F}_h(a) = \mathcal{SUP}\, h(a) \geqslant \mathcal{SUP}\, \Theta,$$

which implies that $x\gamma a \in \mathcal{S}[h;\Theta]$. Similarly, we have $a\gamma x \in \mathcal{S}[h;\Theta]$. Hence, $\mathcal{S}[h;\Theta]$ is an ideal of M, that is, h is a Θ-$\mathcal{SUP}$-HFI of M. Therefore, h is a $\mathcal{SUP}$-HFI of M. □

Theorem 3.9. *A HFS h on M is a $\mathcal{SUP}$-HFI of M if and only if*

$$\mathcal{SUP}\, h(x\gamma y) \geqslant \max\{\mathcal{SUP}\, h(x), \mathcal{SUP}\, h(y)\}, \quad \forall x, y \in M, \quad \forall \gamma \in \Gamma$$

Proof. The proof is given by Lemma 3.8. □

For any IFS $A = (\mu_A, \nu_A)$ on X and $\Theta \in \mathcal{P}[0,1]$, we define the HFS H_A^{Θ} on X and the IvFS I_A in X by $\forall x \in X$,

$$H_A^{\Theta}(x) = \left\{ t \in \Theta \,\middle|\, \frac{\nu_A(x)}{2} \leqslant t \leqslant \frac{1 + \mu_A(x)}{2} \right\},$$

and

$$I_A(x) = \left[\frac{1 - \nu_A(x)}{2}, \frac{1 + \mu_A(x)}{2} \right].$$

Theorem 3.10. *Let $A = (\mu_A, \nu_A)$ be an IFS in M. Then the following assertions are equivalent.*

(1) *A is an IFI of M.*

(2) *H_A^{Θ} is a HFI of M, $\forall \Theta \in \mathcal{P}[0,1]$.*

(3) *I_A is an IvFI of M.*

Proof.
(1) $\Rightarrow$ (2) Assume that A is an IFI of M and $\Theta \in \mathcal{P}[0,1]$. Let $x, y \in M, \gamma \in \Gamma$, and $t \in H_A^{\Theta}(x) \cup H_A^{\Theta}(y)$. If $t \in H_A^{\Theta}(x)$, then $t \in \Theta$ and $\frac{\nu_A(x)}{2} \leqslant t \leqslant \frac{1+\mu_A(x)}{2}$. By assumption, we have

$$\frac{\nu_A(x\gamma y)}{2} \leqslant \frac{\min\{\nu_A(x), \nu_A(y)\}}{2} \leqslant \frac{\nu_A(x)}{2} \leqslant t \leqslant \frac{1+\mu_A(x)}{2} \leqslant \max\{\frac{1+\mu_A(x)}{2}, \frac{1+\mu_A(y)}{2}\}$$
$$= \frac{1+\max\{\mu_A(x), \mu_A(y)\}}{2} \leqslant \frac{1+\mu_A(x\gamma y)}{2}.$$

Hence, $t \in H_A^{\Theta}(x\gamma y)$. In the case that $t \in H_A^{\Theta}(y)$ can be proven that $t \in H_A^{\Theta}(x\gamma y)$. Therefore,

$$H_A^{\Theta}(x) \cup H_A^{\Theta}(y) \subseteq H_A^{\Theta}(x\gamma y).$$

Consequently, H_A^{Θ} is a HFI of M.
(2) $\Rightarrow$ (1) If the condition (IFI1) is false, then there are $x, y \in M$ and $\gamma \in \Gamma$ such that

$$\mu_A(x\gamma y) < \max\{\mu_A(x), \mu_A(y)\}.$$

Taking

$$t = \frac{1}{4}\left(\mu_A(x\gamma y) + \max\{\mu_A(x), \mu_A(y)\}\right),$$

we have $\frac{1}{2} + t \in [0,1]$ and

$$\frac{\mu_A(x\gamma y)}{2} < t < \frac{\max\{\mu_A(x), \mu_A(y)\}}{2}.$$

Then

$$\frac{\max\{\nu_A(x), \nu_A(y)\}}{2} \leqslant \frac{1}{2} < \frac{1}{2} + t < \frac{1+\max\{\mu_A(x), \mu_A(y)\}}{2},$$

which implies $\frac{1}{2} + t \in H_A^{[0,1]}(x)$ or $\frac{1}{2} + t \in H_A^{[0,1]}(y)$. By assumption (2), we have $H_A^{[0,1]}$ is a HFI of M and so $\frac{1}{2} + t \in H_A^{[0,1]}(x\gamma y)$. Hence, we have $\frac{1}{2} + t \leqslant \frac{1+\mu_A(x\gamma y)}{2}$ and then

$$\mu_A(x\gamma y) = 2\left(\frac{1+\mu_A(x\gamma y)}{2}\right) - 1 \geqslant 2\left(\frac{1}{2} + t\right) - 1 = 2t > \mu_A(x\gamma y),$$

it is a contradiction. Therefore, the condition (IFI1) is true. The proof of the condition (IFI2) is similar to the case (IFI1), we omit the proof.
(1) $\Rightarrow$ (3) Let $x, y \in M$ and $\gamma \in \Gamma$. By using assumption (1), we have

$$\frac{1-\nu_A(x\gamma y)}{2} \geqslant \frac{1-\min\{\nu_A(x), \nu_A(y)\}}{2} = \max\left\{\frac{1-\nu_A(x)}{2}, \frac{1-\nu_A(y)}{2}\right\},$$

and

$$\frac{1+\mu_A(x\gamma y)}{2} \geqslant \frac{1+\max\{\mu_A(x), \mu_A(y)\}}{2} = \max\left\{\frac{1+\mu_A(x)}{2}, \frac{1+\mu_A(y)}{2}\right\}.$$

Hence, we have $\mathrm{rmax}\{I_A(x), I_A(y)\} \preceq I_A(x\gamma y)$. Therefore, I_A is an IvFI of M.
(3) $\Rightarrow$ (1) Let $x, y \in M$ and $\gamma \in \Gamma$. By assumption (3), we have $\mathrm{rmax}\{I_A(x), I_A(y)\} \preceq I_A(x\gamma y)$. Then

$$\frac{1-\nu_A(x\gamma y)}{2} \geqslant \frac{1-\min\{\nu_A(x), \nu_A(y)\}}{2},$$

and

$$\frac{1+\mu_A(x\gamma y)}{2} \geqslant \frac{1+\max\{\mu_A(x), \mu_A(y)\}}{2}.$$

Thus $\nu_A(x\gamma y) \leqslant \min\{\nu_A(x), \nu_A(y)\}$ and $\mu_A(x\gamma y) \geqslant \max\{\mu_A(x), \mu_A(y)\}$. Therefore, A is an IFI of M. □

From Lemmas 3.3 and 3.4, and Theorem 3.10, we get Corollary 3.11.

Corollary 3.11. *Let $A = (\mu_A, \nu_A)$ be an IFI of M. Then the following assertions are true.*

(1) H_A^{Θ} *is a* $\mathcal{SUP}$*-HFI of* M, $\forall\Theta \in \mathcal{P}[0,1]$.

(2) I_A *is a* $\mathcal{SUP}$*-HFI of* M.

For every HFS h on X, the HFS h^*, defined by $h^*(x) = \{1 - \mathcal{SUP}\, h(x)\}$, $\forall x \in X$, is said to be the supremum complement [9] of h on X. Then $\mathcal{SUP}\, h^*(x) = 1 - \mathcal{SUP}\, h(x)$, $\forall x \in X$ and it is clear that (F_h, F_{h^*}) is an IFS in X.

Theorem 3.12. *A HFS h on M is a $\mathcal{SUP}$-HFI of M if and only if $(\mathcal{F}_h, \mathcal{F}_{h^*})$ is an IFI of M.*

Proof. Let h be a $\mathcal{SUP}$-HFI of M. By Lemma 3.8, we have

$$\mathcal{SUP}\, h(x\gamma y) = \mathcal{F}_h(x\gamma y) \geqslant \max\{\mathcal{F}_h(x), \mathcal{F}_h(y)\} = \max\{\mathcal{SUP}\, h(x), \mathcal{SUP}\, h(y)\},$$

and then

$$\begin{aligned} \mathcal{F}_{h^*}(x\gamma y) &= 1 - \mathcal{SUP}\, h(x\gamma y) \\ &\leqslant 1 - \max\{\mathcal{SUP}\, h(x), \mathcal{SUP}\, h(y)\} \\ &= \min\{1 - \mathcal{SUP}\, h(x), 1 - \mathcal{SUP}\, h(y)\} \\ &= \min\{\mathcal{F}_{h^*}(x), \mathcal{F}_{h^*}(y)\}. \end{aligned}$$

Hence, $(\mathcal{F}_h, \mathcal{F}_{h^*})$ is an IFI of M.

Conversely, suppose that $(\mathcal{F}_h, \mathcal{F}_{h^*})$ is an IFI of M. Then $\mathcal{F}_h$ is a FI of M. From Lemma 3.8, it can be seen that h is a $\mathcal{SUP}$-HFI of M. □

For every HFS h on X and $t \in [0,1]$, the sets

$$U_{\mathcal{SUP}}(h; t) = \{x \in X \mid \mathcal{SUP}\, h(x) \geqslant t\}, \quad \text{and} \quad L_{\mathcal{SUP}}(h; t) = \{x \in X \mid \mathcal{SUP}\, h(x) \leqslant t\},$$

are called a $\mathcal{SUP}$-upper t-level subset and a $\mathcal{SUP}$-lower t-level subset [9] of h, respectively.

Theorem 3.13. *A HFS h on M is a $\mathcal{SUP}$-HFI of M if and only if $U_{\mathcal{SUP}}(h; t)$ is either empty or an ideal of M, $\forall t \in [0,1]$.*

Proof. Suppose that h is a $\mathcal{SUP}$-HFI of M. Let $t \in [0,1]$ be such that $U_{\mathcal{SUP}}(h;t) \neq \emptyset$. Choose $\Theta = \{t\}$, we have $\mathcal{S}[h;\Theta] = U_{\mathcal{SUP}}(h;t) \neq \emptyset$. By assumption, we obtain that $U_{\mathcal{SUP}}(h;t) = \mathcal{S}[h;\Theta]$ is an ideal of M.

Conversely, suppose that $U_{\mathcal{SUP}}(h;t)$ is either empty or an ideal of M, $\forall t \in [0,1]$. Let $\Theta \in \mathcal{P}[0,1]$ be such that $\mathcal{S}[h;\Theta] \neq \emptyset$. Choose $t = \mathcal{SUP}\,\Theta$, we get $U_{\mathcal{SUP}}(h;t) = \mathcal{S}[h;\Theta] \neq \emptyset$. By assumption, we have $\mathcal{S}[h;\Theta] = U_{\mathcal{SUP}}(h;t)$ is an ideal of M, that is, h is a Θ-$\mathcal{SUP}$-HFI of M. Therefore, h is a $\mathcal{SUP}$-HFI of M. □

Theorem 3.14. *Let h be a HFS on M. Then h^* is a $\mathcal{SUP}$-HFI of M if and only if $L_{\mathcal{SUP}}(h;t)$ is either empty or an ideal of M, $\forall t \in [0,1]$.*

Proof. Suppose that h^* is a $\mathcal{SUP}$-HFI of M. Let $t \in [0,1]$ be such that $L_{\mathcal{SUP}}(h;t) \neq \emptyset$. Choose $\Psi = \{1-t\}$, we get $\mathcal{S}[h^*;\Psi] = L_{\mathcal{SUP}}(h;t) \neq \emptyset$. By assumption, we obtain that $L_{\mathcal{SUP}}(h;t) = \mathcal{S}[h^*;\Psi]$ is an ideal of M.

Conversely, suppose that $L_{\mathcal{SUP}}(h;t)$ is either empty or an ideal of M, $\forall t \in [0,1]$. Let $\Psi \in \mathcal{P}[0,1]$ be such that $\mathcal{S}[h^*;\Psi] \neq \emptyset$. Choose $t = 1 - \mathcal{SUP}\,\Psi$, we get $L_{\mathcal{SUP}}(h;t) = \mathcal{S}[h^*;\Psi] \neq \emptyset$. By assumption, we obtain that $\mathcal{S}[h^*;\Psi] = L_{\mathcal{SUP}}(h;t)$ is an ideal of M, that is, h^* is a Ψ-$\mathcal{SUP}$-HFI of M. Therefore, h^* is a $\mathcal{SUP}$-HFI of M. □

Let I be a subset of X. The characteristic interval-valued fuzzy set (CIvFS) CI_I and the characteristic hesitant fuzzy set (CHFS) CH_I of I in X are defined by

$$CI_I\colon X \to \mathcal{D}[0,1], x \mapsto \begin{cases} \bar{1} & \text{if } x \in I, \\ \bar{0} & \text{otherwise,} \end{cases}$$

and

$$CH_I\colon X \to \mathcal{P}[0,1], x \mapsto \begin{cases} [0,1] & \text{if } x \in I, \\ \emptyset & \text{otherwise.} \end{cases}$$

For any $\Theta, \Psi \in \mathcal{P}[0,1]$ with $\mathcal{SUP}\,\Theta < \mathcal{SUP}\,\Psi$, define a function $H_I^{(\Theta,\Psi)}$ [6] as follows:

$$H_I^{(\Theta,\Psi)}\colon X \to \mathcal{P}[0,1], x \mapsto \begin{cases} \Psi & \text{if } x \in I, \\ \Theta & \text{otherwise.} \end{cases}$$

Then $H_I^{(\Theta,\Psi)}$ is a HFS on X, which is called the $\mathcal{SUP}\,(\Theta,\Psi)$-characteristic hesitant fuzzy set ($\mathcal{SUP}\,(\Theta,\Psi)$-CHFS) of I on X. The $\mathcal{SUP}\,(\Theta,\Psi)$-CHFS with $\Theta = \emptyset$ and $\Psi = [0,1]$ is the CHFS of I, that is, $H_I^{(\emptyset,[0,1])} = CH_I$. The $\mathcal{SUP}\,(\Theta,\Psi)$-CHFS with $\Theta = \bar{0}$ and $\Psi = \bar{1}$ is the CIvFS of I, that is, $H_I^{(\bar{0},\bar{1})} = CI_I$.

Theorem 3.15. *Let* I *be a nonempty subset of* M *and* $\Theta, \Psi \in \mathcal{P}[0,1]$ *with* $\mathcal{SUP}\,\Theta < \mathcal{SUP}\,\Psi$. *Then* I *is an ideal of* M *if and only if* $H_I^{(\Theta,\Psi)}$ *is a* $\mathcal{SUP}$*-HFI of* M.

Proof. Suppose that $\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x\gamma y) < \max\{\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x), \mathcal{SUP}\,H_I^{(\Theta,\Psi)}(y)\}$ for some $x, y \in M$ and $\gamma \in \Gamma$. Then $\max\{\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x), \mathcal{SUP}\,H_I^{(\Theta,\Psi)}(y)\} = \mathcal{SUP}\,\Psi$, which implies that $x \in I$ or $y \in I$. Since I is an ideal of M, we have $x\gamma y \in I$ and so $\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x\gamma y) = \mathcal{SUP}\,\Psi = \max\{\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x), \mathcal{SUP}\,H_I^{(\Theta,\Psi)}(y)\}$, it is a contradiction. Hence, $\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x\gamma y) \geqslant \max\{\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x), \mathcal{SUP}\,H_I^{(\Theta,\Psi)}(y)\}$, $\forall x, y \in M, \forall \gamma \in \Gamma$. From Theorem 3.9, it can be seen that $H_I^{(\Theta,\Psi)}$ is a $\mathcal{SUP}$-HFI of M.

Conversely, let $a \in I, x \in M$ and $\gamma \in \Gamma$. Then $H_I^{(\Theta,\Psi)}(a) = \Psi$. Since $H_I^{(\Theta,\Psi)}$ is a $\mathcal{SUP}$-HFI of M and by using Theorem 3.9, we have $\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(a\gamma x) \geqslant \max\{\mathcal{SUP}\,H_I^{(\Theta,\Psi)}(a), \mathcal{SUP}\,H_I^{(\Theta,\Psi)}(x)\} = \mathcal{SUP}\,\Psi$, which implies that $a\gamma x \in I$. Similarly, we have $x\gamma a \in I$. Hence, I is an ideal of M. □

From Theorem 3.15, we get Corollary 3.16.

Corollary 3.16. *For any nonempty subset* I *of* M, *the following assertions are equivalent.*

(1) I *is an ideal of* M.

(2) CI_I *is a* $\mathcal{SUP}$*-HFI of* M.

(3) CH_I *is a* $\mathcal{SUP}$*-HFI of* M.

4. $\mathcal{SUP}$-hesitant fuzzy translations

In this section, $\mathcal{SUP}$-hesitant fuzzy translations of $\mathcal{SUP}$-HFIs of Γ-semigroups are studied and the concepts of extensions and intensions of $\mathcal{SUP}$-HFIs are discussed in relation to the previous concepts.

Provided a HFS h on X, let

$$\top_h := 1 - \sup\{\mathcal{SUP}\,h(x) \mid x \in X\}.$$

Let $t \in [0, \top_h]$ and we say that a HFS g on X is a $\mathcal{SUP}$-hesitant fuzzy t^+-translation ($\mathcal{SUP}$-HFT$_{t^+}$) [6] of h if $\mathcal{SUP}\,g(x) = \mathcal{SUP}\,h(x) + t\ \forall x \in X$. Then h is a $\mathcal{SUP}$-HFT$_{0^+}$ of h, and in the case that g_1 and g_2 are $\mathcal{SUP}$-HFTs$_{t^+}$ of h, we see that $\mathcal{SUP}\,g_1(x) = \mathcal{SUP}\,g_2(x)\ \forall x \in X$ but g_1 maybe not equal to g_2.

Theorem 4.1. *Let* h *be a* $\mathcal{SUP}$*-HFI of* M *and* $t \in [0, \top_h]$. *Then every* $\mathcal{SUP}$*-HFT*$_{t^+}$ *of* h *is a* $\mathcal{SUP}$*-HFI of* M.

Proof. Assume that g is a $\mathcal{SUP}$-HFT$_{t^+}$ of h. Then $\forall a, b \in M, \forall \gamma \in \Gamma$,

$$\begin{aligned} \mathcal{SUP}\, g(a\gamma b) &= \mathcal{SUP}\, h(a\gamma b) + t \\ &\geqslant \max\{\mathcal{SUP}\, h(a), \mathcal{SUP}\, h(b)\} + t \\ &= \max\{\mathcal{SUP}\, h(a) + t, \mathcal{SUP}\, h(b) + t\} \\ &= \max\{\mathcal{SUP}\, g(a), \mathcal{SUP}\, g(b)\}. \end{aligned}$$

From Theorem 3.9, it can be seen that g is a $\mathcal{SUP}$-HFI of M. □

Theorem 4.2. *Let h be a HFS on M such that its $\mathcal{SUP}$-HFT$_{t^+}$ is a $\mathcal{SUP}$-HFI of M for some $t \in [0, \top_h]$. Then h is a $\mathcal{SUP}$-HFI of M.*

Proof. Assume that a $\mathcal{SUP}$-HFT$_{t^+}$ g of h is a $\mathcal{SUP}$-HFI of M when $t \in [0, \top_h]$. Then $\forall a, b \in M, \forall \gamma \in \Gamma$,

$$\begin{aligned} \mathcal{SUP}\, h(a\gamma b) &= \mathcal{SUP}\, g(a\gamma b) - t \\ &\geqslant \max\{\mathcal{SUP}\, g(a), \mathcal{SUP}\, g(b)\} - t \\ &= \max\{\mathcal{SUP}\, g(a) - t, \mathcal{SUP}\, g(b) - t\} \\ &= \max\{\mathcal{SUP}\, h(a), \mathcal{SUP}\, h(b)\}. \end{aligned}$$

From Theorem 3.9, it can be seen that h is a $\mathcal{SUP}$-HFI of M. □

Theorem 4.3. *Let h be a HFS on M and $t \in [0, \top_h]$. Then a $\mathcal{SUP}$-HFT$_{t^+}$ of h is a $\mathcal{SUP}$-HFI of M if and only if $U_{\mathcal{SUP}}(h; m - t)$ is either empty or an ideal of M, $\forall m \in [t, 1]$.*

Proof.
($\Rightarrow$) The proof is given by Theorem 3.13.
($\Leftarrow$) Let g be a $\mathcal{SUP}$-HFT$_{t^+}$ of h, $\gamma \in \Gamma$ and $a, b \in M$. Taking $m := \max\{\mathcal{SUP}\, g(a), \mathcal{SUP}\, g(b)\}$, we have

$$\begin{aligned} m - t &= \max\{\mathcal{SUP}\, g(a), \mathcal{SUP}\, g(b)\} - t \\ &= \max\{\mathcal{SUP}\, g(a) - t, \mathcal{SUP}\, g(b) - t\} \\ &= \max\{\mathcal{SUP}\, h(a), \mathcal{SUP}\, h(b)\}. \end{aligned}$$

Thus $a \in U_{\mathcal{SUP}}(h; m - t)$ or $b \in U_{\mathcal{SUP}}(h; m - t)$. By assumption, we have $a\gamma b \in U_{\mathcal{SUP}}(h; m - t)$. Hence,

$$\mathcal{SUP}\, g(a\gamma b) = \mathcal{SUP}\, h(a\gamma b) + t \geqslant m = \max\{\mathcal{SUP}\, g(a), \mathcal{SUP}\, g(b)\}.$$

From Theorem 3.9, it can be seen that g is a $\mathcal{SUP}$-HFI of M. □

Definition 4.4. If h and g are HFSs on X such that $\mathcal{SUP}\, h(x) \leqslant \mathcal{SUP}\, g(x)$ $\forall x \in X$, then we say that g is a $\mathcal{SUP}$-hesitant fuzzy extension ($\mathcal{SUP}$-HFEx) of h and say that h is a $\mathcal{SUP}$-hesitant fuzzy intension ($\mathcal{SUP}$-HFIn) of g.

Example 4.5. Let $M = \{0, 1, 2, 3\}$ and $\Gamma = \{\gamma\}$ be two nonempty sets. Then M is a Γ-semigroup with respect to the operation define below:

γ	0	1	2	3
0	0	0	0	0
1	0	0	0	2
2	0	0	0	2
3	2	2	2	2

We define HFSs g_1, g_2, g_3 and h as follows:
$g_1(0) = \{0.1, 0.2, 0.6, 0.8\}$, $g_1(1) = \{0.1, 0.8\}$, $g_1(2) = [0.1, 0.8]$, and $g_1(3) = \{0.1, 0.6\}$,
$g_2(0) = \{[0.2, 0.4, 0.6, 0.8\}$, $g_2(1) = \{0.3, 0.5, 0.8\}$, $g_2(2) = [0.1, 0.8]$, and $g_2(3) = \{0.4, 0.6\}$,
$g_3(0) = [0.1, 0.6]$, $g_3(1) = \{0.2, 0.4, 0.6, 0.8\}$, $g_3(2) = \{0.5, 0.6, 0.7\}$, and $g_3(3) = \emptyset$,
$h(0) = [0.1, 0.6]$, $h(1) = \{0.1, 0.4, 0.5, 0.6\}$, $h(2) = \{0.1, 0.2, 0.3, 0.4, 0.6\}$, and $h(3) = \{0.1, 0.4\}$.
Then the following assertions are true.

(1) $\mathfrak{h}$ is a $\mathcal{SUP}$-HFI of M.

(2) $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\mathcal{SUP}$-HFTs$_{0.2^+}$ of $\mathfrak{h}$ and $\mathcal{SUP}$-HFIs of M.

(3) $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\mathcal{SUP}$-HFEx of $\mathfrak{g}_3$ but not a $\mathcal{SUP}$-HFT$_{t^+}$ of $\mathfrak{g}_3$ $\forall t \in [0, 0.2]$.

(4) $\mathfrak{g}_3$ is a $\mathcal{SUP}$-HFIn of $\mathfrak{g}_1$.

Proposition 4.6. *Let $\mathfrak{h}$ be a HFS on M and $t \in [0, \top_{\mathfrak{h}}]$. Then the following assertions are true.*

(1) *Every $\mathcal{SUP}$-HFT$_{t^+}$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFEx of $\mathfrak{h}$.*

(2) *If $t_1 \in [0, \top_{\mathfrak{h}}]$ and $t_1 \geqslant t$, then every $\mathcal{SUP}$-HFT$_{t_1^+}$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFEx of a $\mathcal{SUP}$-HFT$_{t^+}$ of $\mathfrak{h}$.*

(3) *If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\mathcal{SUP}$-HFTs$_{t^+}$ of $\mathfrak{h}$ and $\mathfrak{g}_3$ is a $\mathcal{SUP}$-HFEx of $\mathfrak{g}_1$, then $\mathfrak{g}_3$ is a $\mathcal{SUP}$-HFEx of $\mathfrak{g}_2$.*

Proof. Straightforward. □

Definition 4.7. Let $\mathfrak{h}$ and $\mathfrak{g}$ be HFSs on M. Then $\mathfrak{g}$ is called a $\mathcal{SUP}$-hesitant fuzzy extension of $\mathfrak{h}$ based on an ideal of M ($\mathcal{SUP}$-HFExI) if the following assertions are valid.

(1) $\mathfrak{g}$ is a $\mathcal{SUP}$-HFEx of $\mathfrak{h}$.

(2) If $\mathfrak{h}$ is a $\mathcal{SUP}$-HFI of M, then so is $\mathfrak{g}$.

Example 4.8. Let $M = \{0, 1, 2, 3\}$ be a Γ-semigroup defined in Example 4.5. We define HFSs $\mathfrak{g}$ and $\mathfrak{h}$ on M as follows:

$$\mathfrak{g}(0) = [0.3, 0.5], \quad \mathfrak{g}(1) = \{0.1, 0.2, 0.4\}, \quad \mathfrak{g}(2) = \{0.1, 0.3, 0.5\}, \quad \mathfrak{g}(3) = \{0.1, 0.2\},$$

and

$$\mathfrak{h}(0) = \{0.1, 0.3\}, \quad \mathfrak{h}(1) = [0.1, 0.3], \quad \mathfrak{h}(2) = \{0.1, 0.2, 0.3\}, \quad \mathfrak{h}(3) = \{0.1, 0.2\}.$$

Then $\mathfrak{g}$ is a $\mathcal{SUP}$-HFExI of $\mathfrak{h}$.

From Proposition 4.6 and Theorem 4.1, we get Theorem 4.9.

Theorem 4.9. *If $\mathfrak{h}$ is a $\mathcal{SUP}$-HFI of M, then its $\mathcal{SUP}$-HFT$_{t^+}$ is a $\mathcal{SUP}$-HFExI of $\mathfrak{h}$, $\forall t \in [0, \top_{\mathfrak{h}}]$.*

From Proposition 4.6 and Theorem 4.9, we get Theorem 4.10.

Theorem 4.10. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFI of M and $t_1, t_2 \in [0, \top_{\mathfrak{h}}]$. If $t_1 \geqslant t_2$, then a $\mathcal{SUP}$-HFT$_{t_1^+}$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFExI of a $\mathcal{SUP}$-HFT$_{t_2^+}$ of $\mathfrak{h}$.*

From Theorem 4.10, we get Theorem 4.11.

Theorem 4.11. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFI of M and $t \in [0, \top_{\mathfrak{h}}]$. If $\mathcal{G}$ is a $\mathcal{SUP}$-HFExI of a $\mathcal{SUP}$-HFT$_{t^+}$ of $\mathfrak{h}$, then $\mathcal{G}$ is a $\mathcal{SUP}$-HFExI of a $\mathcal{SUP}$-HFT$_{k^+}$ of $\mathfrak{h}$ for some $k \in [t, \top_{\mathfrak{h}}]$.*

Provided a HFS $\mathfrak{h}$ on X, let

$$\perp_{\mathfrak{h}} := \inf\{\mathcal{SUP}\, \mathfrak{h}(x) \mid x \in X\}.$$

For any $t \in [0, \perp_{\mathfrak{h}}]$, a HFS $\mathfrak{g}$ of X is called a $\mathcal{SUP}$-hesitant fuzzy t^--translation ($\mathcal{SUP}$-HFT$_{t^-}$) of $\mathfrak{h}$ if $\mathcal{SUP}\, \mathfrak{g}(x) = \mathcal{SUP}\, \mathfrak{h}(x) - t$ $\forall x \in X$. Note that $\mathfrak{h}$ is a $\mathcal{SUP}$-HFT$_{0^-}$ of $\mathfrak{h}$.

Theorem 4.12. *Let $\mathfrak{h}$ be a $\mathcal{SUP}$-HFI of M and $t \in [0, \perp_{\mathfrak{h}}]$. Then every $\mathcal{SUP}$-HFT$_{t^-}$ of $\mathfrak{h}$ is a $\mathcal{SUP}$-HFI of M.*

Proof. It can be proved in a similar way as in the proof of Theorem 4.1. □

Theorem 4.13. *Let h be a HFS on M such that its $\mathcal{SUP}$-HFT$_{t^-}$ is a $\mathcal{SUP}$-HFI of M for some $t \in [0, \perp_h]$. Then h is a $\mathcal{SUP}$-HFI of M.*

Proof. It can be proved in a similar way as in the proof of Theorem 4.2. □

Proposition 4.14. *Let h be a HFS on M and $t \in [0, \perp_h]$. Then the following assertions are true.*

(1) *Every $\mathcal{SUP}$-HFT$_{t^-}$ of h is a $\mathcal{SUP}$-HFIn of h.*

(2) *If $t_1 \in [0, \perp_h]$ and $t_1 \geqslant t$, then every $\mathcal{SUP}$-HFT$_{t_1^-}$ of h is a $\mathcal{SUP}$-HFIn of a $\mathcal{SUP}$-HFT$_{t^-}$ of h.*

(3) *If g_1 and g_2 are $\mathcal{SUP}$-HFTs$_{t^-}$ of h and g_3 is a $\mathcal{SUP}$-HFIn of g_1, then g_3 is a $\mathcal{SUP}$-HFIn of g_2.*

(4) *Every $\mathcal{SUP}$-HFT$_{t^-}$ of h is a $\mathcal{SUP}$-HFIn of a $\mathcal{SUP}$-HFT$_{k^+}$ of h when $k \in [0, \top_h]$.*

Proof. Straightforward. □

Definition 4.15. Let h and g be HFSs on M. Then g is called a $\mathcal{SUP}$-hesitant fuzzy intension of h based on an ideal of M ($\mathcal{SUP}$-HFInI) if the following assertions are valid.

(1) g is a $\mathcal{SUP}$-HFIn of h.

(2) If h is a $\mathcal{SUP}$-HFI of M, then so is g.

Example 4.16. Let $\Gamma = \mathbb{Z}^-$. Then $\mathbb{Z}^-$ is a Γ-semigroup with respect to usual multiplication. Define HFSs g and h on $\mathbb{Z}^-$ by $h(x) = [0, 1 + \frac{1}{x}]$ and $g(x) = [0, 0.5 + \frac{1}{2x}]$, $\forall x \in \mathbb{Z}^-$. Then g is a $\mathcal{SUP}$-HFInI of h.

From Proposition 4.14 and Theorem 4.12, we get Theorem 4.17.

Theorem 4.17. *If h is a $\mathcal{SUP}$-HFI of M, then its $\mathcal{SUP}$-HFT$_{t^-}$ is a $\mathcal{SUP}$-HFInI of h, $\forall t \in [0, \perp_h]$.*

From Proposition 4.14 and Theorem 4.17, we get Theorem 4.18

Theorem 4.18. *Let h be a $\mathcal{SUP}$-HFI of M and $t_1, t_2 \in [0, \perp_h]$. If $t_1 \geqslant t_2$, then a $\mathcal{SUP}$-HFT$_{t_1^-}$ of h is a $\mathcal{SUP}$-HFInI of a $\mathcal{SUP}$-HFT$_{t_2^-}$ of h.*

From Theorem 4.18, we get Theorem 4.19.

Theorem 4.19. *Let h be a $\mathcal{SUP}$-HFI of a Γ-semigroup and $t \in [0, \perp_h]$. If $\mathcal{G}$ is a $\mathcal{SUP}$-HFInI of a $\mathcal{SUP}$-HFT$_{t^-}$ of h, then $\mathcal{G}$ is a $\mathcal{SUP}$-HFInI of a $\mathcal{SUP}$-HFT$_{k^-}$ of h for some $k \in [0, t]$.*

From Theorems 4.2 and 4.12, we get Theorem 4.20.

Theorem 4.20. *Let h be a HFS on M. If there is $t \in [0, \top_h]$ such that one of $\mathcal{SUP}$-HFTs$_{t^+}$ of h is a $\mathcal{SUP}$-HFI of M, then every $\mathcal{SUP}$-HFT$_{k^-}$ of h is a $\mathcal{SUP}$-HFI of M, $\forall k \in [0, \perp_h]$.*

From Theorems 4.1 and 4.13, we have Theorem 4.21.

Theorem 4.21. *Let h be a HFS on M. If there is $t \in [0, \perp_h]$ such that one of $\mathcal{SUP}$-HFTs$_{t^-}$ of h is a $\mathcal{SUP}$-HFI of M, then every $\mathcal{SUP}$-HFT$_{t_1^+}$ of h is a $\mathcal{SUP}$-HFI of M, $\forall t_1 \in [0, \top_h]$.*

Let g and h be two HFSs on X such that g is a $\mathcal{SUP}$-HFIn of h, we define the IvFS $[g, h]$ on X by $[g : h](x) = [\mathcal{SUP}\, g(x), \mathcal{SUP}\, h(x)]$ $\forall x \in X$. If g_1 is a $\mathcal{SUP}$-HFT$_{t_1^-}$ of h and g_2 is a $\mathcal{SUP}$-HFT$_{t_2^+}$ of h when $t_1 \in [0, \perp_h]$ and $t_2 \in [0, \top_h]$, then $[g_1 : h]$, $[h : g_2]$ and $[g_1 : g_2]$ are IvFSs on X.

From Theorems 3.9, 4.2, 4.9, and 4.17, we get Theorem 4.22.

Theorem 4.22. *Let h be a HFS on M, $t_1 \in [0, \perp_h]$ and $t_2 \in [0, \top_h]$. Then the following assertions are equivalent.*

(1) $\mathfrak{h}$ *is a* $\mathcal{SUP}$*-HFI of* M.

(2) $[\mathfrak{g} : \mathfrak{h}]$ *is an IvFI of* M *for every* $\mathcal{SUP}$*-HFT*$_{t_1^-}$ $\mathfrak{g}$ *of* $\mathfrak{h}$.

(3) $[\mathfrak{h} : \mathfrak{g}]$ *is an IvFI of* M *for every* $\mathcal{SUP}$*-HFT*$_{t_2^+}$ $\mathfrak{g}$ *of* $\mathfrak{h}$.

(4) $[\mathfrak{g}_1 : \mathfrak{g}_2]$ *is an IvFI of* M *for every* $\mathcal{SUP}$*-HFT*$_{t_1^-}$ $\mathfrak{g}_1$ *and every* $\mathcal{SUP}$*-HFT*$_{t_2^-}$ $\mathfrak{g}_2$ *of* $\mathfrak{h}$.

(5) $[\mathfrak{g} : \mathfrak{h}]$ *is an IvFI of* M *for every* $\mathcal{SUP}$*-HFInI* $\mathfrak{g}$ *of* $\mathfrak{h}$.

(6) $[\mathfrak{h} : \mathfrak{g}]$ *is an IvFI of* M *for every* $\mathcal{SUP}$*-HFExI* $\mathfrak{g}$ *of* $\mathfrak{h}$.

5. Conclusions

In this paper, we have introduced the concept of $\mathcal{SUP}$-HFIs of Γ-semigroups, which is a generalization of IvFIs and HFIs of Γ-semigroups and investigated some characterizations of $\mathcal{SUP}$-HFIs in terms of sets, FSs, IFSs, IvFSs, and HFSs. Further, we have discussed the relation between ideals and generalizations of the CIvFSs and the CHFSs. Finally, $\mathcal{SUP}$-HFTs of $\mathcal{SUP}$-HFIs of Γ-semigroups are discussed and their relations are investigated.

In the future, we will study $\mathcal{SUP}$-HFIs over hypersemigroups (see [12]) and over left almost Γ-semihypergroups (see [2]), and examine some characterizations of $\mathcal{SUP}$-HFIs in terms of sets, FSs, IFSs, IvFSs, and HFSs. Moreover, we are interested in extending the ideas from this paper to $\mathcal{SUP}$-cubic fuzzy ideals of UP-algebras (see [14]).

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