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# Convergence results for modified SP-iteration in uniformly convex metric spaces

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## Abstract

In this paper, we prove a strong convergence theorem of a Modified SP-iteration for finding a common fixed point of the combination of a finite family of nonexpansive mappings in a convex metric space. Moreover, we give some numerical example for supporting our main theorem and compare convergence rate between the modified SP-iteration and the Ishikawa iteration.

**Keywords:** Convergence theorem, SP-iteration, convex metric space.

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## 1. Introduction

Let $C$ be a nonempty closed convex subset of a metric space $(X, d)$. Let $T : C \to C$ be a mapping. The set of all fixed points of a mapping $T$ is denoted by $\text{Fix}(T)$, that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to C$ is called *nonexpansive* if

$$d(Tx, Ty) \leqslant d(x, y)$$

for all $x, y \in C$.

Let $(X, d)$ be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on $X$ if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(z, W(x, y, \lambda)) \leqslant \lambda d(z, x) + (1 - \lambda)d(z, y)$$

for all $z \in X$. A metric space $(X, d)$ together with a convex structure $W$ is called a *convex metric space* which is denoted by $(X, d, W)$. A nonempty subset $C$ of $X$ is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. The concept of convex metric spaces was introduced by Takahashi [18]. Takahashi [18] also studied some fixed point theorems for nonexpansive mappings in convex metric spaces. Note that a normed space and each of its convex subsets are convex metric spaces, but the converse does not hold. Many authors have increasingly investigated a convex metric space; see for instance [3, 4, 11, 15] and references therein.

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In 1996, Shimizu and Takahashi [16] introduced the concept of uniform convexity in convex metric spaces. A convex metric space $(X, d, W)$ is said to be *uniformly convex* if for any $\epsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, 1]$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leqslant r$, $d(z, y) \leqslant r$ and $d(x, y) \geqslant r\varepsilon$ imply that

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leqslant (1 - \delta)r.$$

Noticeably, uniformly convex Banach spaces and CAT(0) spaces are examples of a uniformly convex metric space [8]. Fixed point theorem has been an essential tool in both theoretical and applied mathematics, as well as in computational, economical, modeling and engineering problems. Its importance is a motivation for many mathematicians to do further research and expand its application. A popular way of approximating fixed points of nonlinear mappings is an iterative method. Throughout the last decade years, many researchers investigated various iterative methods for several classes of contractive and nonexpansive mappings, such as Mann iteration [9], Halpern iteration [5], Ishikawa iteration [6], three-step iterative scheme (or Noor iteration) [10], and SP-iteration [13]. In 2011, Pheungrattana and Suantai [13] introduced the *SP-iteration* is defined by the sequence $\{x_n\}$ as:

$$\begin{cases} x_1 = x \in C, \\ z_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1 - \beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in the interval $[0, 1]$. They also showed that the rate of convergence of the Mann, Ishikawa, Noor and SP-iteration are equivalent for nonexpansive mapping and SP-iteration converges better than the other schemes for the class of continuous and nondecreasing functions.

In 2011, Phuengrattana and Suantai [14] introduced a modified Halpern iteration for finding a common fixed point of a countable infinite family of nonexpansive mappings in a convex metric space as follows:

$$\begin{cases} y_n = W(u, S_n x_n, \alpha_n), \\ x_{n+1} = W(y_n, S_n y_n, \beta_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $S_n$ is the $W$-mapping generated by a countable infinite family of nonexpansive mapping and real numbers in convex metric space. They also proved a strong convergence of the proposed iterative scheme in a convex metric space under suitable conditions.

In 2018, Kangtunyakarn [7] introduced the following Ishikawa iteration for finding a common fixed point of a finite family of nonexpansive mappings in a convex metric space as follows.

$$\begin{cases} y_n = W(x_n, Sx_n, \beta_n), \\ x_{n+1} = W(x_n, Sy_n, \gamma_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $S$ is the $S$-mapping generated by a finite family of nonexpansive mapping and real numbers in convex metric space. Under suitable conditions, he also established that the sequence $\{x_n\}$ generated by (1.1) converge to $z \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$, where $\{T_i\}_{i=1}^{N}$ is a finite family of nonexpansive mappings of $C$ into itself.

In 2019, Siriyan and Kangtunyakarn [17] proposed a new mapping generated by the combination of a finite family of nonexpansive mappings and real numbers in a convex metric space. By using the proposed mapping, they also introduced the Ishikawa iteration for finding a common fixed point of the combination of a finite family of nonexpansive mappings in a convex metric space as follows:

$$\begin{cases} y_n = W(x_n, Sx_n, \beta_n), \\ x_{n+1} = W(x_n, Sy_n, \alpha_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of $C$ into itself with $\bigcap_i^N F(T_i) \neq \emptyset$, $a_1, a_2, \ldots, a_N \in [0,1]$ with $\sum_{i=1}^N a_i = 1$ and $S$ is the mapping generated by $T_1, T_2, \ldots, T_N$ and $a_1, a_2, \ldots, a_N$. Under some mild assumptions, Siriyan and Kangtunyakarn [17] proved that the sequence $\{x_n\}$ generated by (1.2) converges to $z \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.

In this paper, motivated by the previous works, we extend the result of Siriyan and Kangtunyakarn [17] from Ishikawa iteration to modified SP-iteration. By using the mapping generated by the combination of a finite family of nonexpansive mappings and real numbers in convex metric space, we prove a strong convergence theorem of the modified SP-iteration for finding a set of solution of a common fixed point of a finite family of nonexpansive mappings in a convex metric space. Moreover, we give some numerical examples for supporting our main theorem and compare convergence rate between the modified SP-iteration and the Ishikawa iteration. This paper is organized as follows. In Section 2, we recall some definitions and preliminary results used in the paper. Section 3 deals with analyzing the convergence of the proposed iteration. Finally, Section 4, presents several numerical examples to illustrate the behavior of the proposed iteration.

## 2. Preliminary

The purpose of our main results is to extend the results in the literature by using the idea of a convex metric space. So we recollect some needed definitions in this section.

**Lemma 2.1** ([1, 18]). *Let $(X, d, W)$ be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$, we have the following:*

(i) $W(x,x,\lambda) = x, W(x,y,0) = y$ *and* $W(x,y,1) = x$;
(ii) $d(x, W(x,y,\lambda)) = (1-\lambda)\, d(x,y)$ *and* $d(y, W(x,y,\lambda)) = \lambda d(x,y)$;
(iii) $d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda), y)$;
(iv) $|\lambda_1 - \lambda_2| d(x,y) \leqslant d(W(x,y,\lambda_1), W(x,y,\lambda_2))$.

**Lemma 2.2** ([13]). *We say that convex metric space $(X, d, W)$ has the property:*

(C) *if* $W(x,y,\lambda) = W(y,x,1-\lambda)$ *for all* $x, y \in X$ *and* $\lambda \in [0,1]$;
(I) *if* $d(W(x,y,\lambda_1), W(x,y,\lambda_2)) \leqslant |\lambda_1 - \lambda_2|\, d(x,y)$ *for all* $x, y \in X$ *and* $\lambda_1, \lambda_2 \in [0,1]$;
(H) *if* $d(W(x,y,\lambda), W(x,z,\lambda)) \leqslant (1-\lambda)\, d(y,z)$ *for all* $x, y, z \in X$ *and* $\lambda \in [0,1]$;
(S) *if* $d(W(x,y,\lambda), W(z,w,\lambda)) \leqslant \lambda d(x,z) + (1-\lambda)\, d(y,w)$ *for all* $x, y, z, w \in X$ *and* $\lambda \in [0,1]$.

**Lemma 2.3** ([13]). *Property (C) in Lemma 2.2 holds in uniformly convex metric space.*

*Remark* 2.4 ([13]). From Lemma 2.3, a uniformly convex metric space $(X, d, W)$ with the property (H) has the property (S) and the convex structure $W$ is also continuous.

**Lemma 2.5** ([16]). *Let $(X, d, W)$ be a uniformly convex metric space with continuous convex structure. Then, for arbitrary positive number $\varepsilon$, there exists $\eta = \eta(\varepsilon) > 0$ such that*

$$d(z, W(x,y,\lambda)) \leqslant (1 - 2\min\{\lambda, 1-\lambda\}\eta)\, r,$$

*for all $r > 0$ and $x, y, z \in X$, $d(z,x) \leqslant r, d(z,y) \leqslant r, d(x,y) \geqslant r\varepsilon$ and $\lambda \in [0,1]$.*

**Lemma 2.6** ([17]). *Let $(X, d, W)$ be a uniformly convex metric space with continuous convex structure. If $x, y, z \in X$ are such that $d(z, W(x,y,\lambda)) = d(z,x) = d(z,y)$, where $\lambda \in (0,1)$, then $x = y$.*

**Definition 2.7** ([17]). Let $C$ be a nonempty closed and convex subset of a uniformly convex metric $(X, d)$ with continuous convex structure $W$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonlinear mapping of $C$ into itself and let $a_1, a_2, \ldots, a_N \in [0,1]$ with $\sum_{i=1}^N a_i = 1$, where $N \geqslant 2$. We define the mapping $S : C \to C$ as follows:

$$U_2 x = W(T_1 x, T_2 x, a_1),$$

$$U_3x = W\left(T_1x, W\left(T_2x, T_3x, \frac{a_2}{a_2+a_3}\right), a_1\right),$$
$$U_4x = W\left(T_1x, W\left(T_2x, W\left(T_3x, T_4x, \frac{a_3}{a_3+a_4}\right), \frac{a_2}{a_2+a_3+a_4}\right), a_1\right),$$
$$U_5x = W\left(T_1x, W\left(T_2x, W\left(T_3x, W\left(T_4x, T_5x, \frac{a_4}{\sum_{i=4}^{5} a_i}\right), \frac{a_3}{\sum_{i=3}^{5} a_i}\right), \frac{a_2}{\sum_{i=2}^{5} a_i}\right), a_1\right),$$
$$\vdots$$
$$U_{N-1}x = W\left(T_1x, W\left(T_2x, \dots, W\left(T_{N-2}x, T_{N-1}, \frac{a_{N-2}}{\sum_{i=N-2}^{N-1} a_i}\right), \dots, \frac{a_2}{\sum_{i=2}^{N-1} a_i}\right), a_1\right),$$
$$S_x = U_N x = W\left(T_1x, W\left(T_2x, \dots, W\left(T_{N-1}x, T_Nx, \frac{a_{N-1}}{\sum_{i=N-1}^{N} a_i}\right), \dots, \frac{a_2}{\sum_{i=2}^{N} a_i}\right), a_1\right),$$

for all $x \in C$.

**Lemma 2.8** ([17]). *Let $C$ be a nonempty, closed and convex subset of a uniformly convex metric space $(X, d)$ with continuous convex structure $W$ and property $(H)$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mapping of $C$ into itself with $\bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$ and let $S : C \to C$ be a mapping defined as Definition 2.7. Then $\text{Fix}(S) = \bigcap_{i=1}^{N} \text{Fix}(T_i)$. Moreover, $S$ is nonexpansive mapping.*

**Lemma 2.9** ([2, 12]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1-\delta_n)\, a_n + b_n, n \geqslant 1.$$

*If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

## 3. Main results

In this section, we study the convergence theorem of the Modified SP-iteration for finding a common fixed point of the combination of a finite family of nonexpansive mapping in framework of convex metric spaces.

**Theorem 3.1.** *Let $C$ be a nonempty compact closed convex subset of a complete uniformly convex metric space $(X, d)$ with continuous convex structure $W$ and with property $(H)$. Let $T : C \to C$ be a nonexpansive mapping. Let $x_1 \in C$ and $\{x_n\}$ be the sequences generated by*

$$\begin{cases} x_{n+1} = W(y_n, Ty_n, \alpha_n), \\ y_n = W(z_n, Tz_n, \beta_n), \\ z_n = W(x_n, Tx_n, \gamma_n), \end{cases} \tag{3.1}$$

*for all $n \geqslant 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are the sequences in $[0, 1]$ satisfying $\sum_{n=1}^{\infty} \gamma_n (1-\gamma_n) = \infty$. Then, the sequence $\{x_n\}$ converges to $z \in \text{Fix}(T)$.*

*Proof.* Firstly, we will prove that $\inf_{n\in\mathbb{N}} d(x_n, Tx_n) = 0$ by assuming that $\inf_{n\in\mathbb{N}} d(x_n, Tx_n) = r' > 0$ and reach contradiction.

Let $p \in \text{Fix}(T)$, then we have $p = Tp$. From (3.1) and $T$ being nonexpansive, we have

$$\begin{aligned} d(p, x_{n+1}) &= d(p, W(y_n, Ty_n, \alpha_n)) \\ &\leqslant \alpha_n d(p, y_n) + (1-\alpha_n)\, d(p, Ty_n) \\ &\leqslant \alpha_n d(p, y_n) + (1-\alpha_n)\, d(p, y_n) \end{aligned}$$

$$
\begin{aligned}
&= d(p, y_n) \\
&= d(p, W(z_n, Tz_n, \beta_n)) \\
&\leqslant \beta_n d(p, z_n) + (1-\beta_n) d(p, Tz_n) \\
&\leqslant \beta_n d(p, z_n) + (1-\beta_n) d(p, z_n) \\
&= d(p, z_n) \\
&= d(p, W(x_n, Tx_n, \gamma_n)) \\
&\leqslant \gamma_n d(p, x_n) + (1-\gamma_n) d(p, Tx_n) \\
&\leqslant \gamma_n d(p, x_n) + (1-\gamma_n) d(p, x_n) \\
&= d(p, x_n).
\end{aligned}
$$

From Lemma 2.9, we obtain that $\lim_{n\to\infty} d(p, x_n)$ exists. Then, we have $\lim_{n\to\infty} d(p, x_n) = r'' > 0$. From nonexpansiveness of $T$, we have $d(p, Tx_n) \leqslant d(p, x_n)$. Since $d(p, x_{n+1}) \leqslant d(p, x_n)$, then $\{d(p, x_n)\}$ is nonincreasing and since $\inf_{n\in\mathbb{N}} d(x_n, Tx_n) = r' > 0$, we have

$$
d(x_n, Tx_n) \geqslant r' = \frac{r'}{d(p, x_n)} d(p, x_n) \geqslant \frac{r'}{d(p, x_1)} d(p, x_n) > 0,
$$

for all $n \in \mathbb{N}$. From Lemma 2.5, there exists $\eta = \eta\left(\frac{r'}{d(p,x_1)}\right) > 0$ such that

$$
\begin{aligned}
d(p, x_{n+1}) \leqslant d(p, W(x_n, Tx_n, \gamma_n)) &\leqslant (1 - 2\min\{\gamma_n, 1-\gamma_n\}\eta) d(p, x_n) \\
&\leqslant d(p, x_n) - 2d\gamma_n(1-\gamma_n)\eta d(p, x_n).
\end{aligned}
$$

It follows that

$$
2\gamma_n(1-\gamma_n)\eta d(p, x_n) \leqslant d(p, x_n) - d(p, x_{n+1}),
$$

which implies that

$$
2\gamma_n(1-\gamma_n)\eta r'' \leqslant 2\gamma_n(1-\gamma_n)\eta d(p, x_n) \leqslant d(p, x_n) - d(p, x_{n+1}). \tag{3.2}
$$

From (3.2), we have

$$
2\eta r'' \sum_{n=1}^{k} \gamma_n(1-\gamma_n) \leqslant d(p, x_1) - d(p, x_{k+1}), \quad \forall k \geqslant 1. \tag{3.3}
$$

By takting $k \to \infty$ in (3.3), and with the condition $\sum_{n=1}^{\infty} \gamma_n(1-\gamma_n) = \infty$, we have $\infty \leqslant d(p, x_1) - r'' < \infty$, which is a contradiction. Hence, $\inf_{n\in\mathbb{N}} d(x_n, Tx_n) = 0$. Then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} d(x_{n_i}, Tx_{n_i}) = 0$. Since $C$ is compact and $p \in C$, then there exists a subsequence $\left\{x_{n_{i_j}}\right\}$ of $\{x_{n_i}\}$ such that $\lim_{j\to\infty} x_{n_{i_j}} = p$. From nonexpansiveness of $T$, we have

$$
\begin{aligned}
d(p, Tp) \leqslant d\left(p, Tx_{n_{i_j}}\right) + d\left(Tx_{n_{i_j}}, Tp\right) &\leqslant d\left(p, x_{n_{i_j}}\right) + d\left(x_{n_{i_j}}, Tx_{n_{i_j}}\right) + d\left(Tx_{n_{i_j}}, Tp\right) \\
&\leqslant d\left(p, x_{n_{i_j}}\right) + d\left(x_{n_{i_j}}, Tx_{n_{i_j}}\right) + d\left(x_{n_{i_j}}, p\right) \\
&= d\left(p, x_{n_{i_j}}\right) + d\left(x_{n_{i_j}}, Tx_{n_{i_j}}\right) + d\left(p, x_{n_{i_j}}\right) \\
&\leqslant 2d\left(p, x_{n_{i_j}}\right) + d\left(x_{n_{i_j}}, Tx_{n_{i_j}}\right).
\end{aligned}
$$

By taking $j \to \infty$, we have $d(p, Tp) = 0$, and it follows that $p = Tp$, which means $p \in Fix(T)$. Since $\lim_{n\to\infty} d(p, x_n)$ exists, we get that $\{x_n\}$ converges to $p \in Fix(T)$. This completes the proof. □

**Theorem 3.2.** *Let $C$ be a nonempty compact closed convex subset of a complete uniformly convex metric space $(X, d)$ with continuous convex structure $W$ and property $(H)$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of $C$ into itself with $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $a_1, a_2, \ldots, a_N \in [0, 1]$ with $\sum_{i=1}^{N} a_i = 1$. Let $S : C \to C$ be a nonexpansive* *mapping defined as in Definition 2.7. Let* $x_1 \in C$ *and* $\{x_n\}$ *be the sequences generated by*

$$\begin{cases} x_{n+1} = W(y_n, Sy_n, \alpha_n), \\ y_n = W(z_n, Sz_n, \beta_n), \\ z_n = W(x_n, Sx_n, \gamma_n), \end{cases} \tag{3.4}$$

*for all* $n \geqslant 1$, *where* $\{\gamma_n\}$ *is the sequence in* $[0,1]$ *satisfying* $\sum_{n=1}^{\infty} \gamma_n(1-\gamma_n) = \infty$. *Then, the sequence* $\{x_n\}$ *converges to* $z \in \bigcap_{i=1}^{N} \text{Fix}(T_i)$.

*Proof.* Using Theorem 3.1 and Lemma 2.8, we obtain the desired conclusion. □

From Theorem 3.2, we deduce immediately the following result. When $\gamma_n = 1$ in Theorem 3.2, we obtain Theorem 3.2 of Siriyan and Kangtunyakarn [17].

**Corollary 3.3.** *Let* $C$ *be a nonempty compact closed convex subset of a complete uniformly convex metric space* $(X, d)$ *with continuous convex structure* $W$ *and property* $(H)$. *Let* $\{T_i\}_{i=1}^{N}$ *be a finite family of nonexpansive mappings of* $C$ *into itself with* $\bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$. *Let* $a_1, a_2, \ldots, a_N \in [0,1]$ *with* $\sum_{i=1}^{N} a_i = 1$. *Let* $S : C \to C$ *be a nonexpansive mapping defined as in Definition 2.7. Let* $x_1 \in C$ *and* $\{x_n\}$ *be the sequences generated by*

$$\begin{cases} x_{n+1} = W(y_n, Sy_n, \alpha_n), \\ y_n = W(x_n, Sx_n, \beta_n), \end{cases}$$

*for all* $n \geqslant 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the sequences in* $[0,1]$ *satisfying* $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$. *Then, the sequence* $\{x_n\}$ *converges to* $z \in \bigcap_{i=1}^{N} \text{Fix}(T_i)$.

*Proof.* Put $\gamma_n = 1$ in Theorem 3.2. By Theorem 3.2, we can conclude the desired result. □

**Corollary 3.4.** *Let* $C$ *be a nonempty compact closed convex subset of a complete uniformly convex metric space* $(X, d)$ *with continuous convex structure* $W$ *and property* $(H)$. *Let* $\{T_i\}_{i=1}^{N}$ *be a finite family of nonexpansive mappings of* $C$ *into itself with* $\bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$. *Let* $a_1, a_2, \ldots, a_N \in [0,1]$ *with* $\sum_{i=1}^{N} a_i = 1$. *Let* $S : C \to C$ *be a nonexpansive mapping defined as in Definition 2.7. Let* $x_1 \in C$ *and* $\{x_n\}$ *be the sequences generated by*

$$x_{n+1} = W(x_n, Sx_n, \alpha_n)$$

*for all* $n \geqslant 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the sequences in* $[0,1]$ *satisfying* $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. *Then, the sequence* $\{x_n\}$ *converges to* $z \in \bigcap_{i=1}^{N} \text{Fix}(T_i)$.

*Proof.* Put $\beta_n = \gamma_n = 1$ in Theorem 3.2. By Theorem 3.2, we can conclude the desired result. □

## 4. Numerical examples

In this section, we give numerical examples to support our main theorem. We first present the numerical example for supporting Theorem 3.1. The following example is motivated by [17], we compare the rate of convergence of the Modified SP-iteration (3.4) and the Ishikawa iteration (1.2) by Siriyan and Kangtunyakarn [17] in the framework of convex metric spaces.

*Example* 4.1. Let $\mathbb{R}$ be the set real numbers. Let $d(x,y) = |x-y|, \forall x, y \in \mathbb{R}$ and $W(x,y,\lambda) = \lambda x + (1-\lambda) y$, $\forall x, y \in \mathbb{R}$ and $\lambda \in [0,1]$. For every $i = 1, 2, 3, \ldots, N$, let $T_i : [-50, 50] \to [-50, 50]$ be mapping defined by $T_i x = \frac{x+i}{i+1}, \forall x \in [-50, 50]$ and let $a_i = \frac{3}{4^i} + \frac{1}{N4^N}$. We can rewrite mapping $S$ in Theorem 3.2 as follows:

$$Sx = a_1 T_1 x + a_2 T_2 x + a_3 T_3 x + \cdots + a_N T_N x.$$

Let $x_1 \in [-50, 50]$ and $\{x_n\}$ be a sequence generated by (3.4), where $\alpha_n = \frac{4n-1}{10n}$, $\beta_n = \frac{1}{5n}$, and $\gamma_n = \frac{2}{7n}$.

By the definition of $T_i$, we have $\{1\} = \bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i)$. For every $n \in \mathbb{N}$, we can rewrite (3.4) as follows:

$$\begin{aligned} x_{n+1} &= \left(\frac{4n-1}{10n}\right) y_n + \left(\frac{6n+1}{10n}\right) S y_n, \\ y_n &= \left(\frac{1}{5n}\right) z_n + \left(\frac{5n-1}{5n}\right) S z_n, \\ z_n &= \left(\frac{2}{7n}\right) x_n + \left(\frac{7n-2}{7n}\right) S x_n. \end{aligned}$$

It is obvious that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ and $\sum_{i=1}^{N} a_i$ satisfy all the conditions of Theorem 3.2. From Theorem 3.2, we can conclude that the sequence $\{x_n\}$ converges strongly to 1.

The following Table 1, Table 2, and Figure 1, show the comparison of the rate of convergence of the sequence $\{x_n\}$ by using the Modified SP-iteration (3.4) and the Ishikawa iteration (1.2), where $x_1 = 10$ and $n = N = 30$.

Table 1: The convergence of $\{x_n\}$, $\{y_n\}$, and $\epsilon$ with $x_1 = 10$ and $n = N = 30$ using the Ishikawa iteration (1.2).

| $n$ | $x_n$ | $y_n$ | $\epsilon = \lvert x_n - x_{n-1} \rvert$ |
|---|---|---|---|
| 1 | 10.000000000 | 6.055731060 | - |
| 2 | 5.300287562 | 3.180101229 | 4.699712438 |
| 3 | 3.145876360 | 2.048702391 | 2.154411202 |
| 4 | 2.087152656 | 1.521371916 | 1.058723704 |
| 5 | 1.555030053 | 1.263138347 | 0.532122603 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 25 | 1.000001095 | 1.000000500 | 0.000001008 |
| 26 | 1.000000570 | 1.000000260 | 0.000000525 |
| 27 | 1.000000297 | 1.000000135 | 0.000000273 |
| 28 | 1.000000155 | 1.000000071 | 0.000000143 |
| 29 | 1.000000081 | 1.000000037 | 0.000000074 |
| 30 | 1.000000042 | 1.000000019 | 0.000000039 |

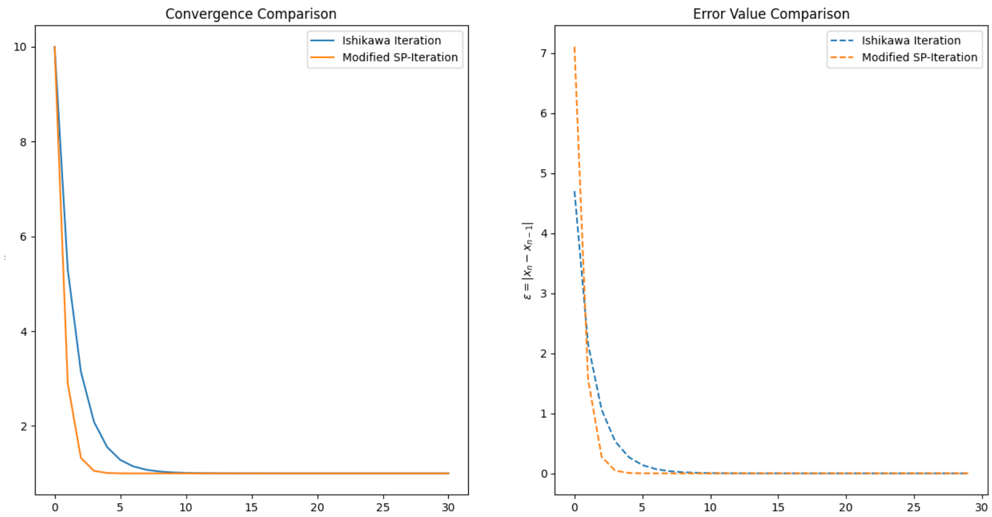


Figure 1: Comparison of the rate of convergence between using the Modified SP-iteration (3.4) and the Ishikawa iteration [17] for the given mapping in Example 4.1.

Table 2: The convergence of $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\epsilon$ with $x_1 = 10$ and $n = N = 30$ using the Modified SP-iteration (3.4).

| $n$ | $x_n$ | $y_n$ | $z_n$ | $\epsilon = \lvert x_n - x_{n-1}\rvert$ |
|---|---|---|---|---|
| 1 | 10.000000000 | 4.077441081 | 6.478331303 | - |
| 2 | 2.897332930 | 1.510225770 | 2.006429198 | 7.102667070 |
| 3 | 1.328544662 | 1.080980540 | 1.165704043 | 1.568788268 |
| 4 | 1.052884375 | 1.012460747 | 1.025982862 | 0.275660287 |
| 5 | 1.008194281 | 1.001878322 | 1.003961890 | 0.044689994 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 13 | 1.000000002 | 1.000000000 | 1.000000001 | 0.000000010 |
| 14 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000002 |
| 15 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 25 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 26 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 27 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 28 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 29 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |
| 30 | 1.000000000 | 1.000000000 | 1.000000000 | 0.000000000 |

*Remark* 4.2. In Example 4.1, by comparing the convergence behavior of the Modified SP-iteration (3.4) and the Ishikawa iteration (1.2) in the framework of convex metric spaces, we observe that:

1. Tables 1, 2, and Figure 1 show that $\{x_n\}$ converge to 1, where $1 \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. The convergence of $\{x_n\}$ of Example 4.1 can be guaranteed by Theorem 3.2;
2. in Tables 1, 2, and Figure 1, we see that the Modified SP-iteration (3.4) is faster than the Ishikawa iteration (1.2);
3. in Figures 1, we plot the error value of iterations, we see that the errors of the Modified SP-iteration (3.4) decrease faster than the Ishikawa iteration (1.2).

Next, we give the numerical example for supporting our main theorem.

*Example* 4.3. Let $\mathbb{R}$ be the set real numbers. Let $d(x,y) = |x-y|$, $\forall x,y \in \mathbb{R}$ and $W(x,y,\lambda) = \lambda x + (1-\lambda)y$, $\forall x,y \in \mathbb{R}$ and $\lambda \in [0,1]$. For every $i = 1,2,3,\dots,N$, let $T_i : [-2000,2000] \to [-2000,2000]$ be mapping defined by $T_i x = \frac{x}{2i}$, $\forall x \in [-2000,2000]$ and let $a_i = \frac{2}{3^i} + \frac{1}{N3^N}$. We can rewrite mapping $S$ in Theorem 3.2 as follows:

$$Sx = a_1T_1x + a_2T_2x + a_3T_3x + \cdots + a_NT_Nx.$$

Let $x_1 \in [-2000,2000]$ and $\{x_n\}$ be a sequence generated by (3.4), where $\alpha_n = \frac{3n-2}{9n}$, $\beta_n = \frac{2}{3n}$, and $\gamma_n = \frac{4}{5n}$. By the definition of $T_i$, we have $\{0\} = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. For every $n \in \mathbb{N}$, we can rewrite (3.4) as follows:

$$\begin{aligned} x_{n+1} &= \left(\frac{3n-2}{9n}\right) y_n + \left(\frac{6n+2}{9n}\right) Sy_n, \\ y_n &= \left(\frac{2}{3n}\right) z_n + \left(\frac{3n-2}{3n}\right) Sz_n, \\ z_n &= \left(\frac{4}{5n}\right) x_n + \left(\frac{5n-4}{5n}\right) Sx_n. \end{aligned}$$

It is obvious that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ and $\sum_{i=1}^{N} a_i$ satisfy all the conditions of Theorem 3.2. From Theorem 3.2, we can conclude that the sequence $\{x_n\}$ converges strongly to 0.

The following Table 3 and Figure 2, show the convergence of the sequences $\{x_n\}$ of the Modified SP-iteration (3.4), where $x_1 = 20$, $n = 15$, and $N = 40$.

Table 3: The convergence of $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\epsilon$ with $x_1 = 20$, $n = 15$, and $N = 40$ using the Modified SP-iteration (3.4).

| $n$ | $x_n$ | $y_n$ | $z_n$ | $\epsilon = \lvert x_n - x_{n-1}\rvert$ |
|---|---|---|---|---|
| 1 | 20.000000000 | 14.129590137 | 17.621860432 | - |
| 2 | 6.662448497 | 2.587103140 | 4.285813639 | 13.337551503 |
| 3 | 1.390785184 | 0.421688189 | 0.784413617 | 5.271663313 |
| 4 | 0.235978306 | 0.062433764 | 0.123740437 | 1.154806879 |
| 5 | 0.035625560 | 0.008644704 | 0.017833824 | 0.200352746 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 13 | 0.000000002 | 0.000000000 | 0.000000001 | 0.000000016 |
| 14 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000002 |
| 15 | 0.000000000 | 0.000000000 | 0.000000000 | 0.000000000 |

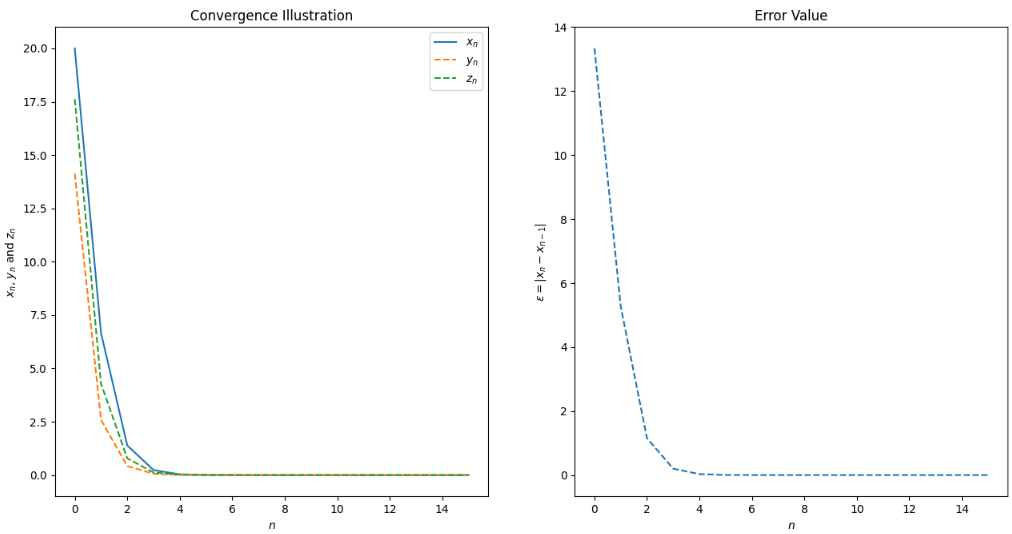


Figure 2: The convergence of $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ and error value with initial value $x_1 = 20$, $n = 15$, and $N = 40$ using the Modified SP-iteration (3.4).

*Remark* 4.4. In Table 3 and Figure 2, by testing the convergence behavior of the Modified SP-iteration (3.4), we observe that $\{x_n\}$ converges to 0, where $0 \in \bigcap_{i=1}^{N} \mathrm{Fix}\,(T_i)$. The convergence of $\{x_n\}$ of Example 4.3 can be guaranteed by Theorem 3.2.

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