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$\mathfrak{F}$-Bipolar metric spaces and fixed point theorems with applications

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Abstract

In this paper, we propose a new generalization of metric spaces by the unification of two novel notions, namely $\mathfrak{F}$-metric spaces and bipolar metric spaces, under the name $\mathfrak{F}$-bipolar metric spaces. Further, in this newly generalized notion we provide a binary topology and prove some fixed point results. As applications of our result, we prove the existence and uniqueness of solution of integral equation and the existence of a unique solution in homotopy theory. We also give some non-trivial examples to vindicate our claims. Our fixed point results extend several results in the existing literature.

Keywords: $\mathfrak{F}$-Bipolar metric spaces, fixed point, completeness.

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1. Introduction and Preliminaries

In 1906, metric space theory was initiated by Fréchet [6]. Since then, many authors have generalized the notion of metric spaces, by weakening some conditions and modifying the metric function (see, for instance, [1, 3–5, 10, 18]). In metric space and its generalizations we consider the distance between points of a single set, however the question of distance can arise between elements of two different sets. Such problems of measuring distance can be encountered in various fields of mathematics and other sciences. In 2016, to encounter such cases, the concept of bipolar metric space was introduced by Mutlu and Gürdal [11] and after that some fixed point and coupled fixed point theorems were tested under contractive conditions for covariant and contravariant mappings (see, for instance, [11–17]). Recently, Kishore et al. [9] proved some common fixed point theorems in bipolar metric spaces along with some applications.

Definition 1.1 ([11])**.** Let $X, Y \neq \phi$ and $d : X \times Y \to [0, +\infty)$ be a mapping. Then d is called a bipolar metric on $X \times Y$ if the following properties are satisfied for all $(x, y), (x', y') \in X \times Y$:

(B_1) $x = y$, if $d(x, y) = 0$;
(B_2) $d(x, y) = 0$, if $x = y$;
(B_3) $d(x, y) = d(y, x)$, if $x, y \in X \cap Y$;

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(B_4) $d(x,y) \leqslant d(x,y^{'}) + d(x^{'},y^{'}) + d(x^{'},y)$.

Then the triple (X,Y,d) is called a bipolar metric space.

Example 1.2. Let X be the class of all singleton subsets of $\mathbb{R}$ and Y be the class of all nonempty compact subsets of $\mathbb{R}$. We define $d : X \times Y \to \mathbb{R}$ as $d(x,A) = |x - \inf(A)| + |x - \sup(A)|$. Then the triple (X,Y,d) is a complete bipolar metric space.

Definition 1.3 ([11])**.** Let (X_1,Y_1,d_1) and (X_2,Y_2,d_2) be bipolar metric spaces. A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a covariant mapping, if $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$. Similarly, a function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a contravariant mapping, if $f(X_1) \subseteq Y_2$ and $f(X_2) \subseteq Y_1$. These mappings are denoted as $f : (X_1,Y_1,d_1) \rightrightarrows (X_2,Y_2,d_2)$ and $f : (X_1,Y_1,d_1) \rightleftarrows (X_2,Y_2,d_2)$, respectively.

Definition 1.4 ([11])**.** Let (X,Y,d) be a bipolar metric space. The points of the sets X, Y, and $X \cap Y$ are called left points, right points, and central points, respectively.

A sequence of left points is called a left sequence and a sequence of right points is called a right sequence. A sequence is a simple term for a left or a right sequence.

A sequence (a_n) is convergent to y, if $\lim\limits_{n\to\infty} d(a_n,y) = 0$, where (a_n) is a left sequence and y is a right point. Similarly, a sequence (b_n) is convergent to x, if $\lim\limits_{n\to\infty} d(x,b_n) = 0$, where (b_n) is a right sequence and x is a left point.

A sequence of the form (x_n,y_n) on the set $X \times Y$ is called a bisequence on (X,Y,d). A bisequence is called convergent, if both the left sequence (x_n) and the right sequence (y_n) converge. If (x_n) and (y_n) converge to a common point, then (x_n,y_n) is called biconvergent.

A bisequence (x_n,y_n) such that $\lim\limits_{n,m\to\infty} d(x_n,y_m) = 0$ is called a Cauchy bisequence. If every Cauchy bisequence converges in a bipolar metric space, then it is called a complete bipolar metric space.

In 2018, the concept of $\mathfrak{F}$-metric spaces was proposed by Jleli et al. [7]. To coin the notion of such abstract spaces the authors used a particular class of auxiliary functions which is defined below.

Definition 1.5 ([7])**.** Let $\mathfrak{F}$ be the set of functions $f : (0,\infty) \to \mathbb{R}$ satisfying the following:

(F_1) f is non-decreasing, i.e., $0 < s < t \Rightarrow f(s) \leqslant f(t)$;
(F_2) for every sequence $t_n \subset (0,\infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \Leftrightarrow \lim_{n\to\infty} f(t_n) = -\infty.$$

Using such functions, the authors generalized the notion of metric spaces and initiated the notion of $\mathfrak{F}$-metric spaces as follows:

Definition 1.6 ([7])**.** Let X be a non-empty set, and let $D : X \times X \to [0,\infty)$ be a given mapping. Suppose that there exists $(f,\alpha) \in \mathfrak{F} \times [0,\infty)$, such that for all $(x,y) \in X \times X$:

(a) $D(x,y) = 0 \Leftrightarrow x = y$;
(b) $D(x,y) = D(y,x)$;
(c) for every $N \in \mathbb{N}, N \geqslant 2$, and for every $(\mu_i)_{i=1}^{N} \subset X$ with $(\mu_1,\mu_N) = (x,y)$, we have

$$D(x,y) > 0 \Rightarrow f(D(x,y)) \leqslant f\left(\sum_{i=1}^{N-1} D(\mu_i,\mu_{i+1})\right) + \alpha.$$

Then D is said to be an $\mathfrak{F}$-metric on X, and the pair (X,D) is said to be an $\mathfrak{F}$-metric space.

The definitions of completeness, convergence, and Cauchy sequence in this setting can be found in [7]. In 2011, Jothi et al. [8] introduced the notion of binary topology to define a topology between two sets.

In this article, we unify the notions of $\mathfrak{F}$-metric spaces and bipolar metric spaces and introduce a new generalized notion named as, $\mathfrak{F}$-bipolar metric space. We also show that every bipolar metric space and $\mathfrak{F}$-metric space is an $\mathfrak{F}$-bipolar metric space but the converse is not true in general. Next, we define a topology $\tau_{\mathfrak{F}_b}$ on $\mathfrak{F}$-bipolar metric spaces using the concept of balls. Further, we give some fixed point theorems which are extensions and generalizations of the Banach contraction principle [2] in the setting of $\mathfrak{F}$-bipolar metric spaces, along with an application to integral equation.

2. $\mathfrak{F}$-bipolar metric space

Now we introduce the definition of $\mathfrak{F}$-bipolar metric space.

Definition 2.1. Let X and Y be two non-empty sets and $d : X \times Y \to [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathfrak{F} \times [0, \infty)$, such that for all $(x, y) \in X \times Y$:

(D_1) $d(x, y) = 0 \Leftrightarrow x = y$;
(D_2) $d(x, y) = d(y, x)$ if $x, y \in X \cap Y$;
(D_3) for every $N \in \mathbb{N}, N \geqslant 2$, and for every $(\mu_i)_{i=1}^N \subset X$ and $(\nu_i)_{i=1}^N \subset Y$ with $(\mu_1, \nu_N) = (x, y)$, we have

$$d(x, y) > 0 \Rightarrow f(d(x, y)) \leqslant f\left[\sum_{i=1}^{N-1} d(\mu_{i+1}, \nu_i) + \sum_{i=1}^{N} d(\mu_i, \nu_i)\right] + \alpha.$$

Then d is said to be an $\mathfrak{F}$-bipolar metric on the pair (X, Y) and (X, Y, d) is said to be an $\mathfrak{F}$-bipolar metric space.

Remark 2.2. Taking $Y = X$, $N = 2n$, $\mu_i = u_{2i-1}$, and $\nu_i = u_{2i}$ in the above definition we get a sequence $(u_i)_{i=1}^{2n} \in X$ with $(u_1, u_{2n}) = (x, y)$, such that condition (c) of Definition 1.6 holds. Thus every $\mathfrak{F}$-metric space is an $\mathfrak{F}$-bipolar metric space but the converse is not true in general.

Remark 2.3. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Throughout this paper, we shall use Definitions 1.3 and 1.4 in a similar way as used in bipolar metric spaces.

Example 2.4. Let $X = \{1, 2\}$ and $Y = \{2, 7\}$. Define a metric $d : X \times Y \to [0, \infty)$ by

$$d(1, 7) = 10,\ d(1, 2) = 6,\ d(2, 7) = 2,\ d(2, 2) = 0.$$

(X, Y, d) is not a bipolar metric space (since $10 = d(1, 7) > d(1, 2) + d(2, 2) + d(2, 7) = 8$, therefore, ($D_3$) is not satisfied). It can be easily seen that d satisfies (D_1) and (D_2). Now, we consider only three cases for (D_3).

Case-I:

$$d(1, 2) > 0 \Rightarrow \ln(d(1, 2)) \leqslant \ln(d(1, 7) + d(2, 7) + d(2, 2)) = \ln 6 \leqslant \ln 12,$$

(D_3) is satisfied with $\alpha = 0$ and $f(t) = \ln t \in \mathfrak{F}$.

Case-II:

$$d(2, 7) > 0 \Rightarrow \ln(d(2, 7)) \leqslant \ln(d(2, 2) + d(1, 2) + d(1, 7)) = \ln 2 \leqslant \ln 16,$$

(D_3) is satisfied with $\alpha = 0$ and $f(t) = \ln(t) \in \mathfrak{F}$.

Case-III:

$$d(1, 7) > 0 \Rightarrow \ln(d(1, 7)) \leqslant \ln(d(1, 2) + d(2, 2) + d(2, 7)) = \ln 10 \leqslant \ln 8 + \alpha,$$

(D_3) is satisfied with $\alpha > 1$ and $f(t) = \ln(t) \in \mathfrak{F}$.

Therefore, d satisfies all the properties of an $\mathfrak{F}$-bipolar metric. Hence, (X, Y, d) is an $\mathfrak{F}$-bipolar metric space.

Remark 2.5. The above example shows that an $\mathfrak{F}$-bipolar metric space need not to be a bipolar metric space. Clearly, $\mathfrak{F}$-bipolar metric space is a generalization of bipolar metric space.

3. Topology on $\mathfrak{F}$-bipolar metric spaces

Definition 3.1. Let X and Y be any two non-empty sets. A binary topology from X to Y is a binary structure $M \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$ that satisfies:

(a) (ϕ, ϕ) and $(X, Y) \in M$;
(b) $(A_1) \cap A_2, B_1 \cap B_2) \in M$ whenever $(A_1, B_1), (A_2, B_2) \in M$;
(c) if $\{(A_\alpha, B_\alpha) : \alpha \in \Delta\}$ is a family of members of M, then $(\cup_{\alpha\in\Delta} A_\alpha, \cup_{\alpha\in\Delta} B_\alpha) \in M$.

Definition 3.2. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Let $x \in X$ be an arbitrary element. Then for $r > 0$, $B_R(x, r) = \{y \in Y : d(x, y) < r\}$ is called a **right ball** in (X, Y, d).

Definition 3.3. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Let $y \in Y$ be an arbitrary element. Then for $r > 0$, $B_L(y, r) = \{x \in X : d(x, y) < r\}$ is called a **left ball** in (X, Y, d).

Definition 3.4. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. A subset P of Y is said to be **right $\mathfrak{F}_b$-open** if for every $y \in P$, there is some $r > 0$, such that:

$$y \in B_R(x, r) \subseteq P.$$

Definition 3.5. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. A subset O of X is said to be **left $\mathfrak{F}_b$-open** if for every $x \in O$, there is some $r > 0$, such that:

$$x \in B_L(y, r) \subseteq O.$$

Definition 3.6. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Let $A \subseteq X$ and $B \subseteq Y$. Then $A \times B$ is called **$\mathfrak{F}_b$-open** if A is left $\mathfrak{F}_b$-open and B is right $\mathfrak{F}_b$-open.

We say that a subset $C \times D$ of $X \times Y$ is $\mathfrak{F}_b$-closed if $(X\backslash C) \times (Y\backslash D)$ is $\mathfrak{F}_b$-open. Let us denote by $\tau_{\mathfrak{F}_b}$ the family of all $\mathfrak{F}_b$-open subsets of $X \times Y$. Then it is easy to prove the following result.

Proposition 3.7. *Let (X, Y, d) be a $\mathfrak{F}$-bipolar metric space. Then $\tau_{\mathfrak{F}_b}$ is a binary topology on $X \times Y$.*

Proposition 3.8. *Let (X, Y, d) be a $\mathfrak{F}$-bipolar metric space. Then for any non-empty subset $A \times B$ of $X \times Y$, the following statements are equal.*

(a) *$A \times B$ is $\mathfrak{F}_b$-closed.*
(b) *For any sequence $(x_n, y_n) \in A \times B$, we have*

$$\lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(x, y_n) = 0\,, \ (x, y) \in X \times Y \quad \Longrightarrow \ (x, y) \in A \times B. \tag{3.1}$$

Proof. Assume that $A \times B$ is $\mathfrak{F}_b$-closed and equation (3.1) holds. Since, $A \times B$ is $\mathfrak{F}_b$-closed $\Longrightarrow$ $(X \setminus A) \times (Y \setminus B)$ is $\mathfrak{F}_b$-open, i.e., $X \setminus A$ is left $\mathfrak{F}_b$-open and $Y \setminus B$ is right $\mathfrak{F}_b$-open. So, for every $x \in X \setminus A$ there exists some $r_1 > 0$ and some $y \in Y$, such that $B_L(y, r_1) \subseteq X \setminus A$.

Similarly for every $y \in Y \setminus B$ there exists some $r_2 > 0$ and some $x \in X$, such that $B_R(x, r_2) \subseteq Y \setminus B$.

$$\Longrightarrow \ B_L(y, r_1) \cap A = B_R(x, r_2) \cap B = \phi.$$

On the other hand by equation (3.1), there exists some $N_1, N_2 \in \mathbb{N}$, such that $d(x_n, y) < r_1$ for all $n \geqslant N_1$ and $d(x, y_n) < r_2$ for all $n \geqslant N_2$. Taking $N = \max\{N_1, N_2\}$, we get $x_n \in B_L(y, r_1)$ and $y_n \in B_R(x, r_2)$ for all $n \geqslant N$, which implies that $(x_N, y_N) \in [B_L(y, r_1) \cap A] \times [B_R(x, r_2) \cap B]$, which is a contradiction. Hence, we deduce that $(x, y) \in A \times B$, i.e. (a) $\Longrightarrow$ (b).

Conversely, assume that (b) is satisfied. Let $(x, y) \in (X \setminus A) \times (Y \setminus B)$. We have to prove that there are some $r_1, r_2 > 0$, such that

$$B_L(y, r_1) \subset X \setminus A \text{ and } B_R(x, r_2) \subset Y \setminus B.$$

On the contrary, suppose that for every $r_1, r_2 > 0$, there exists $x_{r_1} \in B_L(y, r_1) \cap A$ and $y_{r_2} \in B_R(x, r_2) \cap B$. $\Longrightarrow$ for any $n \in \mathbb{N}^*$, there exists $x_n \in B_L(y, \frac{1}{n}) \cap A$ and $y_n \in B_R(x, \frac{1}{n}) \cap B$. Then, $(x_n, y_n) \in A \times B$ and $\lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(x, y_n) = 0$. By (b), this implies that $(x, y) \in (A \times B)$ which is a contradiction to $(x, y) \in (X \setminus A) \times (Y \setminus B)$. Hence, $A \times B$ is $\mathfrak{F}_b$-closed, i.e., (b) $\Longrightarrow$ (a). $\square$

Definition 3.9. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Let $\{x_n\}$ be a left sequence in X. We say that $\{x_n\}$ is convergent to $y \in Y$ if for every right $\mathfrak{F}_b$-open subset O_y of Y containing y, there exists some $N \in \mathbb{N}$ such that $x_n \in O_y$, for all $n \geqslant N$. In this case, we say that y is the limit of $\{x_n\}$.

Definition 3.10. Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Let $\{y_n\}$ be a right sequence in Y. We say that $\{y_n\}$ is convergent to $x \in X$ if for every left $\mathfrak{F}_b$-open subset P_x of X containing x, there exists some $N \in \mathbb{N}$ such that $y_n \in P_x$, for all $n \geqslant N$. In this case, we say that x is the limit of $\{y_n\}$.

The following result follows immediately from Definition 3.9 and the definition of $\tau_{\mathfrak{F}_b}$.

Proposition 3.11. *Let (X, Y, d) be an $\mathfrak{F}$-bipolar metric space. Let $\{x_n\}$ be a left (or right) sequence in X and $y \in Y$. Then the following conditions are equivalent.*

(1) *$\{x_n\}$ is $\mathfrak{F}_b$-convergent to y;*
(2) $\lim\limits_{n\to\infty} d(x_n, y) = 0$.

Similarly, we can do for a right sequence.

Remark 3.12. In $\mathfrak{F}$-bipolar metric space a convergent sequence may have multiple limits just as in bipolar metric spaces.

Definition 3.13. In an $\mathfrak{F}$-bipolar metric space a bisequence (x_n, y_n) is said to be Cauchy bisequence, if for each $\epsilon > 0$, there exists a number $n_0 \in \mathbb{N}$, such that for all positive integers $n, m \geqslant n_0$, $d(x_n, y_m) < \epsilon$.

Definition 3.14. An $\mathfrak{F}$-bipolar metric space (X, Y, d) is called complete, if every Cauchy bisequence in this space is convergent.

4. Fixed point theorems

Theorem 4.1. *Let (X, Y, d) be a complete $\mathfrak{F}$-bipolar metric space and let $g : (X, Y, d) \rightrightarrows (X, Y, d)$ be a covariant contraction. Then the mapping $g : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Given that g is a covariant contraction, then there exists $k \in (0, 1)$ such that

$$d(g(x), g(y)) \leqslant kd(x, y), \quad \forall (x, y) \in X \times Y. \tag{4.1}$$

Let $x_0 \in X$ and $y_0 \in Y$. For each $n \in \mathbb{N}$, define $g(x_n) = x_{n+1}$ and $g(y_n) = y_{n+1}$. Then (x_n, y_n) is a bisequence in (X, Y, d). Let $(f, \alpha) \in \mathfrak{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \implies f(t) < f(\epsilon) - \alpha. \tag{4.2}$$

Now,

$$d(x_n, y_n) = d(g(x_{n-1}), g(y_{n-1})) \leqslant kd(x_{n-1}, y_{n-1}) \leqslant \cdots \leqslant k^n d(x_0, y_0). \tag{4.3}$$

Also,

$$d(x_{n+1}, y_n) = d(g(x_n), g(y_{n-1})) \leqslant kd(x_n, y_{n-1}) \leqslant \cdots \leqslant k^n d(x_1, y_0). \tag{4.4}$$

From (4.3) and (4.4), we get

$$\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i) \leqslant \frac{k^n}{1-k} [d(x_0, y_0) + d(x_1, y_0)], \text{ where } m > n.$$

Since $\lim\limits_{n\to\infty} \dfrac{k^n}{1-k} [d(x_0, y_0) + d(x_1, y_0)] = 0$, there exists $N \in \mathbb{N}$, such that

$$0 < \frac{k^n}{1-k} [d(x_0, y_0) + d(x_1, y_0)] < \delta, \quad n \geqslant N.$$

For $m > n \geqslant N$, using $(\mathfrak{F}_1)$ and equation (4.2), we get

$$f\left(\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i)\right) \leqslant f\left(\frac{k^n}{1-k}[d(x_0, y_0) + d(x_1, y_0)]\right) < f(\epsilon) - \alpha. \tag{4.5}$$

From (D_3) and equation (4.5), we get $d(x_n, y_m) > 0$. Therefore

$$f(d(x_n, y_m)) \leqslant f\left(\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i)\right) + \alpha < f(\epsilon).$$

Similarly, for $n > m \geqslant N$, $d(x_n, y_m) > 0$, we have

$$f(d(x_n, y_m)) \leqslant f\left(\sum_{i=m}^{n-1} d(x_{i+1}, y_i) + \sum_{i=m}^{n} d(x_i, y_i)\right) + \alpha < f(\epsilon).$$

Then by $(\mathfrak{F}_1)$, $d(x_n, y_m) < \epsilon$, for all $m, n \geqslant N$. Therefore, (x_n, y_n) is a Cauchy bisequence. Since, (X, Y, d) is complete, (x_n, y_n) converges and thus biconverges to a point $\mu \in X \cap Y$ and $g(y_n) = y_{n+1} \to \mu \in X \cap Y$ guarantees that $\langle g(y_n) \rangle$ has a unique limit. Also continuity of $g \implies \langle g(y_n) \rangle \to g(\mu)$. Thus, $g(\mu) = \mu$, i.e., μ is a fixed point of g. If ν is any fixed point of g, then $g(\nu) = \nu \implies \nu \in X \cap Y$ and by equation (4.1) we have $d(\mu, \nu) = d(g(\mu), g(\nu)) \leqslant kd(\mu, \nu)$, where $0 < k < 1$. Therefore

$$d(\mu, \nu) = 0.$$

Thus, $\mu = \nu$, i.e., g has a unique fixed point. $\square$

Example 4.2. Let $X = \mathbb{N} \cup \{0\}$ and $Y = \frac{1}{n} \cup \{0\}$. Define $d : X \times Y \to [0, \infty]$ as

$$d(x, y) = \begin{cases} 2, & \text{if } (x, y) = (2, 1), \\ |x - y|, & \text{otherwise.} \end{cases}$$

Then the triple (X, Y, d) is an $\mathfrak{F}$-bipolar metric space for $f(t) = \ln t$ and $\alpha > 2$. The covariant mapping $g : X \cup Y \to X \cup Y$, defined as

$$g(z) = \begin{cases} 0, & z \in X \backslash \{0, 1\}, \\ 1, & z \in Y, \end{cases}$$

satisfies the inequality $d(g(x), g(y)) \leqslant kd(x, y)$, for some $k \in (0, 1)$. Hence, by Theorem 4.1, g must have a unique fixed point, which is $12 \in X \cap Y$.

Theorem 4.3. *Let (X, Y, d) be a complete $\mathfrak{F}$-bipolar metric space and let $g : (X, Y, d) \rightleftarrows (X, Y, d)$ be a contravariant contraction. Then the mapping $g : X \cup Y \to X \cup Y$ has a unique fixed point.*

Proof. Given that g is a contravariant contraction $\Rightarrow$ there exists $k \in (0, 1)$ such that

$$d(g(y), g(x)) \leqslant kd(x, y), \quad \forall (x, y) \in X \times Y. \tag{4.6}$$

Let $x_0 \in X$ and for each $n \in \mathbb{N}$, define $g(x_n) = y_n$ and $g(y_n) = x_{n+1}$. Then (x_n, y_n) is a bisequence in (X, Y, d).
Let $(f, \alpha) \in \mathfrak{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \implies f(t) < f(\epsilon) - \alpha. \tag{4.7}$$

Now,

$$\begin{aligned} d(x_n, y_n) &= d(g(y_{n-1}), g(x_n)) \\ &\leqslant kd(x_n, y_{n-1}) \\ &= kd(g(y_{n-1}), g(x_{n-1})) \\ &\leqslant k^2 d(x_{n-1}, y_{n-1}) \\ &\vdots \\ &\leqslant k^{2n} d(x_0, y_0), \quad n \in \mathbb{N}. \end{aligned} \tag{4.8}$$

Also,

$$d(x_{n+1}, y_n) = d(g(y_n), g(x_n)) \leqslant kd(x_n, y_n) \leqslant \cdots \leqslant k^{2n+1} d(x_0, y_0), \quad n \in \mathbb{N}. \tag{4.9}$$

From (4.8) and (4.9), we get

$$\begin{aligned} \sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i) &\leqslant (k^{2n} + k^{2n+2} + \ldots + k^{2m}) d(x_0, y_0) + (k^{2n+1} + k^{2n+3} + \cdots + k^{2m-1}) d(x_0, y_0) \\ &\leqslant k^{2n} \sum_{n=0}^{\infty} k^n d(x_0, y_0) = \frac{k^{2n}}{1-k} d(x_0, y_0), \quad m > n. \end{aligned}$$

Since, $\lim_{n\to\infty} \frac{k^{2n}}{1-k} d(x_0, y_0) = 0 \implies$ there exists $N \in \mathbb{N}$, such that

$$0 < \frac{k^{2n}}{1-k} d(x_0, y_0) < \delta, \quad n \geqslant N.$$

For $m > n \geqslant N$, using $(\mathfrak{F}_1)$ and equation (4.7), we get

$$f\left(\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i)\right) \leqslant f\left(\frac{k^{2n}}{1-k} d(x_0, y_0)\right) < f(\epsilon) - \alpha. \tag{4.10}$$

From (D_3) and equation (4.10), we get that $d(x_n, y_m) > 0$ implies

$$f(d(x_n, y_m)) \leqslant f\left(\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i)\right) + \alpha < f(\epsilon).$$

Similarly, for $n > m \geqslant N$, $d(x_n, y_m) > 0$ implies

$$f(d(x_n, y_m)) \leqslant f\left(\sum_{i=m}^{n-1} d(x_{i+1}, y_i) + \sum_{i=m}^{n} d(x_i, y_i)\right) + \alpha < f(\epsilon).$$

Then by $(\mathfrak{F}_1)$, $d(x_n, y_m) < \epsilon$, for all $m, n \geqslant N$. Therefore, (x_n, y_n) is a Cauchy bisequence. Since, (X, Y, d) is complete, (x_n, y_n) converges and thus biconverges to a point $\mu \in X \cap Y$. So $\langle x_n \rangle \to \mu$, $\langle y_n \rangle \to \mu$ and the contravariant map g is continuous, we get

$$\langle x_n \rangle \to \mu \text{ implies that } \langle y_n \rangle = \langle g(x_n) \rangle \to g(u)$$

and combining the above with $\langle y_n \rangle \to \mu$ gives

$$g(\mu) = \mu.$$

Now to show the uniqueness of the fixed point suppose ν is another fixed point of g, i.e., $g(\nu) = \nu$, which implies that $\nu \in X \cap Y$. By equation (4.6) and (D_3), we get

$$d(\mu, \nu) = d(g(\mu), g(\nu)) \leqslant kd(\nu, \mu) = kd(\mu, \nu), \text{ which implies that } d(\mu, \nu) = 0,$$

which gives that $\mu = \nu$. Therefore, g has a unique fixed point. □

Example 4.4. Let $X = \{7, 8, 17, 19\}$ and $Y = \{2, 4, 9, 17\}$. Define $d : X \times Y \to [0, \infty]$ as the usual metric, i.e., $d(x, y) = |x - y|$. Then the triple (X, Y, d) is an $\mathfrak{F}$-bipolar metric space. The contravariant mapping $g : X \cup Y \to X \cup Y$, defined as

$$g(z) = \begin{cases} 17, & z \in X \cup \{9\}, \\ 19, & \text{otherwise}, \end{cases}$$

satisfies the inequality $d(g(y), g(x)) \leqslant kd(x, y)$, for some $k \in (0, 1)$. Hence, by Theorem 4.3, g must have a unique fixed point, which is $17 \in X \cap Y$.

Theorem 4.5. *Let (X, Y, d) be a complete $\mathfrak{F}$-bipolar metric space and given two contravariant contractions $F, G : (X, Y, d) \rightleftarrows (X, Y, d)$ satisfying*

$$d(Fy, Gx) \leqslant k[d(x, Gx) + d(Fy, y)], \textit{ for all } (x, y) \in X \times Y,$$

where $k \in (0, \frac{1}{2})$. Then the mapping $F, G : X \cup Y \to X \cup Y$ has a unique common fixed point.

Proof. Let $x_0 \in X$ and $y_0 \in Y$, then for each non-negative integer n, define

$$Fx_{2n} = y_{2n}, \;\; Gx_{2n+1} = y_{2n+1}, \;\; Fy_{2n} = x_{2n+1}, \;\; Gy_{2n+1} = x_{2n+2}.$$

Then (x_n, y_n) is a bisequence in (X, Y, d). Let $(f, \alpha) \in \mathfrak{F} \times [0, \infty)$ be such that (D_3) is satisfied. Let $\epsilon > 0$ be fixed. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \implies f(t) < f(\epsilon) - \alpha. \tag{4.11}$$

Now,

$$\begin{aligned} d(x_{2n+1}, y_{2n+1}) = d(Fy_{2n}, Gx_{2n+1}) &\leqslant k[d(x_{2n+1}, Gx_{2n+1}) + d(Fy_{2n}, y_{2n})] \\ &\leqslant k[d(x_{2n+1}, y_{2n+1}) + d(x_{2n+1}, y_{2n})], \\ \text{Hence, } d(x_{2n+1}, y_{2n+1}) &\leqslant \frac{k}{1-k} d(x_{2n+1}, y_{2n}). \end{aligned}$$

Also,

$$\begin{aligned} d(x_{2n+1}, y_{2n}) = d(Fy_{2n}, Fx_{2n}) \leqslant k[d(x_{2n}, Fx_{2n}) + d(Fy_{2n}, y_{2n})] \leqslant k[d(x_{2n}, y_{2n}) + d(x_{2n+1}, y_{2n})], \\ \text{So, } d(x_{2n+1}, y_{2n}) \leqslant \frac{k}{1-k} d(x_{2n}, y_{2n}). \end{aligned}$$

Since $k \in (0, \frac{1}{2})$ and $\frac{k}{1-k} = \lambda$ (say), then $\lambda \in (0, 1)$. So, $d(x_{2n+1}, y_{2n+1}) \leqslant \lambda^{4n+2} d(x_0, y_0)$ and $d(x_{2n+1}, y_{2n}) \leqslant \lambda^{4n+1} d(x_0, y_0)$. Now, we can get that for any $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1}, y_{n+1}) \leqslant \lambda^{2n+2} d(x_0, y_0) \text{ and } d(x_{n+1}, y_n) &\leqslant \lambda^{2n+1} d(x_0, y_0) \\ \implies d(x_n, y_n) &\leqslant \lambda^{2n} d(x_0, y_0), \qquad (4.12) \\ \text{and } d(x_{n+1}, y_n) &\leqslant \lambda^{2n+1} d(x_0, y_0). \qquad (4.13) \end{aligned}$$

From (4.12) and (4.13), we get

$$\begin{aligned} \sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i) &= (\lambda^{2n+1} + \lambda^{2n+3} + \ldots + \lambda^{2m-1}) d(x_0, y_0) + (\lambda^{2n} + \lambda^{2n+2} + \cdots + \lambda^{2m}) d(x_0, y_0) \\ &= \lambda^{2n}[1 + \lambda + \lambda^2 + \cdots + \lambda^{2m-2n-1}] d(x_0, y_0) \\ &\leqslant \frac{\lambda^{2n}}{1-\lambda} d(x_0, y_0), \quad m > n. \end{aligned}$$

Since, $\lim_{n\to\infty} \frac{\lambda^{2n}}{1-\lambda} d(x_0, y_0) = 0$, there exists $N \in \mathbb{N}$, such that

$$0 < \frac{\lambda^{2n}}{1-\lambda} d(x_0, y_0) < \delta\,, \;\; n \geqslant N.$$

For $m > n \geqslant N$, using $(\mathfrak{F}_1)$ and equation (4.11), we get

$$f\left(\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i)\right) \leqslant f\left(\frac{\lambda^{2n}}{1-\lambda} d(x_0, y_0)\right) < f(\epsilon) - \alpha. \tag{4.14}$$

From (D_3) and equation (4.14), we get $d(x_n, y_m) > 0$

$$\implies f(d(x_n, y_m)) \leqslant f\left(\sum_{i=n}^{m-1} d(x_{i+1}, y_i) + \sum_{i=n}^{m} d(x_i, y_i)\right) + \alpha < f(\epsilon).$$

Similarly, for $n > m \geqslant N$, $d(x_n, y_m) > 0$

$$\implies f(d(x_n, y_m)) \leqslant f\left(\sum_{i=m}^{n-1} d(x_{i+1}, y_i) + \sum_{i=m}^{n} d(x_i, y_i)\right) + \alpha < f(\epsilon).$$

Then by $(\mathfrak{F}_1)$, $d(x_n, y_m) < \epsilon$, for all $m, n \geqslant N$. Therefore, (x_n, y_n) is a Cauchy bisequence. Since, (X, Y, d) is complete, the bisequence converges and thus biconverges to some $\mu \in X \cap Y$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \mu$. $\langle Fx_{2n} \rangle = \langle y_{2n} \rangle \to \mu \in X \cap Y$ implies that $\langle Fx_{2n} \rangle$ has a unique limit μ. $\langle x_n \rangle \to \mu$ implies that $\langle x_{2n} \rangle \to \mu$. Now, continuity of F implies that $\langle Fx_{2n} \rangle \to F\mu$. Therefore $F\mu = \mu$.

Similarly, $\langle Gy_{2n+1} \rangle = \langle x_{2n+1} \rangle \to \mu \in X \cap Y$ (since $\langle x_n \rangle \to \mu$) $\implies$ $\langle Gy_{2n+1} \rangle$ has a unique limit. $\langle y_n \rangle \to \mu$ implies that $\langle y_{2n+1} \rangle \to \mu$. Now, continuity of G implies that $\langle Gx_{2n+1} \rangle \to G\mu$. Therefore $G\mu = \mu$. So, $F\mu = G\mu = \mu$, i.e., G and F have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. Let $\nu \in X \cup Y$ be another common fixed point of F and G such that $F\nu = G\nu = \nu \in X \cap Y$, then

$$d(\mu, \nu) = d(F\mu, G\nu) \leqslant k[d(\nu, G\nu) + d(F\mu, \mu)] = k[d(\nu, \nu) + d(\mu, \mu)] = 0,$$

that implies $d(\mu, \nu) = 0$, i.e., $\mu = \nu$. Therefore, F and G have a unique common fixed point. □

Corollary 4.6. *Let (X, Y, d) be a complete $\mathfrak{F}$-bipolar metric space and given a contravariant contraction $F : (X, Y, d) \rightleftarrows (X, Y, d)$ satisfying*

$$d(Fy, Fx) \leqslant k[d(x, Fx) + d(Fy, y)], \textit{ for all } (x, y) \in X \times Y,$$

where $k \in (0, \frac{1}{2})$. Then the mapping $F : X \cup Y \to X \cup Y$ has a unique fixed point.

Proof. By taking $F = G$ in Theorem 4.5, we get that F has a unique fixed point. □

5. Applications

5.1. Integral equations

We prove the existence and uniqueness of solution of integral equation.

Theorem 5.1. *Let us consider the integral equation*

$$\phi(x) = f(x) + \int_{X\cup Y} P(x, y, \phi(y))dy\,, \textit{ where } x \in X \cup Y,$$

where $X \cup Y$ is a Lebesgue measurable set. Suppose that:

(i) $P : (X^2 \cup Y^2) \times [0, \infty) \to [0, \infty)$ *and* $f \in L^\infty(X) \cup L^\infty(Y)$;

(ii) *there is a continuous function* $\gamma : X^2 \cup Y^2 \to [0, \infty)$ *such that*

$$|P(x, y, \phi(y)) - P(x, y, \psi(y))| \leqslant k.\gamma(x, y)|\phi(y) - \psi(y)|, \text{ for } (x, y) \in X^2 \cup Y^2;$$

(iii) $\| \int_{X\cup Y} \gamma(x, y)dy\| \leqslant 1$, *i.e.* $\sup_{x\in X\cup Y} \int_{X\cup Y} |\gamma(x, y)|dy \leqslant 1$.

Then the integral equation has a unique solution in $L^\infty(X) \cup L^\infty(Y)$.

Proof. Let $A = L^\infty(X)$ and $B = L^\infty(Y)$ be two normed linear spaces, where X, Y are Lebesgue measurable set and $m(X \cup Y) < \infty$. Consider $d : A \times B \to [0, \infty)$ to be defined by $d(g, h) = \|g - h\|_\infty$, for all $g, h \in A \times B$. Then (A, B, d) is a complete $\mathfrak{F}$-bipolar metric space. Define the covariant mapping $I : L^\infty(X) \cup L^\infty(Y) \to L^\infty(X) \cup L^\infty(Y)$ by

$$I(\phi(x)) = \int_{X\cup Y} P(x, y, \phi(y))dy + f(x) \text{ , where } x \in X \cup Y.$$

Now, we have

$$\begin{aligned}
d(I(\phi(x)), I(\psi(x))) &= \|I(\phi(x)) - I(\psi(x))\| \\
&= \left| \int_{X\cup Y} P(x, y, \phi(y))dy - \int_{X\cup Y} P(x, y, \psi(y))dy \right| \\
&\leqslant \int_{X\cup Y} |P(x, y, \phi(y)) - P(x, y, \psi(y))|dy \\
&\leqslant k \int_{X\cup Y} \gamma(x, y)|\phi(y) - \psi(y)|dy \\
&\leqslant k\|\phi(x) - \psi(x)\| \int_{X\cup Y} \gamma(x, y)dy \\
&\leqslant k\|\phi(x) - \psi(x)\| \sup_{x\in X\cup Y} \int_{X\cup Y} \gamma(x, y)dy \\
&\leqslant k\|\phi(x) - \psi(x)\| \\
&= k.d(\phi(x) - \Psi(x)).
\end{aligned}$$

Therefore it follows from Theorem 4.1 that I has a unique fixed point in $A \cup B$. □

5.2. Homotopy

Now, we study the existence of a unique solution in homotopy theory.

Theorem 5.2. *Let* $(\mathcal{C}, \mathcal{D}, d)$ *be a complete* $\mathfrak{F}$*-bipolar metric space,* $(\mathcal{A}, \mathcal{B})$ *be an open subset of* $(\mathcal{C}, \mathcal{D})$ *and* $(\overline{\mathcal{A}}, \overline{\mathcal{B}})$ *be a closed subset of* $(\mathcal{C}, \mathcal{D})$ *s.t.* $(\mathcal{A}, \mathcal{B}) \subseteq (\overline{\mathcal{A}}, \overline{\mathcal{B}})$. *Suppose* $\mathcal{H} : (\overline{\mathcal{A}} \cap \overline{\mathcal{B}}) \times [0, 1] \to \mathcal{C} \cup \mathcal{D}$ *be an operator satisfying the following conditions:*

(i) $x \neq \mathcal{H}(x, \kappa)$ *for each* $x \in \partial\mathcal{A} \cup \partial\mathcal{B}$ *and* $\kappa \in [0, 1]$ *(here* $\partial\mathcal{A} \cup \partial\mathcal{B}$ *is the boundary of* $\mathcal{A} \cup \mathcal{B}$ *in* $\mathcal{C} \cup \mathcal{D}$*);*

(ii) $d(\mathcal{H}(x, \kappa), \mathcal{H}(y, \kappa)) \leqslant \alpha d(x, y)$ *for all* $x \in \overline{\mathcal{A}}, y \in \overline{\mathcal{B}}, \kappa \in [0, 1]$ *and* $\alpha \in (0, 1)$*;*

(iii) *there exists* $M \geqslant 0$ *s.t.* $d(\mathcal{H}(x, \rho), \mathcal{H}(y, \sigma)) \leqslant M|\rho - \sigma|$ *for all* $x \in \overline{\mathcal{A}}, y \in \overline{\mathcal{B}}$ *and* $\rho, \sigma \in [0, 1]$.

Then $\mathcal{H}(., 0)$ *has a fixed point iff* $\mathcal{H}(., 1)$ *has a fixed point.*

Proof. Let $\mathcal{X} = \{\rho \in [0, 1] : x = \mathcal{H}(x, \rho) \text{ f.s } x \in \mathcal{A}\}$, $\mathcal{Y} = \{\sigma \in [0, 1] : y = \mathcal{H}(y, \sigma) \text{ f.s } y \in \mathcal{B}\}$. Since $\mathcal{H}(., 0)$ has a fixed point in $\mathcal{A} \cup \mathcal{B}$, we have $0 \in \mathcal{X} \cap \mathcal{Y}$. Thus, $\mathcal{X} \cap \mathcal{Y}$ is a non-empty set. Now, we'll show $\mathcal{X} \cap \mathcal{Y}$ is both closed and open in $[0, 1]$ and so by connectedness $\mathcal{X} = \mathcal{Y} = [0, 1]$. Let $(\{\rho_n\}_{n=1}^\infty, \{\sigma_n\}_{n=1}^\infty) \subseteq (\mathcal{X}, \mathcal{Y})$

with $(\rho_n, \sigma_n) \to (\rho, \sigma) \in [0,1]$ as $n \to \infty$. We must show $\rho = \sigma \in \mathcal{X} \cap \mathcal{Y}$. Since $(\rho_n, \sigma_n) \in (\mathcal{X}, \mathcal{Y})$ for $n = 1, 2, 3, \ldots$, there exists $(x_n, y_n) \in (\mathcal{A}, \mathcal{B})$ such that $x_{n+1} = \mathcal{H}(x_n, \rho_n)$ and $y_{n+1} = \mathcal{H}(y_n, \sigma_n)$. Now,

$$d(x_{n+1}, y_n) = d(\mathcal{H}(x_n, \rho_n), \mathcal{H}(y_{n-1}, \sigma_{n-1})) \leqslant \alpha d(x_n, y_{n-1}) \leqslant \cdots \leqslant \alpha^n d(x_1, y_0).$$

Also,

$$d(x_n, y_n) = d(\mathcal{H}(x_{n-1}, \rho_{n-1}), \mathcal{H}(y_{n-1}, \sigma_{n-1})) \leqslant \alpha d(x_{n-1}, y_{n-1}) \leqslant \cdots \leqslant \alpha^n d(x_0, y_0).$$

By the same process as used in Theorem 4.1 we can easily prove that (x_n, y_n) is a Cauchy bisequence in $(\mathcal{A}, \mathcal{B})$. By completeness there exists $\mu \in \mathcal{A}$ and $\nu \in \mathcal{B}$ such that

$$\lim_{n\to\infty} x_n = \nu \text{ and } \lim_{n\to\infty} y_n = \mu. \tag{5.1}$$

Consider

$$\begin{aligned} d(\mathcal{H}(\mu, \rho), \nu) &\leqslant d(\mathcal{H}(\mu, \rho), y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, \nu) \\ &\leqslant d(\mathcal{H}(\mu, \rho), \mathcal{H}(y_n, \sigma_n)) + d(\mathcal{H}(x_n, \rho_n), \mathcal{H}(y_n, \sigma_n)) + d(x_{n+1}, \nu) \\ &\leqslant \alpha d(\mu, y_n) + M|\rho_n - \sigma_n| + d(x_{n+1}, \nu) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $\mathcal{H}(\mu, \rho) = \nu$. Similarly, we get $\mathcal{H}(\nu, \sigma) = \mu$. Now, from equation (5.1) we get

$$d(\mu, \nu) = d(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = \lim_{n\to\infty} d(x_n, y_n) = 0.$$

So, $\mu = \nu$. Thus $\rho = \sigma \in \mathcal{X} \cap \mathcal{Y}$ which further implies that $\mathcal{X} \cap \mathcal{Y}$ is closed in $[0,1]$. Now, we have to prove that $\mathcal{X} \cap \mathcal{Y}$ is open in $[0,1]$. Let $(\rho_0, \sigma_0) \in (\mathcal{X}, \mathcal{Y})$, then there exists a bisequence (x_0, y_0) such that

$$x_0 = \mathcal{H}(x_0, \rho_0) \text{ and } y_0 = \mathcal{H}(y_0, \sigma_0).$$

As $\mathcal{A} \cup \mathcal{B}$ is open, there exists $r > 0$ such that $B_d(x_0, r) \subseteq \mathcal{A} \cup \mathcal{B}$ and $B_d(r, y_0) \subseteq \mathcal{A} \cup \mathcal{B}$. Pick $\rho \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$, and $\sigma \in (\rho_0 - \epsilon, \rho_0 + \epsilon)$ such that $|\rho - \sigma_0| \leqslant \frac{1}{M^n} < \frac{\epsilon}{2}$, $|\sigma - \rho_0| \leqslant \frac{1}{M^n} < \frac{\epsilon}{2}$, and $|\rho_0 - \sigma_0| \leqslant \frac{1}{M^n} < \frac{\epsilon}{2}$. Then we have $y \in \overline{B_{\mathcal{X}\cup\mathcal{Y}}(x_0, r)} = \{y \in \mathcal{B} | d(x_0, y) \leqslant r + d(x_0, y_0) \text{ for some } y_0 \in \mathcal{B}\}$ and $x \in \overline{B_{\mathcal{X}\cup\mathcal{Y}}(r, y_0)} = \{x \in \mathcal{A} | d(x, y_0) \leqslant r + d(x_0, y_0) \text{ for some } x_0 \in \mathcal{A}\}$. Additionally

$$\begin{aligned} d(\mathcal{H}(x, \rho), y_0) &= d(\mathcal{H}(x, \rho), \mathcal{H}(y_0, \sigma_0)) \\ &\leqslant d(\mathcal{H}(x, \rho), \mathcal{H}(y, \sigma_0)) + d(\mathcal{H}(x_0, \rho), \mathcal{H}(y, \sigma_0)) + d(\mathcal{H}(x_0, \rho), \mathcal{H}(y_0, \sigma_0)) \\ &\leqslant 2M|\rho - \sigma_0| + d(\mathcal{H}(x_0, \rho), \mathcal{H}(y, \sigma_0)) \\ &\leqslant 2M|\rho - \sigma_0| + \alpha d(x_0, y) \\ &\leqslant \frac{2}{M^{n-1}} + d(x_0, y). \end{aligned}$$

On letting $n \to \infty$, we get

$$d(\mathcal{H}(x, \rho), y_0) \leqslant d(x_0, y) \leqslant r + d(x_0, y_0).$$

By similar procedure, we get $d(x_0, \mathcal{H}(y, \sigma)) \leqslant d(x, y_0) \leqslant r + d(x_0, y_0)$. But

$$d(x_0, y_0) = d(\mathcal{H}(x_0, \rho_0), \mathcal{H}(y_0, \sigma_0)) \leqslant M|\rho_0 - \sigma_0| \leqslant \frac{1}{M^{n-1}} \to 0 \text{ as } n \to \infty,$$

which implies $x_0 = y_0$. Therefore, for each fixed σ, where $\sigma = \rho \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$ and $\mathcal{H}(., \rho) : \overline{B_{\mathcal{X}\cup\mathcal{Y}}(x_0, r)} \to \overline{B_{\mathcal{X}\cup\mathcal{Y}}(x_0, r)}$. Since all the hypothesis of Theorem 4.1 hold, $\mathcal{H}(., \rho)$ has a fixed point in $\overline{\mathcal{A}} \cap \overline{\mathcal{B}}$ but the fixed point must be in $\mathcal{A} \cap \mathcal{B}$ as (i) holds. Therefore $\rho = \sigma \in \mathcal{X} \cap \mathcal{Y}$ for each $\sigma \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$. Thus $(\sigma_0 - \epsilon, \sigma_0 + \epsilon) \in \mathcal{X} \cap \mathcal{Y}$, which gives $\mathcal{X} \cap \mathcal{Y}$ is open in $[0,1]$. Now we get the required result from the connectedness of $[0,1]$ because $\mathcal{X} \cap \mathcal{Y} = [0,1]$.

We use a similar process for converse part. □

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