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Robust passivity analysis of uncertain neutral-type neural networks with distributed interval time-varying delay under the effects of leakage delay

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Abstract

This paper deals with the problem of delay-range-dependent robust passivity analysis of uncertain neutral-type neural networks with distributed interval time-varying delay under the effects of leakage delay. The uncertainties under consideration are norm-bounded uncertainties and the restriction on the derivative of the discrete and distributed interval time-varying delays is removed, which means that a fast interval time-varying delay is allowed. By applying a novel Lyapunov-Krasovskii functional approach, improved integral inequalities, Leibniz-Newton formula and utilization of zero equation, then a new delay-range-dependent passivity criterion of neutral-type neural networks with distributed interval time-varying delay under the effects of leakage delay is established in terms of linear matrix inequalities (LMIs). Furthermore, some less conservative delay-dependent passivity criteria are obtained. Moreover, we derived a robust passivity criterion for uncertain neutral-type neural networks with distributed interval time-varying delay under the effects of leakage delay. Besides, a less conservative delay-dependent robust passivity criterion is obtained. Finally, five numerical examples are given to show the effectiveness and less conservativeness of the proposed methods.

Keywords: Neutral type, neural network, leakage delay, Lyapunov-Krasovskii functional, linear matrix inequality, delay-range-dependent passivity.

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1. Introduction

In the past several years, neural networks (NNs) have found a wide range of applications in a variety of areas such as combinatorial optimization [10], signal processing, pattern recognition [45], communication, statistic image processing [13], fix-point computation, associative memory [5, 32, 53], and other scientific areas see [2, 14, 18, 19, 38, 39, 43]. Many scholars have paid their attentions to NNs which possess many advantages, including paralel computation, learning ability, function approximation, fault tolerance, etc. Most of these applications mainly depend on the dynamical behaviors of the considered NNs and their equilibrium points. Therefore, the study of dynamical behaviors of the delayed NNs is an active research

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topic and has received considerable attention in recent years [1, 6, 9, 17, 21, 31, 34–36, 46, 54]. On the other hand, neutral-type time delay in the system models have been intensively studied since neutral systems can be found in many industrial systems such as population ecology [23], water pipes, distributed networks containing lossless transmission lines [8], chemical reactors, heat exchangers, robots in contact with rigid environments [33], etc. Moreover, the NNs containing the information of past state derivatives are called neutral-type neural networks (NTNNs). The existing works on the state estimator of NTNNs with mixed delays have been presented in [28, 56]. It is known that time delays in the response of neurons may result in instability or oscillation of NTNNs. Thus, stability analysis of NTNNs with time delays has been widely studied and so on. Currently, Rakkiyappan et al. [37] studied the new global exponential stability results for neutral-type neural networks with distributed time delays. Samidurai et al. [40] discussed the global exponential stability of neutral-type impulsive neural networks with discrete and distributed delays. In addition, Manivannan et al. [30] studied on the delay-dependent stability criteria for neutral-type neural networks with interval time-varying delay signals under the effects of leakage delay and reference therein see [3, 11, 15, 28].

The concept of passivity has played an important role in the analysis of the stability of dynamical systems, nonlinear control, and other research area [12, 29, 47, 48]. The main idea of the passivity theory is that the passive properties of a system can keep the system internal stability. Thus, the passivity theory has received a great deal of attention from the control community since 1970s [20, 50]. In [7, 11, 15, 22, 28, 30, 37, 40, 41, 44, 55, 56], the authors studied the passivity of NTNNs with time delay and obtained some passivity conditions. Hence, passivity analysis of NTNNs have been considered in recent years. Furthermore, the stability and passivity analysis of NTNNs with time delay in leakage (or forgetting) term have become one of impressive research topics and have been widely studied by many researchers. In fact, the leakage term also has great impact on the dynamical behavior of NTNNs. For example, Li et al. [25] investigated the delay-dependent stability of neural networks of neutral-type with time delays in the leakage term. Balasubramaniam et al. [4] discussed the passivity analysis for neural networks of neutral-type with Markovian jumping parameters and time delay in the leakage term mentioned in [30] and references therein. However, there is no result has been obtained for passive condition of NTNNs with distributed interval time-varying delay under the effects of leakage delay. The challenge of this paper is studying the new results on robust passivity analysis of NTNNs and NNs with non-differentiable discrete and distributed interval time-varying delays which mean that this works can be used for various systems with fast interval time-varying delays compared with previous works considered on differentiable delay ($\dot{d}(t) \leqslant \mu$). This motivates our research.

In this paper, we will present a passivity criterion of NTNNs with discrete, neutral, distributed interval time-varying delays and leakage delay. By constructing a novel Lyapunov-Krasovskii functional, using new integral inequalities in derivative of Lyapunov functional, Leibniz-Newton formula and utilization of zero equation, then a passivity criterion of the considered system is established in terms of LMIs. Furthermore, we obtained some passivity criteria of NNs with discrete and distributed time-varying delays under the effect of leakage delay and NNs with discrete time-varying delay, respectively. Moreover, a robust passivity criterion of uncertain NTNNs with discrete, neutral and distributed interval time-varying delays under the effect of leakage delay is derived. Then, a robust passivity criterion of uncertain NTNNs with discrete, neutral and distributed time-varying delays is presented. Lastly, five numerical examples are given to demonstrate the effectiveness of the proposed results.

2. Network model and mathematics preliminaries

Notation: Throughout this paper, the notations are standard. R^n denotes the n-dimensional space with the vector norm $\|\cdot\|$; $\|x\|$ denotes the Euclidean vector norm of $x \in R^n$; $R^{n\times r}$ denotes the set $n \times r$ real matrices; A^T denotes the transpose of the matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; matrix A is called semi-positive definite ($A \geqslant 0$) if $x^T Ax \geqslant 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $x^T Ax > 0$ for all $x \neq 0$; matrix B is called semi-negative definite ($B \leqslant 0$) if $x^T Bx \leqslant 0$, for all

$x \in R^n$; B is negative definite ($B < 0$) if $x^T Bx < 0$ for all $x \neq 0$; $C([-d_2, 0], R^n)$ denotes the space of all continuous vector functions mapping $[-d_2, 0]$ into R^n; $x_t = x(t+s), s \in [-d_2, 0]$; $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the following continuous NTNNs with discrete, neutral, distributed interval time-varying delays and leakage delay.

$$\begin{cases} \dot{\xi}(t) = -A\xi(t-\delta) + W_1 f(\xi(t)) + W_2 f(\xi(t-d(t))) + W_3\dot{\xi}(t-r(t)) + W_4\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t), \\ z(t) = C_1 f(\xi(t)) + C_2 f(\xi(t-d(t))) + C_3\dot{\xi}(t-r(t)) + C_4 u(t), \\ \xi(t) = \phi(t), \quad t \in [-\tau_{max}, 0], \quad \tau_{max} = \max\{d_2, \rho_2, r_2\}, \end{cases} \tag{2.1}$$

where $\xi(t) = [\xi_1(t), \xi_2(t), \dots, \xi_n(t)] \in \mathcal{R}^n$ is the neural state vector. The diagonal matrix A is a self-feedback connection weight matrix, W_1, W_2, W_3 and W_4 are the connection weight matrices between neurons with appropriate dimensions, C_1, C_2, C_3 and C_4 are given real matrixes, $f(\cdot) = (f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot))^T$ represent the activation functions, $u(t)$ and $z(t)$ represent the input and output vectors, respectively; $\phi(t)$ is an initial condition. Where $\delta \geqslant 0$ denotes the constant leakage delay, the variable $d(t)$ is the discrete interval time-varying delay, $\rho(t)$ is the distributed interval time-varying delay and $r(t)$ is the neutral interval time-varying delay, satisfying

$$d_1 \leqslant d(t) \leqslant d_2, \quad \rho_1 \leqslant \rho(t) \leqslant \rho_2, \quad r_1 \leqslant r(t) \leqslant r_2, \quad 0 \leqslant \dot{r}(t) \leqslant r_d, \tag{2.2}$$

where d_M and r_M are positive real constants. The neural activation functions $f_k(\cdot), k = 1, 2, \dots, n$ satisfy $f_k(0) = 0$ and for $s_1, s_2 \in \mathcal{R}, s_1 \neq s_2$,

$$l_k^- \leqslant \frac{f_k(s_1) - f_k(s_2)}{s_1 - s_2} \leqslant l_k^+, \tag{2.3}$$

where l_k^-, l_k^+, are known real scalars. Moreover, we denote $L^+ = \mathrm{diag}(l_1^+, l_2^+, \dots, l_n^+), L^- = \mathrm{diag}(l_1^-, l_2^-, \dots, l_n^-)$. In addition, if the constant matrices are extended to be the norm-bounded uncertainties of the form:

$$A = A + \Delta A(t), \quad W_1 = W_1 + \Delta B(t), \quad W_2 = W_2 + \Delta C(t), \quad W_3 = W_3 + \Delta D_1(t), \quad W_4 = W_4 + \Delta D_2(t),$$

then system (2.1) extends to the following system:

$$\begin{cases} \dot{\xi}(t) = -[A + \Delta A(t)]\xi(t-\delta) + [W_1 + \Delta B(t)]f(\xi(t)) + [W_2 + \Delta C(t)]f(\xi(t-d(t))) \\ \qquad\quad +[W_3 + \Delta D_1(t)]\dot{\xi}(t-r(t)) + [W_4 + \Delta D_2(t)]\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t), \\ z(t) = C_1 f(\xi(t)) + C_2 f(\xi(t-d(t))) + C_3\dot{\xi}(t-r(t)) + C_4 u(t), \\ \xi(t) = \phi(t), \quad t \in [-\tau_{max}, 0], \quad \tau_{max} = \max\{d_2, \rho_2, r_2\}. \end{cases} \tag{2.4}$$

The uncertain matrices $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, $\Delta D_1(t)$ and $\Delta D_2(t)$ are norm bounded and can be described as

$$\begin{bmatrix} \Delta A(t) & \Delta B(t) & \Delta C(t) & \Delta D_1(t) & \Delta D_2(t) \end{bmatrix} = E\Delta(t) \begin{bmatrix} G_1 & G_2 & G_3 & G_4 & G_5 \end{bmatrix}, \tag{2.5}$$

where E, G_1, G_2, G_3, G_4 and G_5 are constant matrices with appropriate dimensions. The uncertain matrix $\Delta(t)$ satisfies

$$\Delta(t) = F(t)[I - JF(t)]^{-1}, \tag{2.6}$$

which is said to be admissible where J is a known matrix satisfying

$$I - JJ^T > 0. \tag{2.7}$$

The uncertain matrix $F(t)$ satisfies

$$F(t)^T F(t) \leqslant I. \tag{2.8}$$

Definition 2.1 ([27]). The system (2.1) is said to be passive if there exist a scalar γ such that for all $t_f \geqslant 0$, $2\int_0^{t_f} y^T(s)u(s)ds \geqslant -\gamma \int_0^{t_f} u^T(s)u(s)ds$, and for all solution of (2.1) with $x(t,0)$.

Lemma 2.2 ([16]). *For any positive definite matrix* $M \in R^{m\times n}$, *scalars* $h_2 > h_1 > 0$, *and a vector function* $w : [h_1, h_2] \to R^n$ *such that the integrations concerned are well defined, we have the inequality*

$$-[h_2 - h_1]\int_{t-h_2}^{t-h_1} w^T(s)Mw(s)ds \leqslant -(\int_{t-h_2}^{t-h_1} w(s)ds)^T M(\int_{t-h_2}^{t-h_1} w(s)ds).$$

Lemma 2.3 ([26]). *Suppose that* $\Delta(t)$ *is given by* (2.6)-(2.8). *Let* M, S *and* N *be real matrices of appropriate dimension with* $M = M^T$. *Then, the inequality*

$$M + S\Delta(t)N + N^T\Delta(t)^T S^T < 0,$$

holds if and only if, for any scalar $\sigma > 0$,

$$\begin{pmatrix} M & S & \sigma N^T \\ S^T & -\sigma I & \sigma J^T \\ \sigma N & \sigma J & -\sigma I \end{pmatrix} < 0.$$

Lemma 2.4 ([24]). *For a positive matrix* M, *the following inequality holds:*

$$-(\alpha-\beta)\int_\beta^\alpha \dot{x}^T(s)M\dot{x}(s)ds \leqslant \begin{bmatrix} x(\alpha) \\ x(\beta) \end{bmatrix}^T \begin{bmatrix} -M & M \\ * & -M \end{bmatrix} \begin{bmatrix} x(\alpha) \\ x(\beta) \end{bmatrix}.$$

Lemma 2.5 ([24]). *For a positive matrix* M, *the following inequality holds:*

$$-\frac{(\alpha-\beta)^2}{2}\int_\beta^\alpha\int_s^\alpha x^T(u)Mx(u)duds \leqslant -\Big(\int_\beta^\alpha\int_s^\alpha x(u)duds\Big)^T M\Big(\int_\beta^\alpha\int_s^\alpha x(u)duds\Big).$$

Lemma 2.6 ([24]). *For a positive matrix* M, *the following inequality holds:*

$$-\frac{(\alpha-\beta)^3}{6}\int_\beta^\alpha\int_s^\alpha\int_u^\alpha x^T(\lambda)Mx(\lambda)d\lambda duds \leqslant -\Big(\int_\beta^\alpha\int_s^\alpha\int_u^\alpha x(\lambda)d\lambda duds\Big)^T M\Big(\int_\beta^\alpha\int_s^\alpha\int_u^\alpha x(\lambda)d\lambda duds\Big).$$

Lemma 2.7 ([42]). *For any constant symmetric positive definite matrix* $Q \in R^{n\times n}$, $d(t)$ *is discrete time-varying delays with* (2.2), *vector function* $\omega : [-d_2, 0] \to R^n$ *such that the integrations concerned are well defined, then*

$$-[d_2 - d_1]\int_{-d_2}^{-d_1} \omega^T(s)Q\omega(s)ds \leqslant -\int_{-d(t)}^{-d_1}\omega^T(s)dsQ\int_{-d(t)}^{-d_1}\omega(s)ds - \int_{-d_2}^{-d(t)}\omega^T(s)dsQ\int_{-d_2}^{-d(t)}\omega(s)ds.$$

Lemma 2.8 ([42]). *For any constant matrices* $Q_1, Q_2, Q_3 \in R^{n\times n}$, $Q_1 \geqslant 0, Q_3 > 0$, $\begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \geqslant 0$, $d(t)$ *is discrete time-varying delays with* (2.2) *and vector function* $\dot{x} : [-d_2, 0] \to R^n$ *such that the following integration is well defined, then*

$$\begin{aligned} &-[d_2 - d_1]\int_{t-d_2}^{t-d_1} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} ds \\ &\leqslant \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \\ \int_{t-d(t)}^{t-d_1} x(s)ds \\ \int_{t-d_2}^{t-d(t)} x(s)ds \end{bmatrix}^T \begin{bmatrix} -Q_3 & Q_3 & 0 & -Q_2^T & 0 \\ * & -Q_3 - Q_3^T & Q_3 & Q_2^T & -Q_2^T \\ * & * & -Q_3 & 0 & Q_2^T \\ * & * & * & -Q_1 & 0 \\ * & * & * & * & -Q_1 \end{bmatrix} \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \\ \int_{t-d(t)}^{t-d_1} x(s)ds \\ \int_{t-d_2}^{t-d(t)} x(s)ds \end{bmatrix}. \end{aligned}$$

Lemma 2.9 ([42]). *Let $x(t) \in R^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any constant matrices $X, M_i \in R^{n\times n}, i = 1,2,\dots,5$ and $d(t)$ is discrete time-varying delays with* (2.2),

$$
\begin{aligned}
&-\int_{t-d_2}^{t-d_1} \dot{x}^T(s)X\dot{x}(s)ds \\
&\leqslant \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix}^T \begin{bmatrix} M_1+M_1^T & -M_1^T+M_2 & 0 \\ * & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2 \\ * & * & -M_2-M_2^T \end{bmatrix} \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix} \\
&\quad + [d_2-d_1] \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix}^T \begin{bmatrix} M_3 & M_4 & 0 \\ * & M_3+M_5 & M_4 \\ * & * & M_5 \end{bmatrix} \begin{bmatrix} x(t-d_1) \\ x(t-d(t)) \\ x(t-d_2) \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{bmatrix} X & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0.
$$

3. Main results

3.1. *Delay-range-dependent pasivity criteria*

We will present the passivity criteria dependent on interval time-varying delays of system (2.1) via LMIs approach. We introduce the following notations for later use.

$$
\sum = [\Sigma_{i,j}]_{21\times 21},
$$

where $\Sigma_{i,j} = \Sigma_{j,i}^T$, $i,j = 1,2,\dots,21$,

$$
\begin{aligned}
\Sigma_{1,1} &= -P_1A - A^TP_1 + P_2 + \delta^2P_3 + P_4^TG - Q_1^TA + G^TP_4 + (d_2-d_1)^2P_8 + (d_2-d_1)^2R_4 \\
&\quad - (d_2-d_1)^2P_{11} - \frac{d_1^4P_{12}}{4} - \frac{d_2^4P_{13}}{4} - 2H_1\epsilon_1 \\
\Sigma_{1,2} &= -Q_1^TA, \quad \Sigma_{1,5} = -P_4^TG, \quad \Sigma_{1,6} = P_4^T - Q_1^T + (d_2-d_1)^2R_5, \\
\Sigma_{1,9} &= P_1W_3 + Q_1^TW_3, \quad \Sigma_{1,10} = P_1W_1 + Q_1^TW_1 + H_1\epsilon_2, \quad \Sigma_{1,11} = P_1W_2 + Q_1^TW_2, \\
\Sigma_{1,12} &= P_1(A^2)^T, \quad \Sigma_{1,13} = P_1W_4 + Q_1^TW_4, \quad \Sigma_{1,15} = (d_2-d_1)P_{11}, \quad \Sigma_{1,17} = -P_4^TG + Q_4, \\
\Sigma_{1,18} &= \frac{d_1^2P_{12}}{2}, \quad \Sigma_{1,19} = \frac{d_2^2P_{13}}{2}, \quad \Sigma_{1,21} = P_1 + Q_1^T, \quad \Sigma_{2,2} = -P_2, \quad \Sigma_{2,5} = -A^TQ_3, \\
\Sigma_{2,6} &= -A^TQ_2, \quad \Sigma_{2,10} = -A^TK_1, \quad \Sigma_{2,13} = A^T, \quad \Sigma_{2,21} = -A^TK_2, \\
\Sigma_{3,3} &= P_5 + R_1 + M_1 + M_1^T + (d_2-d_1)M_3 - R_6 + \frac{(d_2-d_1)^4}{4}P_{10} - \frac{(d_2-d_1)^2}{2}P_{14}, \\
\Sigma_{3,5} &= -M_1^T + M_2 + (d_2-d_1)M_4 + R_6, \quad \Sigma_{3,16} = -R_5^T, \quad \Sigma_{3,20} = \frac{(d_2-d_1)^2}{2}P_{14}, \\
\Sigma_{4,4} &= -P_5 - R_1 - M_2 - M_2^T + (d_2-d_1)M_5 - R_6, \\
\Sigma_{4,5} &= -M_1^T + M_2 + (d_2-d_1)M_4^T + R_6^T, \quad \Sigma_{4,11} = R_5^T, \quad \Sigma_{4,15} = R_5^T, \\
\Sigma_{5,5} &= M_1 + M_1^T - M_2 - M_2^T + (d_2-d_1)(M_3+M_5) - R_6 - R_6^T - 2H_2\epsilon_1 \\
\Sigma_{5,6} &= -Q_3^T, \quad \Sigma_{5,9} = Q_3^TW_3, \quad \Sigma_{5,10} = Q_3^TW_1, \quad \Sigma_{5,11} = H_2\epsilon_2 - R_5^T + Q_3^TW_2, \\
\Sigma_{5,13} &= Q_3^TW_4, \quad \Sigma_{5,15} = -R_5^T, \quad \Sigma_{5,16} = R_5^T, \quad \Sigma_{5,17} = -Q_4, \quad \Sigma_{5,21} = Q_3^T,
\end{aligned}
$$

$$\Sigma_{6,6} = -Q_2^T - Q_2 + (r_2 - r_1)P_5 + (d_2 - d_1)P_9 + (d_2 - d_1)R_6 + \frac{(d_1)^6}{36}P_{12} + \frac{(d_2)^6}{36}P_{13},$$
$$\Sigma_{6,9} = Q_2^T W_3, \quad \Sigma_{6,10} = Q_2^T W_1 + K1^T, \quad \Sigma_{6,11} = Q_2^T W_2, \quad \Sigma_{6,13} = Q_2^T W_4 + I,$$
$$\Sigma_{6,21} = Q_2^T + K_2^T, \quad \Sigma_{7,7} = P_6 + R_3 + \frac{(d_2 - d_1)^4}{4}P_{11} + \frac{(d_2 - d_1)^6}{36}P_{14}, \quad \Sigma_{8,8} = -P_6 - R_2,$$
$$\Sigma_{9,9} = -(r_2 - r_1)(1 - r_d)P_5, \quad \Sigma_{9,12} = -W_3^T P_1 A^T, \quad \Sigma_{9,13} = -W_3^T,$$
$$\Sigma_{9,21} = -C_3, \quad \Sigma_{10,10} = (\rho_2 - \rho_1)^2 - 2H_1 - K_1 W_1 - W_1^T K_1^T, \quad \Sigma_{10,11} = -K_1 W_2,$$
$$\Sigma_{10,12} = -W_1 P_1 A^T, \quad \Sigma_{10,13} = -W_1^T, \quad \Sigma_{10,21} = -C_1 - K_1 - W_1^T K_2^T,$$
$$\Sigma_{11,11} = -P_8 - R_4 - 2H_2, \quad \Sigma_{11,12} = -W_2^T P_1 A^T, \quad \Sigma_{11,13} = W_2^T,$$
$$\Sigma_{11,21} = -C_2 - W_2^T K_2^T, \quad \Sigma_{12,12} = -P_3, \quad \Sigma_{12,13} = -AP_1 W_4, \quad \Sigma_{12,21} = -AP_1,$$
$$\Sigma_{13,13} = -W_4 - W_4^T, \quad \Sigma_{13,21} = -I, \quad \Sigma_{14,14} = -P_6, \quad \Sigma_{15,15} = -P_{11},$$
$$\Sigma_{16,16} = -P_8 - R_4, \quad \Sigma_{17,17} = -Q_4^T - Q_4, \quad \Sigma_{18,18} = -P_{12},$$
$$\Sigma_{19,19} = -P_{13}, \quad \Sigma_{20,20} = -P_{10} - P_{14}, \quad \Sigma_{21,21} = \gamma - C_4 - C_4^T - K_2 - K_2^T,$$

and the other terms are 0.

Theorem 3.1. *The delayed NTNNs* (2.1) *are passive in Definition 2.1, if there exist positive definite symmetric matrices* Q_1, R_4, R_6 P_i, $i \in \{1, 2, \ldots, 15\}$, *any appropriate dimensional matrices* G, K_1, K_2, R_m, Q_n, M_o *and* $m = 1, 2, \ldots, 6$, $n = 1, 2, \ldots, 4$, $o = 1, 2, \ldots, 5$ *such that the following symmetric linear matrix inequalities hold*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_9 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0, \tag{3.1}$$
$$\sum < 0. \tag{3.2}$$

Proof. Choose the Lyapunov-Krasovskii functional candidate for the system (2.1) of the form

$$V(\xi(t), t) = \sum_{i=1}^{8} V_i(\xi(t), t),$$

where

$$V_1(\xi(t), t) = \Big(\xi(t) - A\int_{t-\delta}^{t} \xi(s)ds\Big)^T P_1 \Big(\xi(t) - A\int_{t-\delta}^{t} \xi(s)ds\Big),$$
$$V_2(\xi(t), t) = \int_{t-\delta}^{t} \xi^T(s)P_2\xi(s)ds + \delta\int_{-\delta}^{0}\int_{t+\theta}^{t} \xi^T(s)P_3\xi(s)dsd\theta,$$
$$V_3(\xi(t), t) = \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \xi(t - d(t)) \\ \int_{t-d(t)}^{t} \dot{\xi}(s)ds \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \xi(t - d(t)) \\ \int_{t-d(t)}^{t} \dot{\xi}(s)ds \end{bmatrix},$$
$$V_4(\xi(t), t) = (r_2 - r_1)\int_{t-r(t)}^{t} \dot{\xi}^T(s)P_5\dot{\xi}(s)ds + (\rho_2 - \rho_1)\int_{-\rho_2}^{-\rho_1}\int_{t+s}^{t} f(\xi(\theta))^T P_6 f(\xi(\theta))d\theta ds,$$
$$V_5(\xi(t), t) = \int_{t-d_2}^{t-d_1} \xi^T(s)P_7\xi(s)ds + \int_{t-d_2}^{t-d_1} \dot{\xi}^T(s)P_8\dot{\xi}(s)ds + \int_{t-d_2}^{t-d_1} \begin{bmatrix} \xi(s) \\ \dot{\xi}(s) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \dot{\xi}(s) \end{bmatrix} ds,$$
$$V_6(\xi(t), t) = (d_2 - d_1)\int_{-d_2}^{-d_1}\int_{t+s}^{t} \xi^T(\theta)P_9\xi(\theta)d\theta ds + \int_{-d_2}^{-d_1}\int_{t+s}^{t} \dot{\xi}^T(\theta)P_{10}\dot{\xi}(\theta)d\theta ds$$

$$+(d_2-d_1)\int_{-d_2}^{-d_1}\int_{t+s}^{t}\begin{bmatrix}\xi(\theta)\\ \dot{\xi}(\theta)\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ * & R_6\end{bmatrix}\begin{bmatrix}\xi(\theta)\\ \dot{\xi}(\theta)\end{bmatrix}d\theta ds,$$

$$V_7(\xi(t),t)=\frac{(d_2-d_1)^2}{2}\int_{t-d_2}^{t-d_1}\int_{s}^{t-d_1}\int_{u}^{t-d_1}\xi^T(\lambda)P_{11}\xi(\lambda)d\lambda du ds$$
$$+\frac{(d_2-d_1)^2}{2}\int_{t-d_2}^{t-d_1}\int_{s}^{t-d_1}\int_{u}^{t-d_1}\dot{\xi}^T(\lambda)P_{12}\dot{\xi}(\lambda)d\lambda du ds,$$

$$V_8(\xi(t),t)=\frac{d_1^3}{6}\int_{t-d_1}^{t}\int_{s}^{t}\int_{u}^{t}\int_{\lambda}^{t}\dot{\xi}^T(\theta)P_{13}\dot{\xi}(\theta)d\theta d\lambda du ds+\frac{d_2^3}{6}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\int_{\lambda}^{t}\dot{\xi}^T(\theta)P_{14}\dot{\xi}(\theta)d\theta d\lambda du ds$$
$$+\frac{(d_2-d_1)^3}{6}\int_{t-d_2}^{t-d_1}\int_{s}^{t-d_1}\int_{u}^{t-d_1}\int_{\lambda}^{t-d_1}\dot{\xi}^T(\theta)P_{15}\dot{\xi}(\theta)d\theta d\lambda du ds.$$

The derivative of $V(t)$ along the trajectory of system (2.1) is given by

$$\dot{V}(\xi(t),t)=\sum_{i=1}^{8}\dot{V}_i(\xi(t),t).$$

The time derivative of $V_1(t)$ can be represented as

$$\begin{aligned}\dot{V}_1(\xi(t),t)=2\big(\xi(t)-A\int_{t-\delta}^{t}\xi(s)ds\big)^TP_1\big(&-A\xi(t)+W_1f(\xi(t))\\ &+W_2f(\xi(t-d(t)))+W_3\dot{\xi}(t-r(t))+W_4\int_{t-\rho(t)}^{t}f(\xi(s))ds+u(t)\big).\end{aligned}\tag{3.3}$$

It is from Lemma 2.2 that we have

$$\begin{aligned}\dot{V}_2(\xi(t),t)&=\xi^T(t)\big(P_2+\delta^2P_3\big)\xi(t)+\xi^T(t-\delta)(-P_2)\xi(t-\delta)-\delta\int_{t-\delta}^{t}\xi^T(s)P_3\xi(s)ds\\ &\leqslant\xi^T(t)\big(P_2+\delta^2P_3\big)\xi(t)+\xi^T(t-\delta)(-P_2)\xi(t-\delta)-\big(\int_{t-\delta}^{t}\xi(s)ds\big)^TP_3\big(\int_{t-\delta}^{t}\xi(s)ds\big).\end{aligned}\tag{3.4}$$

Calculating $\dot{V}_3(\xi(t),t)$ and utilizing the following zero equation

$$0=\xi(t)-\xi(t-d(t))-\int_{t-d(t)}^{t}\dot{\xi}(s)ds,\quad 0=G\xi(t)-G\xi(t-d(t))-G\int_{t-d(t)}^{t}\dot{\xi}(s)ds,\tag{3.5}$$

where $G\in\mathbb{R}^{n\times n}$ will be chosen to guarantee the stability of the system (2.1), leads to

$$\begin{aligned}\dot{V}_3(\xi(t),t)=2&\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\\ \xi(t-d(t))\\ \int_{t-d(t)}^{t}\dot{\xi}(s)ds\end{bmatrix}^T\begin{bmatrix}P_4^T & 0 & 0 & Q_1^T\\ 0 & 0 & 0 & Q_2^T\\ 0 & 0 & 0 & Q_3^T\\ 0 & 0 & 0 & Q_4^T\end{bmatrix}\begin{bmatrix}\dot{\xi}(t)\\ 0\\ 0\\ 0\end{bmatrix}\\ &=2\xi^T(t)P_4^T[\dot{\xi}(t)+G\xi(t)-G\xi(t-d(t))-G\int_{t-d(t)}^{t}\dot{\xi}(s)ds]\\ &\quad+2\xi^T(t)Q_1^T[-\dot{\xi}(t)-A\xi(t-\delta)+W_1f(\xi(t))\\ &\quad+W_2f(\xi(t-d(t)))+W_3\dot{\xi}(t-r(t))+W_4\int_{t-\rho(t)}^{t}f(\xi(s))ds+u(t)]\\ &\quad+2\dot{\xi}^T(t)Q_2^T[-\dot{\xi}(t)-A\xi(t-\delta)+W_1f(\xi(t))\end{aligned}\tag{3.6}$$

$$
\begin{aligned}
&+W_2 f(\xi(t-d(t)))+W_3\dot{\xi}(t-r(t))+W_4\int_{t-\rho(t)}^{t} f(\xi(s))ds+u(t)]\\
&+2\xi^T(t-d(t))Q_3^T[-\dot{\xi}(t)-A\xi(t-\delta)+W_1 f(\xi(t))\\
&+W_2 f(\xi(t-d(t)))+W_3\dot{\xi}(t-r(t))+W_4\int_{t-\rho(t)}^{t} f(\xi(s))ds+u(t)]\\
&+2\int_{t-d(t)}^{t}\dot{\xi}^T(s)dsQ_4^T[\xi(t)-\xi(t-d(t))-\int_{t-d(t)}^{t}\dot{\xi}(s)ds].
\end{aligned}
$$

Using Lemma 2.2, the increments of $\dot{V}_4(\xi(t),t)$ is easily computed as

$$
\begin{aligned}
\dot{V}_4(\xi(t),t)&=(r_2-r_1)\dot{\xi}^T(t)P_5\dot{\xi}(t)-(r_2-r_1)(1-\dot{r}(t))\dot{\xi}^T(t-r(t))P_5\dot{\xi}(t-r(t))\\
&\quad+(\rho_2-\rho_1)^2 f(\xi^T(t))P_6 f(\xi(t))-(\rho_2-\rho_1)\int_{t-\rho_2}^{t-\rho_1} f(\xi^T(s))P_6 f(\xi(s))ds\\
&\leqslant (r_2-r_1)\dot{\xi}^T(t)P_5\dot{\xi}(t)-(r_2-r_1)(1-r_d)\dot{\xi}^T(t-r(t))P_5\dot{\xi}(t-r(t))\\
&\quad+(\rho_2-\rho_1)^2 f(\xi^T(t))P_6 f(\xi(t))-\int_{t-\rho_2}^{t-\rho_1} f(\xi^T(s))dsP_6\int_{t-\rho_2}^{t-\rho_1} f(\xi(s))ds.
\end{aligned}
\tag{3.7}
$$

Taking the time derivative of $V_5(t)$, we have

$$
\begin{aligned}
\dot{V}_5(\xi(t),t)&=\xi^T(t-d_1)P_7\xi(t-d_1)-\xi^T(t-d_2)P_7\xi(t-d_2)\\
&\quad+\dot{\xi}^T(t-d_1)P_8\dot{\xi}(t-d_1)-\dot{\xi}^T(t-d_2)P_8\dot{\xi}(t-d_2)\\
&\quad+\begin{bmatrix}\xi(t-d_1)\\ \dot{\xi}(t-d_1)\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}\xi(t-d_1)\\ \dot{\xi}(t-d_1)\end{bmatrix}-\begin{bmatrix}\xi(t-d_2)\\ \dot{\xi}(t-d_2)\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ R_2^T & R_3\end{bmatrix}\begin{bmatrix}\xi(t-d_2)\\ \dot{\xi}(t-d_2)\end{bmatrix}.
\end{aligned}
\tag{3.8}
$$

Using Lemmas 2.7, 2.8, and 2.9, we obtain

$$
\begin{aligned}
\dot{V}_6(\xi(t),t)&=(d_2-d_1)^2\xi^T(t)P_9\xi(t)-(d_2-d_1)\int_{t-d_2}^{t-d_1}\xi^T(s)P_9\xi(s)ds\\
&\quad+(d_2-d_1)\dot{\xi}^T(t)P_{10}\dot{\xi}(t)-\int_{t-d_2}^{t-d_1}\dot{\xi}^T(s)P_{10}\dot{\xi}(s)ds\\
&\quad+(d_2-d_1)^2\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}+(d_2-d_1)\int_{t-d_2}^{t-d_1}\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}\\
&\leqslant (d_2-d_1)^2\xi^T(t)P_9\xi(t)+(d_2-d_1)\dot{\xi}^T(t)P_{10}\dot{\xi}(t)+(d_2-d_1)^2\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ R_5^T & R_6\end{bmatrix}\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\end{bmatrix}\\
&\quad-\int_{t-d(t)}^{t-d_1}\xi^T(s)dsP_9\int_{t-d(t)}^{t-d_1}\xi(s)ds-\int_{t-d_2}^{t-d(t)}\xi^T(s)dsP_9\int_{t-d_2}^{t-d(t)}\xi(s)ds\\
&\quad+\begin{bmatrix}\xi(t-d_1)\\ \xi(t-d(t))\\ \xi(t-d_2)\end{bmatrix}^T\begin{bmatrix}M_1+M_1^T & -M_1^T+M_2 & 0\\ -M_1+M_2^T & M_1+M_1^T-M_2-M_2^T & -M_1^T+M_2\\ 0 & -M_1+M_2^T & -M_2-M_2^T\end{bmatrix}\begin{bmatrix}\xi(t-d_1)\\ \xi(t-d(t))\\ \xi(t-d_2)\end{bmatrix}\\
&\quad+(d_2-d_1)\begin{bmatrix}\xi(t-d_1)\\ \xi(t-d(t))\\ \xi(t-d_2)\end{bmatrix}^T\begin{bmatrix}M_3 & M_4 & 0\\ M_4^T & M_3+M_5 & M_4\\ 0 & M_4^T & M_5\end{bmatrix}\begin{bmatrix}\xi(t-d_1)\\ \xi(t-d(t))\\ \xi(t-d_2)\end{bmatrix}\\
&\quad+\begin{bmatrix}\xi(t-d_1)\\ \xi(t-d(t))\\ \xi(t-d_2)\\ \int_{t-d(t)}^{t-d_1}\xi(s)ds\\ \int_{t-d_2}^{t-d(t)}\xi(s)ds\end{bmatrix}^T\begin{bmatrix}-R_6 & R_6 & 0 & -R_5^T & 0\\ R_6^T & -R_6-R_6^T & R_6 & R_5^T & -R_5^T\\ 0 & R_6^T & -R_6 & 0 & R_5^T\\ -R_5 & R_5 & 0 & -R_4 & 0\\ 0 & -R_5 & R_5 & 0 & -R_4\end{bmatrix}\begin{bmatrix}\xi(t-d_1)\\ \xi(t-d(t))\\ \xi(t-d_2)\\ \int_{t-d(t)}^{t-d_1}\xi(s)ds\\ \int_{t-d_2}^{t-d(t)}\xi(s)ds\end{bmatrix}.
\end{aligned}
\tag{3.9}
$$

By Lemma 2.5, we get

$$\begin{aligned}\dot{V}_7(\xi(t),t) &= \frac{(d_2-d_1)^4}{4}\xi^T(t-d_1)P_{11}\xi(t-d_1)\int_{t-d_2}^{t-d_1}\int_u^t \xi^T(\lambda)d\lambda du P_{11}\int_{t-d_2}^{t-d_1}\int_u^t \xi(\lambda)d\lambda du \\ &\quad + \frac{(d_2-d_1)^4}{4}\dot{\xi}^T(t-d_1)P_{12}\dot{\xi}(t-d_1) - \int_{t-d_2}^{t-d_1}\int_u^t \dot{\xi}^T(\lambda)d\lambda du P_{12}\int_{t-h_2}^{t-h_1}\int_u^t \dot{\xi}(\lambda)d\lambda du \\ &\leqslant \frac{(d_2-d_1)^4}{4}\xi^T(t-d_1)P_{11}\xi(t-d_1) \\ &\quad - \int_{t-d_2}^{t-d_1}\int_u^t \xi^T(\lambda)d\lambda du P_{11}\int_{t-d_2}^{t-d_1}\int_u^t \xi(\lambda)d\lambda du + \frac{(d_2-d_1)^4}{4}\dot{\xi}^T(t-d_1)P_{12}\dot{\xi}(t-d_1) \\ &\quad - \Big[(d_2-d_1)\xi^T(t) - \int_{t-d_2}^{t-d_1}\xi^T(u)du\Big]P_{12}\Big[(d_2-d_1)\xi(t) - \int_{t-d_2}^{t-d_1}\xi(u)du\Big].\end{aligned} \tag{3.10}$$

By Lemma 2.6 and calculating $\dot{V}_8(t)$, we have

$$\begin{aligned}\dot{V}_8(\xi(t),t) &\leqslant \frac{d_1^6}{36}\dot{\xi}^T(t)P_{13}\dot{\xi}(t) + \frac{d_2^6}{36}\dot{\xi}^T(t)P_{14}\dot{\xi}(t) + \frac{(d_2-d_1)^6}{36}\dot{\xi}^T(t-d_1)P_{15}\dot{\xi}(t-d_1) \\ &\quad - \int_{t-d_1}^{t}\int_u^t\int_\lambda^t \dot{\xi}^T(\theta)d\theta d\lambda du P_{13}\int_{t-d_1}^{t}\int_u^t\int_\lambda^t \dot{\xi}(\theta)d\theta d\lambda du \\ &\quad - \int_{t-d_2}^{t}\int_u^t\int_\lambda^t \dot{\xi}^T(\theta)d\theta d\lambda du P_{14}\int_{t-d_2}^{t}\int_u^t\int_\lambda^t \dot{\xi}(\theta)d\theta d\lambda du \\ &\quad - \int_{t-d_2}^{t-d_1}\int_u^{t-d_1}\int_\lambda^{t-d_1} \dot{\xi}^T(\theta)d\theta d\lambda du P_{15}\int_{t-d_2}^{t-d_1}\int_u^{t-d_1}\int_\lambda^{t-d_1} \dot{\xi}(\theta)d\theta d\lambda du \\ &= \frac{d_1^6}{36}\dot{\xi}^T(t)P_{13}\dot{\xi}(t) + \frac{d_2^6}{36}\dot{\xi}^T(t)P_{14}\dot{\xi}(t) + \frac{(d_2-d_1)^6}{36}\dot{\xi}^T(t-d_1)P_{15}\dot{\xi}(t-d_1) \\ &\quad - \Big[\frac{d_1^2}{2}\xi^T(t) - \int_{t-d_1}^{t}\int_u^t \xi^T(\lambda)d\lambda du\Big]P_{13}\Big[\frac{d_1^2}{2}\xi(t) - \int_{t-d_1}^{t}\int_u^t \xi(\lambda)d\lambda du\Big] \\ &\quad - \Big[\frac{d_2^2}{2}\xi^T(t) - \int_{t-d_2}^{t}\int_u^t \xi^T(\lambda)d\lambda du\Big]P_{14}\Big[\frac{d_2^2}{2}\xi(t) - \int_{t-d_2}^{t}\int_u^t \xi(\lambda)d\lambda du\Big] \\ &\quad - \Big[\frac{(d_2-d_1)^2}{2}\xi^T(t-d_1) - \int_{t-d_2}^{t-d_1}\int_u^{t-d_1} \xi^T(\lambda)d\lambda du\Big]P_{15} \\ &\quad \times \Big[\frac{(d_2-d_1)^2}{2}\xi(t-d_1) - \int_{t-d_2}^{t-d_1}\int_u^{t-d_1} \xi(\lambda)d\lambda du\Big].\end{aligned}$$

From (2.1), we have

$$\begin{aligned}&2\int_{t-\rho(t)}^{t} f(\xi(s))ds \times [\dot{\xi}(t) + A\xi(t-\delta) - W_1 f(\xi(t)) - W_2 f(\xi(t-d(t))) - W_3\dot{\xi}(t-r(t)) \\ &\quad - W_4\int_{t-\rho(t)}^{t} f(\xi(s))ds - u(t)] = 0.\end{aligned} \tag{3.11}$$

From (2.3), we obtain for any positive real constants ϵ_1 and ϵ_2,

$$\begin{aligned}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix}^T \begin{bmatrix}-2H_1\epsilon_1 & H_1\epsilon_2\\ \epsilon_2^T H_1^T & -2H_1\end{bmatrix}\begin{bmatrix}\xi(t)\\ f(\xi(t))\end{bmatrix} &\geqslant 0, \\ \begin{bmatrix}\xi(t-d(t))\\ f(\xi(t-d(t)))\end{bmatrix}^T \begin{bmatrix}-2H_2\epsilon_1 & H_2\epsilon_2\\ \epsilon_2^T H_2^T & -2H_2\end{bmatrix}\begin{bmatrix}\xi(t-d(t))\\ f(\xi(t-d(t)))\end{bmatrix} &\geqslant 0.\end{aligned} \tag{3.12}$$

By utilization of zero equation, the following equations are true for any real constant matrices K_i, $i = 1, 2$ with appropriate dimensions

$$\Big[2K_1 f^T(\xi(t)) + 2K_2 u^T(t)\Big]\Big[\dot{\xi}(t) + A\xi(t-\delta) - W_1 f(\xi(t)) - W_2 f(\xi(t-d(t))) - W_3\dot{\xi}(t-r(t)) - W_4\int_{t-\rho(t)}^{t} f(\xi(s))ds - u(t)\Big] = 0. \tag{3.13}$$

According to (3.3)-(3.13), it is straightforward to see that

$$\dot{V}(\xi(t)) - \gamma u^T(t)u(t) - 2z^T(t)u(t) \leqslant \sigma^T(t)\Sigma\sigma(t),$$

where

$$\begin{aligned}\sigma^T(t) = [&\xi^T(t), \xi^T(t-\delta), \xi^T(t-d_1), \xi^T(t-d_2), \xi^T(t-d(t)), \dot{\xi}^T(t), \dot{\xi}^T(t-d_1), \dot{\xi}^T(t-d_2),\\ &\dot{\xi}^T(t-r(t)), f^T(\xi(t)), f^T(\xi(t-d(t))), \int_{t-\delta}^{t}\xi^T(s)ds, \int_{t-\rho(t)}^{t} f^T(\xi(s))ds, \int_{t-\rho_2}^{t-\rho_1} f^T(\xi(s))ds,\\ &\int_{t-d_2}^{t-d(t)}\xi(s)ds, \int_{t-d(t)}^{t-d_1}\xi^T(s)ds, \int_{t-d(t)}^{t}\dot{\xi}^T(s)ds, \int_{t-d_1}^{t}\int_{u}^{t}\xi^T(\lambda)d\lambda du, \int_{t-d_2}^{t}\int_{u}^{t}\xi^T(\lambda)d\lambda du,\\ &\int_{t-d_2}^{t-d_1}\int_{u}^{t}\xi^T(\lambda)d\lambda du, u(t)].\end{aligned}$$

If the conditions (3.1) hold and $\sum < 0$, then

$$\dot{V}(\xi(t)) - \gamma u^T(t)u(t) - 2z^T(t)u(t) \leqslant 0, \tag{3.14}$$

for any $\sigma(t) \neq 0$. Since $V(\xi(0)) = 0$ under zero initial condition, let $\xi(t) = 0$ for $t \in [\tau_{max,0}]$ after integrating (3.14) with respect to t over the time period from 0 to t_f, we get

$$\begin{aligned}2\int_0^{t_f} z^T(s)u(s)ds &\geqslant V(\xi(t_f)) - V(\xi(0)) - \gamma\int_0^{t_f} u^T(s)u(s)ds\\ &\geqslant -\gamma\int_0^{t_f} u^T(s)u(s)ds.\end{aligned}$$

Thus, the NTNNs (2.1) is passive in the sense of Definition 2.1. This completes the proof. □

Remark 3.2. The proof of Theorem 3.1 shows estimating of integral terms by Lemmas 2.7, 2.8, and 2.9 applying $V_6(\xi(t), t)$. Moreover, we have created these lemmas ourselves, which obtained a tighter upper bound than [7, 41, 44, 49, 51, 52, 55, 57].

Remark 3.3. When $W_3 = 0, C_1 = I, C_2 = C_3 = C_4 = 0$, the system (2.1) without neutral term is reduced to the following NNs

$$\begin{cases}\dot{\xi}(t) = -A\xi(t-\delta) + W_1 f(\xi(t)) + W_2 f(\xi(t-d(t))) + W_4\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t),\\ z(t) = f(\xi(t)),\\ \xi(t) = \phi(t), \quad t\in[-\tau_{\max}, 0], \quad \tau_{\max} = \max\{d_2, \rho_2\}.\end{cases} \tag{3.15}$$

By employing the following Lyapunov-Krasovskii functional as

$$\begin{aligned}V_1(\xi(t), t) &= \big(\xi(t) - A\int_{t-\delta}^{t}\xi(s)ds\big)^T P_1\big(\xi(t) - A\int_{t-\delta}^{t}\xi(s)ds\big),\\ V_2(\xi(t), t) &= \int_{t-\delta}^{t}\xi^T(s)P_2\xi(s)ds + \delta\int_{-\delta}^{0}\int_{t+\theta}^{t}\xi^T(s)P_3\xi(s)dsd\theta,\end{aligned}$$

$$V_3(\xi(t),t) = \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \xi(t-d(t)) \\ \int_{t-d(t)}^{t} \dot{\xi}(s)ds \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \xi(t-d(t)) \\ \int_{t-d(t)}^{t} \dot{\xi}(s)ds \end{bmatrix},$$

$$V_4(\xi(t),t) = \int_{t-d_2}^{t} \xi^T(s)P_7\xi(s)ds + \int_{t-d_2}^{t} \dot{\xi}^T(s)P_8\dot{\xi}(s)ds + \int_{t-d_2}^{t} \begin{bmatrix} \xi(s) \\ \dot{\xi}(s) \end{bmatrix}^T \begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \begin{bmatrix} \xi(s) \\ \dot{\xi}(s) \end{bmatrix} ds,$$

$$V_5(\xi(t),t) = d_2 \int_{-d_2}^{0} \int_{t+s}^{t} \xi^T(\theta)P_9\xi(\theta)d\theta ds + \int_{-d_2}^{0} \int_{t+s}^{t} \dot{\xi}^T(\theta)P_{10}\dot{\xi}(\theta)d\theta ds$$

$$+ d_2 \int_{-d_2}^{0} \int_{t+s}^{t} \begin{bmatrix} \xi(\theta) \\ \dot{\xi}(\theta) \end{bmatrix}^T \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \begin{bmatrix} \xi(\theta) \\ \dot{\xi}(\theta) \end{bmatrix} d\theta ds,$$

$$V_6(\xi(t),t) = \frac{{d_2}^2}{2} \int_{t-d_2}^{t} \int_{s}^{t} \int_{u}^{t} \xi^T(\lambda)P_{11}\xi(\lambda)d\lambda du ds + \frac{{d_2}^2}{2} \int_{t-d_2}^{t} \int_{s}^{t} \int_{u}^{t} \dot{\xi}^T(\lambda)P_{12}\dot{\xi}(\lambda)d\lambda du ds,$$

$$V_7(\xi(t),t) = \frac{d_2^3}{6} \int_{t-d_2}^{t} \int_{s}^{t} \int_{u}^{t} \int_{\lambda}^{t} \dot{\xi}^T(\theta)P_{14}\dot{\xi}(\theta)d\theta d\lambda du ds + \frac{d_2^3}{6} \int_{t-d_2}^{t} \int_{s}^{t} \int_{u}^{t} \int_{\lambda}^{t} \dot{\xi}^T(\theta)P_{15}\dot{\xi}(\theta)d\theta d\lambda du ds,$$

we can obtain the following Corollary 3.4 for the passivity of the above NNs (3.15) by using Theorem 3.1. We introduce the following notations for later use:

$$\tilde{\sum} = [\tilde{\Sigma}_{i,j}]_{20\times 20},$$

where $\tilde{\Sigma}_{i,j} = \tilde{\Sigma}_{j,i}^T = \Sigma_{i,j}, \quad i,j = 1,2,3,\dots,21$, except

$$\tilde{\Sigma}_{1,9} = \tilde{\Sigma}_{5,9} = \tilde{\Sigma}_{6,9} = \tilde{\Sigma}_{9,9} = \tilde{\Sigma}_{12,9} = \tilde{\Sigma}_{13,9} = \tilde{\Sigma}_{21,9} = 0,$$

$$\tilde{\Sigma}_{6,6} = -Q_2^T - Q_2 + d_2P_9 + d_2R_6 + \frac{(d_2)^6}{36}P_{13},$$

$$\tilde{\Sigma}_{11,21} = -W_2^TK_2^T, \quad \tilde{\Sigma}_{21,21} = \gamma - K_2 - K_2^T.$$

Corollary 3.4. *The delayed NNs (3.15) are passive in Definition 2.1, if there exist positive definite matrices* Q_1, R_4, R_6 P_i, $i \in \{1,2,\dots,15\}$, *except* $P_5 = 0$, *any appropriate dimensional matrices* G, K_1, K_2, R_m, Q_n, M_o *and* $m = 1,2,\dots,6$, $n = 1,2,\dots,4$, $o = 1,2,\dots,5$ *such that the following symmetric linear matrix inequalities hold*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_9 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0, \quad \tilde{\sum} < 0.$$

Proof. The proof is similar to that in Theorem 3.1, and so it is omitted. □

Remark 3.5. When $W_3 = W_4 = 0, C_1 = I, C_2 = C_3 = C_4 = 0$ and $\delta = 0$, the system (2.1) without neutral, leakage and distributed term is reduced to the following NNs

$$\begin{cases} \dot{\xi}(t) = -A\xi(t) + W_1f(\xi(t)) + W_2f(\xi(t-d(t))) + u(t), \\ z(t) = f(\xi(t)), \\ \xi(t) = \phi(t), \quad t \in [-d_2, 0]. \end{cases} \tag{3.16}$$

By employing the following Lyapunov-Krasovskii functional as

$$V_1(\xi(t),t) = \xi(t)P_1\xi(t),$$

$$V_2(\xi(t),t) = \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \xi(t-d(t)) \\ \int_{t-d(t)}^{t} \dot{\xi}(s)ds \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix} \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \\ \xi(t-d(t)) \\ \int_{t-d(t)}^{t} \dot{\xi}(s)ds \end{bmatrix},$$

$$V_3(\xi(t),t)=\int_{t-d_2}^{t}\xi^T(s)P_7\xi(s)ds+\int_{t-d_2}^{t}\dot{\xi}^T(s)P_8\dot{\xi}(s)ds+\int_{t-d_2}^{t}\begin{bmatrix}\xi(s)\\ \dot{\xi}(s)\end{bmatrix}^T\begin{bmatrix}R_1 & R_2\\ * & R_3\end{bmatrix}\begin{bmatrix}\xi(s)\\ \dot{\xi}(s)\end{bmatrix}ds,$$

$$V_4(\xi(t),t)=d_2\int_{-d_2}^{0}\int_{t+s}^{t}\xi^T(\theta)P_9\xi(\theta)d\theta ds+\int_{-d_2}^{0}\int_{t+s}^{t}\dot{\xi}^T(\theta)P_{10}\dot{\xi}(\theta)d\theta ds$$
$$+d_2\int_{-d_2}^{0}\int_{t+s}^{t}\begin{bmatrix}\xi(\theta)\\ \dot{\xi}(\theta)\end{bmatrix}^T\begin{bmatrix}R_4 & R_5\\ * & R_6\end{bmatrix}\begin{bmatrix}\xi(\theta)\\ \dot{\xi}(\theta)\end{bmatrix}d\theta ds,$$

$$V_5(\xi(t),t)=\frac{{d_2}^2}{2}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\xi^T(\lambda)P_{11}\xi(\lambda)d\lambda du ds+\frac{{d_2}^2}{2}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\dot{\xi}^T(\lambda)P_{12}\dot{\xi}(\lambda)d\lambda du ds,$$

$$V_6(\xi(t),t)=\frac{d_2^3}{6}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\int_{\lambda}^{t}\dot{\xi}^T(\theta)P_{14}\dot{\xi}(\theta)d\theta d\lambda du ds+\frac{d_2^3}{6}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\int_{\lambda}^{t}\dot{\xi}^T(\theta)P_{15}\dot{\xi}(\theta)d\theta d\lambda du ds,$$

we can obtain the following Corollary 3.6 for the passivity of the above NNs (3.16) by using Theorem 3.1. We introduce the following notations for later use:

$$\hat{\sum}=[\hat{\Sigma}_{i,j}]_{17\times 17},$$

where $\hat{\Sigma}_{i,j}=\hat{\Sigma}^T_{j,i}=\Sigma_{i,j},\quad i,j=1,2,3,\dots,21,$ except

$$\begin{aligned}
&\hat{\Sigma}_{1,9}=\hat{\Sigma}_{5,9}=\hat{\Sigma}_{6,9}=\hat{\Sigma}_{9,9}=\hat{\Sigma}_{12,9}=\hat{\Sigma}_{13,9}=\hat{\Sigma}_{21,9}=0,\\
&\hat{\Sigma}_{6,6}=-Q_2^T-Q_2+d_2P_9+d_2R_6+\frac{(d_2)^6}{36}P_{13},\\
&\hat{\Sigma}_{11,21}=-W_2^TK_2^T,\quad \hat{\Sigma}_{21,21}=\gamma-K_2-K_2^T,\\
&\hat{\Sigma}_{1,12}=\hat{\Sigma}_{9,12}=\hat{\Sigma}_{10,12}=\hat{\Sigma}_{11,12}=\hat{\Sigma}_{12,12}=\hat{\Sigma}_{13,12}=\hat{\Sigma}_{21,12}=0,\\
&\hat{\Sigma}_{1,1}=-P_1A-A^TP_1+P_4^TG-Q_1^TA+G^TP_4+d_2^2P_8+d_2^2R_4+d_2^2P_{11}-\frac{d_2^4P_{13}}{4}-2H_1\epsilon_1,\\
&\hat{\Sigma}_{2,2}=\hat{\Sigma}_{2,5}=\hat{\Sigma}_{2,6}=\hat{\Sigma}_{2,13}=\hat{\Sigma}_{14,14}=0,\\
&\hat{\Sigma}_{1,13}=\hat{\Sigma}_{2,13}=\hat{\Sigma}_{5,13}=\hat{\Sigma}_{6,13}=\hat{\Sigma}_{9,13}=\hat{\Sigma}_{10,13}=\hat{\Sigma}_{11,13}=\hat{\Sigma}_{12,13}=\hat{\Sigma}_{13,13}=\hat{\Sigma}_{13,21}=0.
\end{aligned}$$

Corollary 3.6. *The delayed NNs (3.16) are passive in Definition 2.1, if there exist positive definite matrices Q_1, R_4, R_6 P_i, $i\in\{1,2,\dots,15\}$, except $P_2=P_3=P_5=P_6=0$, any appropriate dimensional matrices G, K_1, K_2 R_m, Q_n, M_o and $m=1,2,\dots,6$, $n=1,2,\dots,4$, $o=1,2,\dots,5$ such that the following symmetric linear matrix inequalities hold*

$$\begin{bmatrix}R_1 & R_2\\ * & R_3\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}R_4 & R_5\\ * & R_6\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}P_9 & M_1 & M_2\\ * & M_3 & M_4\\ * & * & M_5\end{bmatrix}\geqslant 0,\quad \hat{\sum}<0.$$

Proof. The proof is similar to that in Theorem 3.1, and so it is omitted. □

3.2. *Delay-range-dependent robust passivity criteria*

According to Theorem 3.1, we can obtain delay-range-dependent robustly passivity criteria of system (2.4). We introduce the following notations for later use.

$$\overline{\sum}=[\overline{\Sigma}_{i,j}]_{21\times 21},\tag{3.17}$$

where $\overline{\Sigma}_{i,j}=\overline{\Sigma}^T_{j,i}=\Sigma_{i,j}$, $i,j=1,2,3,\dots,21$, $Z_1=P_1^TW_3$, $Z_2=P_1^TW_1$, $Z_3=P_1^TW_2$, $Z_4=P_1^TW_4$, $Z_5=G_1P_1^T$, $Z_6=G_1P_1$ and

$$S_1^T=[E^TP_1\quad 0],$$

$$
\begin{aligned}
S_2^T &= [E^TQ_1 \quad 0],\\
S_3^T &= [0 \quad 0 \quad 0 \quad 0 \quad E^TQ_3 \quad E^TQ_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_4^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_5^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_6^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_7^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -E^TP_1A \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_8^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -E^TP_1^TA^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_9^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_{10}^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_{11}^T &= [E^T \quad 0],\\
S_{12}^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
S_{13}^T &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad E^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_1 &= [-G_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_4 \quad G_2 \quad G_3 \quad 0 \quad G_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_2 &= [-G_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_4 \quad G_2 \quad G_3 \quad 0 \quad G_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_3 &= [0 \quad -G_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_4 \quad G_2 \quad G_3 \quad 0 \quad G_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_4 &= [0 \quad G_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -G_4 \quad -G_2 \quad -G_3 \quad 0 \quad -G_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_5 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -G_1Z_1 \quad -G_1Z_2 \quad -G_1Z_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_6 &= [0 \quad G_1P_1],\\
N_7 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_4 \quad G_2 \quad G_3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_8 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_9 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_1Z_4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_{10} &= [Z_5A^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -G_1P_1^T \quad -G_1P_1^T \quad -G_1P_1^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_{11} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad Z_6A^T + Z_6 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_{12} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad Z_6 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],\\
N_{13} &= [G_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad G_4 \quad G_2 \quad G_3 \quad 0 \quad G_5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].
\end{aligned}
$$

Theorem 3.7. *The delayed uncertain NTNNs (2.4) are robust passivity in Definition 2.1, if there exist positive definite matrices Q_1, R_4, R_6 P_i, $i \in \{1, 2, \dots, 15\}$, any appropriate dimensional matrices G, K_1, K_2 R_m, Z_m Q_n, M_o and $m = 1, 2, \dots, 6$, $n = 1, 2, \dots, 4$, $o = 1, 2, \dots, 5$ such that the following symmetric linear matrix inequalities hold*

$$
\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \geqslant 0, \tag{3.18}
$$

$$
\begin{bmatrix} P_9 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0, \tag{3.19}
$$

$$
\sum_{k=1}^{13} \begin{bmatrix} \sum & S_k & \sigma N_k^T \\ * & -\sigma I & \sigma J^T \\ * & * & -\sigma I \end{bmatrix} < 0. \tag{3.20}
$$

Proof. Replacing A, W_1, W_2, W_3 and W_4 in (3.2) with $A = A + E\Delta(t)G_1$, $W_1 = W_1 + E\Delta(t)G_2$, $W_2 = W_2 + E\Delta(t)G_3$, $W_3 = W_3 + E\Delta(t)G_4$ and $W_4 = W_4 + E\Delta(t)G_5$, respectively, we find that condition (3.2) is equivalent to the following condition

$$
\overline{\sum} + S\Delta(t)N + N^T\Delta(t)^TS^T < 0, \tag{3.21}
$$

where $\overline{\sum}$ are defined in (3.17). By using Lemma 2.3, we can find that (3.21) is equivalent to the LMIs as follows

$$\sum_{k=1}^{13}\begin{bmatrix}\overline{\sum} & S_k & \sigma N_k^T\\ * & -\sigma I & \delta J^T\\ * & * & -\sigma I\end{bmatrix}<0, \tag{3.22}$$

and σ is positive real constant. From Theorem 3.1 and conditions (3.18)-(3.20), the system (2.4) is robust passivity. The proof of theorem is complete. □

Remark 3.8. When $\delta = 0, C_1 = I, C_2 = C_3 = C_4 = 0$, the system (2.4) without leakage term is reduced to the following uncertain NTNNs with parameter uncertainties satisfying (2.5)-(2.8),

$$\begin{cases}\dot{\xi}(t) = -[A+\Delta A(t)]\xi(t) + [W_1+\Delta B(t)]f(\xi(t)) + [W_2+\Delta C(t)]f(\xi(t-d(t)))\\ \qquad\quad +[W_3+\Delta D(t)]\dot{\xi}(t-r(t)) + [W_4+\Delta E(t)]\int_{t-\rho(t)}^{t} f(\xi(s))ds + u(t),\\ z(t) = f(\xi(t)),\\ \xi(t) = \phi(t), \quad t\in[-\tau_{max},0], \quad \tau_{max} = \max\{d_2,\rho_2,r_2\},\end{cases} \tag{3.23}$$

By employing the following Lyapunov-Krasovskii functional as

$$\begin{aligned}
V_1(\xi(t),t) &= \xi(t)P_1\xi(t),\\
V_2(\xi(t),t) &= \begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\\ \xi(t-d(t))\\ \int_{t-d(t)}^{t}\dot{\xi}(s)ds\end{bmatrix}^T \begin{bmatrix}I&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}P_4&0&0&0\\0&0&0&0\\0&0&0&0\\Q_1&Q_2&Q_3&Q_4\end{bmatrix}\begin{bmatrix}\xi(t)\\ \dot{\xi}(t)\\ \xi(t-d(t))\\ \int_{t-d(t)}^{t}\dot{\xi}(s)ds\end{bmatrix},\\
V_3(\xi(t),t) &= (r_2-r_1)\int_{t-r(t)}^{t}\dot{\xi}^T(s)P_5\dot{\xi}(s)ds + (\rho_2-\rho_1)\int_{-\rho_2}^{-\rho_1}\int_{t+s}^{t} f(\xi(\theta))^T P_6 f(\xi(\theta))d\theta ds,\\
V_4(\xi(t),t) &= \int_{t-d_2}^{t}\xi^T(s)P_7\xi(s)ds + \int_{t-d_2}^{t}\dot{\xi}^T(s)P_8\dot{\xi}(s)ds + \int_{t-d_2}^{t}\begin{bmatrix}\xi(s)\\ \dot{\xi}(s)\end{bmatrix}^T\begin{bmatrix}R_1&R_2\\ *&R_3\end{bmatrix}\begin{bmatrix}\xi(s)\\ \dot{\xi}(s)\end{bmatrix}ds,\\
V_5(\xi(t),t) &= d_2\int_{-d_2}^{0}\int_{t+s}^{t}\xi^T(\theta)P_9\xi(\theta)d\theta ds + \int_{-d_2}^{0}\int_{t+s}^{t}\dot{\xi}^T(\theta)P_{10}\dot{\xi}(\theta)d\theta ds\\
&\quad + d_2\int_{-d_2}^{0}\int_{t+s}^{t}\begin{bmatrix}\xi(\theta)\\ \dot{\xi}(\theta)\end{bmatrix}^T\begin{bmatrix}R_4&R_5\\ *&R_6\end{bmatrix}\begin{bmatrix}\xi(\theta)\\ \dot{\xi}(\theta)\end{bmatrix}d\theta ds,\\
V_6(\xi(t),t) &= \frac{d_2^2}{2}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\xi^T(\lambda)P_{11}\xi(\lambda)d\lambda du ds + \frac{{d_2}^2}{2}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\dot{\xi}^T(\lambda)P_{12}\dot{\xi}(\lambda)d\lambda du ds,\\
V_7(\xi(t),t) &= \frac{d_2^3}{6}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\int_{\lambda}^{t}\dot{\xi}^T(\theta)P_{14}\dot{\xi}(\theta)d\theta d\lambda du ds + \frac{d_2^3}{6}\int_{t-d_2}^{t}\int_{s}^{t}\int_{u}^{t}\int_{\lambda}^{t}\dot{\xi}^T(\theta)P_{15}\dot{\xi}(\theta)d\theta d\lambda du ds,
\end{aligned}$$

we can obtain the following Corollary 3.9 for the passivity of the above uncertain NTNNs (3.23) by using Theorem 3.1.

$$\dot{\sum} = [\dot{\Sigma}_{i,j}]_{19\times 19},$$

where $\dot{\Sigma}_{i,j} = \dot{\Sigma}_{j,i}^T = \Sigma_{i,j},\ i,j = 1,2,3,\dots,21$, except

$$\begin{aligned}
\dot{\Sigma}_{1,1} &= -P_1A - A^TP_1 + P_2 + \delta^2P_3 + P_4^TG - Q_1^TA + G^TP_4 + d_2^2P_8\\
&\quad + d_2^2R_4 - d_2^2P_{11} - \frac{d_2^4P_{13}}{4} - 2H_1\epsilon_1 - Q_1^TA - P_2,\\
\dot{\Sigma}_{1,2} &= \dot{\Sigma}_{2,2} = \dot{\Sigma}_{5,2} = \dot{\Sigma}_{6,2} = \dot{\Sigma}_{10,2} = \dot{\Sigma}_{13,2} = \dot{\Sigma}_{21,2} = 0,
\end{aligned}$$

$$\dot{\Sigma}_{1,5} = -P_4^TG - A^TQ_3,$$
$$\dot{\Sigma}_{1,6} = P_4^T - Q_1^T + d_2^2R_5 - A^TQ_2,$$
$$\dot{\Sigma}_{1,10} = P_1W_1 + Q_1^TW_1 + H_1\epsilon_2 + A^TK_1,$$
$$\dot{\Sigma}_{1,13} = P_1W_4 + Q_1^TW_4 + A^T, \quad \dot{\Sigma}_{1,21} = P_1 + Q_1^T + A^TK_2,$$
$$\dot{\Sigma}_{1,12} = \dot{\Sigma}_{9,12} = \dot{\Sigma}_{10,12} = \dot{\Sigma}_{11,12} = \dot{\Sigma}_{12,12} = \dot{\Sigma}_{13,12} = \dot{\Sigma}_{21,12} = 0.$$

Corollary 3.9. *The delayed uncertain NTNNs* (3.23) *are robust passivity in Definition* 2.1*, if there exist positive definite matrices* Q_1*,* R_4*,* R_6 P_i*,* $i \in \{1, 2, \ldots, 15\}$*, except* $P_2 = P_3 = 0$*, any appropriate dimensional matrices* G*,* K_1*,* K_2 R_m*,* Z_m Q_n*,* M_o *and* $m = 1, 2, \ldots, 6$*,* $n = 1, 2, \ldots, 4$*,* $o = 1, 2, \ldots, 5$ *such that the following symmetric linear matrix inequalities hold*

$$\begin{bmatrix} R_1 & R_2 \\ * & R_3 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} R_4 & R_5 \\ * & R_6 \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_9 & M_1 & M_2 \\ * & M_3 & M_4 \\ * & * & M_5 \end{bmatrix} \geqslant 0, \quad \sum_{k=1}^{13} \begin{bmatrix} \dot{\Sigma} & S_k & \delta N_k^T \\ * & -\delta I & \delta J^T \\ * & * & -\delta I \end{bmatrix} < 0.$$

Proof. The proof is similar to that in Theorem 3.7, and so it is omitted. □

4. Numerical examples

Example 4.1. Consider the following continuous NTNNs with discrete, neutral, distributed interval time-varying delays and leakage delay (2.1). We consider passivity analysis of system (2.1) by using Theorem 3.1. The system (2.1) is specified as follow

$$A = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} 0.2 & 1.2 \\ 1.3 & 0.3 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 3 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4.2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1 & 0.3 \\ 0 & 1.3 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3.1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0.5 \\ 0 & 2.5 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$d(t) = |\cos(t)|, \quad \rho(t) = \sin^2(0.6t), \quad r(t) = \cos^2(0.5t), \quad \phi(t) = \begin{bmatrix} -0.3, \\ 0.3 \end{bmatrix}, \quad t \in [-1, 0].$$

The activation function are assumed to be

$$f_k(x_k) = 0.5(|x_k + 1| - |x_k - 1|), k = 1, 2.$$

It is easy to check that the activation functions are satisfied (2.3) with $l_k^- = 0$, $l_k^+ = 1$, $k = 1, 2$. For $\delta = 0.5$, $\gamma = 0.7$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.6$, $d_1 = 0.1$, $d_2 = 0.7$, $\rho_1 = 0.1$, $\rho_2 = 0.3$, $r_1 = 0.1$ $r_2 = 0.6$, and $r_d = 0.5$. By using LMI Toolbox in MATLAB and by solving the LMIs in Theorem 3.1. This example shows that the solutions of LMIs are given as follows:

$$P_1 = \begin{bmatrix} 19.4347 & -1.5847 \\ -1.5847 & 41.6291 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.5271 & 0.0021 \\ 0.0021 & 4.5278 \end{bmatrix} \times 10^4,$$
$$P_3 = \begin{bmatrix} 9.5240 & 0.0402 \\ 0.0402 & 8.4138 \end{bmatrix} \times 10^4, \quad P_4 = \begin{bmatrix} 1.2518 & 0.0015 \\ 0.0015 & 1.2526 \end{bmatrix} \times 10^3,$$
$$P_5 = \begin{bmatrix} 4.1971 & 0.0014 \\ 0.0014 & 4.2147 \end{bmatrix} \times 10^4, \quad P_6 = \begin{bmatrix} 9.8610 & 0.0000 \\ 0.000 & 9.8610 \end{bmatrix} \times 10^4,$$
$$P_7 = \begin{bmatrix} 9.8610 & 0 \\ 0 & 9.8610 \end{bmatrix} \times 10^4, \quad P_8 = \begin{bmatrix} 4.3146 & 0.0023 \\ 0.0023 & 4.3161 \end{bmatrix} \times 10^4,$$

$$P_9 = \begin{bmatrix} 1.5375 & -0.0000 \\ -0.0000 & 4.3161 \end{bmatrix} \times 10^5, \qquad P_{10} = \begin{bmatrix} 905.8027 & 0.4037 \\ 0.4037 & 906.1232 \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} 8.2612 & -0.0071 \\ -0.0071 & 8.3253 \end{bmatrix} \times 10^4, \qquad P_{12} = \begin{bmatrix} 1.4532 & 0.0030 \\ 0.0030 & 1.4583 \end{bmatrix},$$

$$P_{13} = \begin{bmatrix} 3.3852 & 0.0080 \\ 0.0080 & 3.3725 \end{bmatrix} \times 10^3, \qquad P_{14} = \begin{bmatrix} 945.2841 & -0.4866 \\ -0.4866 & 944.5719 \end{bmatrix},$$

$$P_{15} = \begin{bmatrix} 9.8610 & 0 \\ 0 & 9.8610 \end{bmatrix} \times 10^4, \qquad R_1 = \begin{bmatrix} 2.2317 & -0.0000 \\ -0.0000 & 2.2305 \end{bmatrix} \times 10^4,$$

$$R_2 = \begin{bmatrix} 0.0025 & -0.0000 \\ -0.0000 & 0.0025 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 9.8610 & -0.0000 \\ 0.0000 & 9.8610 \end{bmatrix} \times 10^4,$$

$$R_4 = \begin{bmatrix} 6.0027 & 0.0022 \\ 0.0022 & 6.0041 \end{bmatrix} \times 10^4, \qquad R_5 = \begin{bmatrix} -94.6213 & 28.8074 \\ 28.8074 & -11.2297 \end{bmatrix},$$

$$R_6 = \begin{bmatrix} 1.3333 & -0.0001 \\ -0.0001 & 1.3339 \end{bmatrix} \times 10^5, \qquad Q_1 = \begin{bmatrix} 94.8348 & -3.4949 \\ -3.4949 & 75.1631 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -0.1927 & -0.1155 \\ -0.1155 & -2.8088 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 23.6291 & -1.2434 \\ -1.2434 & 24.6800 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 1.9272 & -0.0024 \\ -0.0024 & 1.9292 \end{bmatrix} \times 10^4, \qquad M_1 = \begin{bmatrix} -4.8991 & 0.0013 \\ 0.0013 & -4.9047 \end{bmatrix} \times 10^4,$$

$$M_2 = \begin{bmatrix} -9.5622 & -0.0171 \\ -0.0171 & -0.96267 \end{bmatrix} \times 10^3, \qquad M_3 = \begin{bmatrix} 1.2089 & 0.0004 \\ 0.0004 & 1.2083 \end{bmatrix} \times 10^5,$$

$$M_4 = \begin{bmatrix} -1.1839 & -0.0006 \\ -0.0006 & -1.1829 \end{bmatrix} \times 10^5, \qquad M_5 = \begin{bmatrix} 1.6993 & 0.0008 \\ 0.0008 & 1.6988 \end{bmatrix} \times 10^5,$$

$$H_1 = \begin{bmatrix} 3.9346 & -0.0081 \\ -0.0081 & 3.8945 \end{bmatrix} \times 10^5, \qquad H_2 = \begin{bmatrix} 3.3267 & -0.0007 \\ -0.0007 & 13.3528 \end{bmatrix} \times 10^4,$$

$$G = \begin{bmatrix} 824.2293 & -11.9000 \\ -11.9000 & 814.0333 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} 5.3643 & 0.0110 \\ 0.0110 & 5.3822 \end{bmatrix} \times 10^7,$$

$$K_2 = \begin{bmatrix} 3.5750 & -0.0026 \\ -0.0026 & 3.5942 \end{bmatrix} \times 10^6.$$

The above result show that all the conditions stated in Theorem 3.1 have been satisfied and hence system (2.1) with the above given parameters is passive.

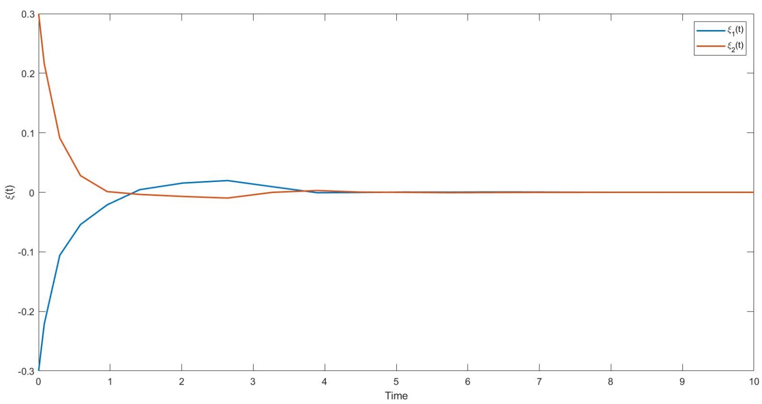

Figure 1: The trajectories of $\xi_1(t)$ and $\xi_2(t)$ with $u(t) = 0$ in Example 4.1.

Example 4.2. Consider neural networks (3.15) with

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.4 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The activation functions are assumed to be

$$f_k(x) = \tanh(x), \quad k = 1, 2.$$

It is easy to check that the activation functions are satisfied (2.3) with $l_k^- = 0.5$, $l_k^+ = 0$, $k = 1, 2$. Using Corollary 3.4 for various leakage delay δ, the maximal upper bounds of d_2 are shown in Table 1. From Table 1, it can be easily seen that the method proposed in this paper is much less conservative than the corresponding method in [41, 57].

Table 1: The Maximal allowable delay d_2 of Example 4.2 for different values of leakage delay δ.

δ	0.01	0.05	0.1
Zhao (2014) [57]	0.6231	0.4341	0.2
Samidurai (2016) [41]	1.0112	0.6213	0.4131
Corollary 3.4	2.1189	1.1010	1.0123

Example 4.3. Consider neural networks (3.16) with

$$A = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The activation functions are assumed to be

$$f_k(x_k) = 0.5(|x_k + 1| - |x_k - 1|), \quad k = 1, 2.$$

Using the MATLAB LMI Toolbox to solve the LMIs in Corollary 3.6, we have that the delayed neural networks in this example, which guarantee the passivity of neural network (3.16), that are list in Table 2. It can be seen that the passivity result we proposed is less conservative than that in [7, 44, 49, 52, 55].

Table 2: The maximum upper bound d_2 of Example 4.3.

Xu (2009) [49]	0.6791
Zeng (2011) [52]	1.3027
Thuan (2016) [44]	2.9068
Zhang (2018) [55]	3.7113
Botmart (2021) [7]	4.1010
Corollary 3.6	4.1124

Example 4.4. Consider the following uncertain NTNNs with discrete, neutral, distributed interval time-varying delays and leakage delay (2.4). We consider passivity analysis of system (2.4) by using Theorem 3.7. The system (2.4) is specified as follow

$$\begin{aligned}
A &= \begin{bmatrix} 1.3 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.3 & -0.2 \\ 0.5 & -0.1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}, \\
W_3 &= \begin{bmatrix} 0.1 & -0.3 \\ 0.2 & 1.2 \end{bmatrix}, \quad W_4 = \begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.05 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 1 & 0.5 \\ 0 & 1.5 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1 & 0.5 \\ 0 & 2.5 \end{bmatrix}, \quad J = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
\end{aligned}$$

$$E = 0.1, \quad G_1 = 0.1, \quad G_2 = 0.2, \quad G_3 = 0.3, \quad G_4 = 0.4, \quad G_5 = 0 \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$d(t) = |\sin(t)|, \quad \rho(t) = \sin^2(0.1t), \quad r(t) = \cos^2(0.3t), \quad \phi(t) = \begin{bmatrix} -0.1, \\ 0.1 \end{bmatrix}, \quad t \in [-1, 0].$$

The activation functions are assumed to be

$$f_k(x_k) = 0.5(|x_k + 1| - |x_k - 1|), k = 1, 2.$$

It is easy to check that the activation functions are satisfied (2.3) with $l_k^- = 0, \quad l_k^+ = 1, \quad k = 1, 2$. For $\delta = 0.5, \ \sigma = 0.5, \ \gamma = 0.7, \ \epsilon_1 = 0.5, \ \epsilon_2 = 0.6, \ d_1 = 0.1, \ d_2 = 2, \ \rho_1 = 0.1, \ \rho_2 = 0.3, \ r_1 = 0.1, \ r_2 = 0.6$, and $r_d = 0.5$, by using MATLAB LMI control toolbox and by solving the LMIs in Theorem 3.7 in our paper we obtain the following feasible solutions:

$$P_1 = \begin{bmatrix} 495.7480 & -2.7398 \\ -2.7398 & 441.6610 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 2.5703 & -0.1695 \\ -0.1695 & 2.6821 \end{bmatrix} \times 10^5,$$

$$P_3 = \begin{bmatrix} 5.8529 & -0.0214 \\ -0.0214 & 6.0117 \end{bmatrix} \times 10^4, \qquad P_4 = \begin{bmatrix} 909.8406 & -10.5743 \\ -10.5743 & 949.1335 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 4.0970 & -0.3902 \\ -0.3902 & 3.3536 \end{bmatrix} \times 10^3, \qquad P_6 = \begin{bmatrix} 2.1797 & -0.0755 \\ -0.0755 & 1.7580 \end{bmatrix} \times 10^4,$$

$$P_7 = \begin{bmatrix} 7.4418 & 0 \\ 0 & 7.4418 \end{bmatrix} \times 10^4, \qquad P_8 = \begin{bmatrix} 1.0714 & -0.0001 \\ -0.0001 & 1.1284 \end{bmatrix} \times 10^4,$$

$$P_9 = \begin{bmatrix} 1.8595 & -0.2772 \\ -0.2772 & 1.4106 \end{bmatrix} \times 10^3, \qquad P_{10} = \begin{bmatrix} 2.0081 & -0.0500 \\ -0.0500 & 1.7264 \end{bmatrix} \times 10^4,$$

$$P_{11} = \begin{bmatrix} 2.0169 & -0.0544 \\ -0.0544 & 1.9221 \end{bmatrix} \times 10^4, \qquad P_{12} = \begin{bmatrix} 0.1044 & -0.0110 \\ -0.0110 & 0.0827 \end{bmatrix} \times 10^{-3},$$

$$P_{13} = \begin{bmatrix} 4.8504 & -0.6067 \\ -0.6067 & 3.7434 \end{bmatrix} \times 10^3, \qquad P_{14} = \begin{bmatrix} 6.4440 & -0.1281 \\ -0.1281 & 5.4473 \end{bmatrix} \times 10^4,$$

$$P_{15} = \begin{bmatrix} 7.4418 & 0 \\ 0 & 7.4418 \end{bmatrix} \times 10^4, \qquad R_1 = \begin{bmatrix} 6.7176 & -0.0644 \\ -0.0644 & 5.7450 \end{bmatrix} \times 10^3,$$

$$R_2 = \begin{bmatrix} 942.2893 & -4.8363 \\ -4.8363 & 831.4867 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 9.3211 & -0.3746 \\ -0.3746 & 7.2046 \end{bmatrix} \times 10^3,$$

$$R_4 = \begin{bmatrix} 2.1523 & -0.0150 \\ -0.0150 & 2.4499 \end{bmatrix} \times 10^4, \qquad R_5 = \begin{bmatrix} -3.5944 & 0.4504 \\ 0.4504 & -4.1657 \end{bmatrix} \times 10^3,$$

$$R_6 = \begin{bmatrix} 2.6206 & -0.4679 \\ -0.4679 & 2.1380 \end{bmatrix} \times 10^3, \qquad Q_1 = \begin{bmatrix} 677.8933 & -28.3527 \\ -28.3527 & 690.1097 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -45.4274 & -16.1953 \\ -16.1953 & 24.6726 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 65.5743 & 49.0451 \\ 49.0451 & 70.2736 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 1.5917 & -0.0014 \\ -0.0014 & 1.5347 \end{bmatrix} \times 10^4, \qquad M_1 = \begin{bmatrix} -973.1174 & 152.7987 \\ 152.7987 & -752.0766 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 969.0124 & -157.8168 \\ -157.8168 & 758.9788 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 2.3028 & -0.1600 \\ -0.1600 & 1.8332 \end{bmatrix} \times 10^3,$$

$$M_4 = \begin{bmatrix} -745.8570 & 139.1913 \\ 139.1913 & -578.8167 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} 3.8545 & -0.3016 \\ -0.3016 & 2.9758 \end{bmatrix} \times 10^3,$$

$$H_1 = \begin{bmatrix} 6.0371 & -0.2713 \\ -0.2713 & 6.4835 \end{bmatrix} \times 10^5, \qquad H_2 = \begin{bmatrix} 1.0162 & 0.1817 \\ 0.1817 & 0.6859 \end{bmatrix} \times 10^5,$$

$$G = \begin{bmatrix} 703.3671 & -95.0924 \\ -95.0924 & 669.5938 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} 9.5808 & -0.2651 \\ -0.2651 & 7.9451 \end{bmatrix} \times 10^4,$$

$$K_2 = \begin{bmatrix} 6.9086 & -0.8192 \\ -0.8192 & 4.9141 \end{bmatrix} \times 10^4, \qquad Z_1 = \begin{bmatrix} -238.1269 & 3.2910 \\ 3.2910 & -212.3648 \end{bmatrix},$$

$$Z_2 = \begin{bmatrix} 1.2429 & -0.8969 \\ -0.8969 & 2.6923 \end{bmatrix} \times 10^3, \qquad Z_3 = \begin{bmatrix} -1.8274 & -0.1978 \\ -0.1978 & -1.3873 \end{bmatrix} \times 10^3,$$

$$Z_4 = \begin{bmatrix} 240.8660 & -3.6914 \\ -3.6914 & 211.6639 \end{bmatrix}, \qquad Z_5 = \begin{bmatrix} 14.1088 & -11.8786 \\ -11.8786 & 44.2535 \end{bmatrix},$$
$$Z_6 = \begin{bmatrix} 0.9210 & 0.0555 \\ 0.0555 & 1.0598 \end{bmatrix}.$$

The above result shows that all the conditions stated in Theorem 3.7 have been satisfied and hence system (2.4) with the above given parameters is passive.

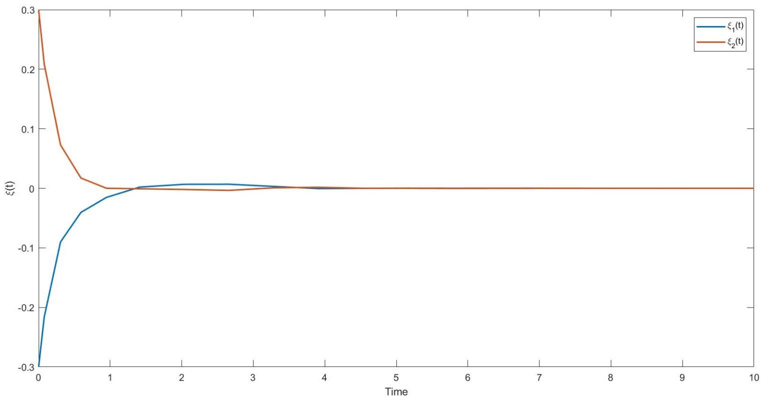

Figure 2: The trajectories of $\xi_1(t)$ and $\xi_2(t)$ with $u(t) = 0$ in Example 4.4.

Example 4.5. Consider the uncertain neutral-type neural networks (3.23) with

$$A = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -1.198 & 0.1 \\ 0.1 & -1.198 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.1 & 0.16 \\ 0.05 & 0.1 \end{bmatrix},$$
$$W_3 = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.2 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0.3 & -0.15 \\ 0.5 & -0.2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = 0.2,$$
$$G_1 = 0.20, \quad G_2 = 0.25, \quad G_3 = 0.15, \quad G_4 = 0.20, \quad G_5 = 0.25.$$

The activation functions are assumed to be

$$f_k(x_k) = 0.5(|x_k + 1| - |x_k - 1|), \quad k = 1, 2.$$

Obviously, (2.3) is satisfied with $l_k^- = 0.5, \quad l_k^+ = 0, \quad k = 1, 2$. Using Corollary 3.9, the maximal upper bounds of d_2 are shown in Table 3. From Table 3, it can be easily seen that the method proposed in this paper is much less conservative than the corresponding method in [41, 51].

Table 3: The maximum upper bound d_2 of Example 4.5.

Zeng (20013) [51]	2.3642
Samidurai (2016) [41]	3.5138
Corollary 3.9	4.1458

Remark 4.6. In this work, the Lyapunov-Krasovskii functional consists of single, double, triple, and quadruple integral terms, which full of the information of the delays $\delta, d_1, d_2, \rho_1, \rho_2, r_1, r_2$ and a state variable $\xi(t)$. Furthermore, we have used various integral inequalities to estimate the derivative of Lyapunov functional, Leibniz-Newton formula and utilization of zero equation. Hence, the construction and the technique for computation of the Lyapunov-Krasovskii functional are the main key to improve results of this work. All of these lead to the improved results in our work as we can see the compared results with some existing works in numerical examples. However, the complex computation of the Lyapunov-Krasovskii functional leads to the LMIs derived in this work which contains many information of the system. It is feasible for NTNNs and NNs with leakage delay which can be solved by using the Matlab LMI toolbox. Consequently, for further work, it is interesting for researchers to improve these technique for a simple Lyapunov-Krasovskii functional and also achieve better results.

5. Conclusions

In this research, We focused on new results for robust passivity analysis of uncertain NTNNs with distributed interval time-varying delay under the effects of leakage delay. By applying a novel Lyapunov-Krasovskii functional approach and using new integral inequalities to estimate the derivative of Lyapunov functional, Leibniz-Newton formula and utilization of zero equation. The new delay-range-dependent criteria for the passivity of the addressed NTNNs, NNs and uncertain NTNNs have been established in term of LMIs, which can be checked by using LMI toolbox in MATLAB. Besides, these results are less conservative than the existing ones and can be an effective method. Five numerical examples have been given to demonstrate the usefulness and the merits of the proposed method.

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