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New distance in cone S-metric spaces and common fixed point theorems

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Abstract

In this work, we define a new distance called c_s-distance in a cone S-metric space with some properties. Then, we prove some common fixed point and fixed point theorems for self-mappings with this distance. After that, we obtain some common fixed point and fixed point results in the setting of cone S-metric spaces. We give some examples to support our work.

Keywords: Cone S-metric spaces, c_s-distance, common fixed points, fixed points.

2020 MSC: 47H10, 54H25.

1. Introduction and preliminaries

Fixed point theory is one of the most arousal research fields in nonlinear analysis. In the last decades, many number of authors have published papers and battened continuously. The application potential is the main reason for this involvement. Fixed point theory has an application in many areas such as chemistry, physics, biology, computer science and many branches of mathematics. Banach contraction mapping principle or Banach fixed point theorem is the most celebrated and pioneer result in a complete metric space. The easily dealing of this theorem, stimulated many authors to introduce new spaces and give various generalizations of the Banach contraction mapping principle, for example the S-metric space.

In 2012, Sedghi et al. [15] introduced the concept of a S-metric space which is different from other spaces and proved fixed point theorems in such spaces. They also give some examples of a S-metric space which shows that the S-metric space is different from other spaces. They built up some topological properties in such spaces and proved some fixed point theorems in the setting of S-metric spaces. Sedghi et al. [15] have done grateful beginning work that open a new field which attracted the authors to study the problems of the fixed point, common fixed point, coupled fixed point and common coupled fixed point by using many contractive conditions for mappings (see [6, 8, 12, 14, 16, 19, 20]).

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In 2016, Souayah and Mlaiki [17] introduced the concept of the S_b-metric space as a generalization of the S-metric space. This work encourage many authors to prove fixed point theorems, as well as common fixed point theorems for two or more mappings on S_b-metric spaces (see [7, 9, 13, 18]).

In 2017, Dhamodharan and Krishnakumar [4] extended and generalized the notion of S-metric spaces to cone S-metric spaces with some properties and proved some fixed point results under the condition of normality for cones. Then, Saluja [10], proved some fixed point theorems on cone S-metric spaces using implicit relation under the condition of normality for cones.

In 2018, Singh and Singh [2] have introduced the concept of a cone S_b-metric space which is a generalization of the S-metric space, the S_b-metric space and the cone S-metric space and they proved some fixed point results (see [3, 11]).

In this paper, we introduce the concept of a c_s-distance in a cone S-metric space with some properties and give some examples. Then, we prove some common fixed point and fixed point theorems for weakly compatible self mappings with this new distance. After that, as a direct application of this new distance, we obtain some common fixed point and fixed point results in the setting of cone S-metric spaces for weakly compatible self mappings with out assumption of the normality for cones.

Let $\mathbb{E}$ be a real Banach space, θ denote the zero element in $\mathbb{E}$ and $\mathbb{P}$ a subset of $\mathbb{E}$. We called $\mathbb{P}$ a cone if satisfies the following three conditions:

1. $\mathbb{P}$ is a nonempty set closed and $\mathbb{P} \neq \{\theta\}$;
2. If a, b are nonnegative real numbers and $\upsilon, \nu \in \mathbb{P}$ then $a\upsilon + b\nu \in \mathbb{P}$;
3. $\upsilon \in \mathbb{P}$ and $-\upsilon \in \mathbb{P}$ imply $\upsilon = \theta$.

For any cone $\mathbb{P} \subset \mathbb{E}$, the partial ordering $\preceq$ with respect to $\mathbb{P}$ is defined by $\upsilon \preceq \nu$ if and only if $\nu - \upsilon \in \mathbb{P}$. The notation of $\prec$ stand for $\upsilon \preceq \nu$ but $\upsilon \neq \nu$. Also, we used $\upsilon \ll y$ to indicate that $\nu - \upsilon \in \mathrm{int}\mathbb{P}$, where $\mathrm{int}\mathbb{P}$ denotes the interior of $\mathbb{P}$. A cone $\mathbb{P}$ is called normal if there exists a number $\mathbb{K}$ such that

$$\theta \preceq \upsilon \preceq \nu \quad \text{implies} \quad \|\upsilon\| \leqslant \mathbb{K}\|\nu\|, \tag{1.1}$$

for all $\upsilon, \nu \in \mathbb{E}$. Equivalently, the cone $\mathbb{P}$ is normal if

$$(\forall n)\ \upsilon_n \preceq \nu_n \preceq \omega_n \text{ and } \lim_{n\to+\infty} \upsilon_n = \lim_{n\to+\infty} \omega_n = \upsilon \quad \text{imply} \quad \lim_{n\to+\infty} \nu_n = \upsilon. \tag{1.2}$$

The least positive number $\mathbb{K}$ satisfying Condition (1.1) is called the normal constant of $\mathbb{P}$ (for more details about cones see [1]).

The next lemma is helpful in our work.

Lemma 1.1 ([5]). *Let $\mathbb{E}$ be a real Banach space with a solid cone $\mathbb{P}$. Suppose that a_n be a sequens in $\mathbb{P}$. Then we have:*

1. *If $a \preceq \lambda a$ where $a \in \mathbb{P}$ and $0 \leqslant \lambda < 1$, then $a = \theta$.*
2. *If $\epsilon \in \mathrm{int}\mathbb{P}$, $\theta \preceq a_n$ and $a_n \longrightarrow \theta$, then there exists a positive integer N such that $a_n \ll \epsilon$ for all $n \geqslant N$.*
3. *If $a \preceq b$ and $b \ll \epsilon$, then $a \ll \epsilon$.*
4. *If $\theta \preceq u \ll \epsilon$ for each $\theta \ll \epsilon$, then $u = \theta$.*

Throughout this paper, we do not assume the normality condition for cones, the only assumptions is that, the cone $\mathbb{P}$ is solid, that is $\mathrm{int}\mathbb{P} \neq \emptyset$.

Definition 1.2 ([4]). Let $\mathbb{X}$ be a nonempty set and $\mathbb{E}$ be a real Banach space equipped and $\preceq$ is the partial ordering with respect to $\mathbb{P}$. A function $S : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{E}$ is said to be a cone S-metric function on $\mathbb{X}$ if it satisfies the following conditions:

1. $\theta \preceq S(\upsilon, \nu, \omega)$ for all $\upsilon, \nu, \omega \in \mathbb{X}$;
2. $S(\upsilon, \nu, \omega) = \theta$ if and only if $\upsilon = \nu = \omega$ for all $\upsilon, \nu, \omega \in \mathbb{X}$;
3. $S(\upsilon, \nu, \omega) \preceq S(\upsilon, \upsilon, a) + S(\nu, \nu, a) + S(\omega, \omega, a)$ for all $\upsilon, \nu, \omega, a \in \mathbb{X}$.

Then S is called a cone S-metric on $\mathbb{X}$ and the pairs $(\mathbb{X}, S)$ is called a cone S-metric space.

Definition 1.3 ([4]). Let $(\mathbb{X}, S)$ be a cone S-metric space, $\{\nu_{n_1}\}$ be a sequence in $\mathbb{X}$ and $\nu \in \mathbb{X}$.

1. For all $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$, if there exists a positive integer N such that $S(\nu_{n_1}, \nu_{n_1}, \nu) \ll \epsilon$ for all $n_1 > N$, then ν_{n_1} is said to be convergent and ν is the limit of $\{\nu_{n_1}\}$. We denote this by $\nu_{n_1} \longrightarrow \nu$.
2. For all $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$, if there exists a positive integer N such that $S(\nu_{n_1}, \nu_{n_1}, \nu_{n_2}) \ll \epsilon$ for all $n_1, n_2 > N$, then $\{\nu_{n_1}\}$ is called a Cauchy sequence in $\mathbb{X}$.
3. A cone S-metric space $(\mathbb{X}, S)$ is called a complete if every Cauchy sequence in $\mathbb{X}$ is convergent.

Lemma 1.4 ([4]). *Let $(\mathbb{X}, S)$ be a cone S-metric space. Then, $S(\upsilon, \upsilon, \nu) = S(\nu, \nu, \upsilon)$ for all $\upsilon, \nu \in \mathbb{X}$.*

Let $f : \mathbb{X} \longrightarrow \mathbb{X}$ and $g : \mathbb{X} \longrightarrow \mathbb{X}$ be two mappings and let $\nu \in \mathbb{X}$. Recall that, if $\omega = g\nu = f\nu$ then we called ν is a coincidence point of mappings and ω is a point of coincidence. If $\nu = g\nu = f\nu$, then we called ν is a common fixed point of f and g. The mappings f and g are called weakly compatible if $gf\nu = fg\nu$ whenever $g\nu = f\nu$.

2. c_s-distance

Now, we gave the idea of a c_s-distance on a cone S-metric space with some properties.

Definition 2.1. Let $(\mathbb{X}, S)$ be a cone S-metric space. A function $\Upsilon : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{E}$ is called a c_s-distance on $\mathbb{X}$ if the following conditions hold:

(Υ1) $\theta \preceq \Upsilon(\upsilon, \nu, \omega)$ for all $\upsilon, \nu, \omega \in \mathbb{X}$;

(Υ2) $\Upsilon(\upsilon, \nu, \omega) \preceq \Upsilon(\upsilon, \upsilon, a) + \Upsilon(\nu, \nu, a) + \Upsilon(a, a, \omega)$ for all $\upsilon, \nu, \omega, a \in \mathbb{X}$;

(Υ3) for each $\upsilon \in \mathbb{X}$ and $n \geqslant 1$, if $\Upsilon(\upsilon, \upsilon, \nu_n) \preceq u$ for some $u = u_x \in \mathbb{P}$, then $\Upsilon(\upsilon, \upsilon, \nu) \preceq u$ whenever $\{\nu_n\}$ is a sequence in $\mathbb{X}$ converging to a point $\nu \in \mathbb{X}$;

(Υ4) for all $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$, there exists $\sigma \in \mathbb{E}$ with $\theta \ll \sigma$ such that $\Upsilon(a, a, \upsilon) \ll \sigma$, $\Upsilon(a, a, \nu) \ll \sigma$ and $\Upsilon(a, a, \omega) \ll \sigma$ imply $S(\upsilon, \nu, \omega) \ll \epsilon$.

Example 2.2. Let $\mathbb{E} = \mathbb{R}^2$ and $\mathbb{P} = \{(\upsilon, \nu) \in \mathbb{E} : \upsilon, \nu \geqslant 0\}$. Let $\mathbb{X} = [0, \infty)$ and define a mapping $S : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{E}$ by $S(\upsilon, \nu, \omega) = (|\upsilon - \omega| + |\nu - \omega|, |\upsilon - \omega| + |\nu - \omega|)$ for all $\upsilon, \nu, \omega \in \mathbb{X}$. Then $(\mathbb{X}, S)$ is a complete cone S-metric space. Define a mapping $\Upsilon : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{E}$ by $\Upsilon(\upsilon, \nu, \omega) = (\omega + \nu, \omega + \nu)$ for all $\upsilon, \nu, \omega \in \mathbb{X}$. It is Clear that, Υ satisfy (Υ1), (Υ2) and (Υ3). Let $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$ be given and put $\sigma = \frac{\epsilon}{4}$. Let $a \in \mathbb{X}$ and suppose that $\Upsilon(a, a, \upsilon) \ll \sigma$, $\Upsilon(a, a, \nu) \ll \sigma$ and $\Upsilon(a, a, \omega) \ll \sigma$. Then we have

$$\begin{aligned}
S(\upsilon, \nu, \omega) &= (|\upsilon - \omega| + |\nu - \omega|, |\upsilon - \omega| + |\nu - \omega|) \\
&\preceq (\upsilon + \nu + \upsilon + \omega, \upsilon + \nu + \upsilon + \omega) \\
&= 2(\upsilon, \upsilon) + (\nu, \nu) + (\omega, \omega) \\
&\preceq 2(\upsilon + a, \upsilon + a) + (\nu + a, \nu + a) + (\omega + a, \omega + a) \\
&= 2\Upsilon(a, a, \upsilon) + \Upsilon(a, a, \nu) + \Upsilon(a, a, \omega) \\
&\ll 2\frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&= \epsilon.
\end{aligned}$$

Hence Υ satisfies (Υ4). Thus, Υ is a c_s-distance on $\mathbb{X}$.

Example 2.3. Let $\mathbb{X} = [0, 1]$ and $\mathbb{E} = C^1_{\mathbb{R}}[0, 1]$ with $\|\upsilon\| = \|\upsilon\|_\infty + \|\upsilon'\|_\infty, \upsilon \in \mathbb{E}$ and let

$$\mathbb{P} = \{\upsilon \in \mathbb{E} : \upsilon(t) \geqslant 0 \text{ on } [0, 1]\}.$$

This cone is solid but it is not normal. Define a cone S-metric $S : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{E}$ by

$$S(\upsilon,\nu,\omega)(t) = (|\upsilon-\omega|+|\nu-\omega|)e^t,$$

for all $\upsilon,\nu,\omega \in \mathbb{X}$. Then $(\mathbb{X}, S)$ is a complete cone S-metric space. Define a mapping $\Upsilon : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{E}$ by $\Upsilon(\upsilon,\nu,\omega) = (\omega+\nu)\cdot e^t$ for all $\upsilon,\nu,\omega \in \mathbb{X}$. It is Clear that Υ satisfy (Υ1), (Υ2) and (Υ3). Let $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$ be given and put $\sigma = \frac{\epsilon}{4}$. Let $a \in \mathbb{X}$ and suppose that $\Upsilon(a,a,\upsilon) \ll \sigma$, $\Upsilon(a,a,\nu) \ll \sigma$ and $\Upsilon(a,a,\omega) \ll \sigma$. Then we have

$$\begin{aligned} S(\upsilon,\nu,\omega)\,(t) &= (|\upsilon-\omega|+|\nu-\omega|)e^t \\ &\preceq (\upsilon+\nu+\upsilon+\omega)e^t \\ &= 2\upsilon e^t+\nu e^t+\omega e^t \\ &\preceq 2(\upsilon+a)e^t+(\nu+a)e^t+(\omega+a)e^t \\ &= 2\Upsilon(a,a,\upsilon)+\Upsilon(a,a,\nu)+\Upsilon(a,a,\omega) \\ &\ll 2\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4} \\ &= \epsilon. \end{aligned}$$

Hence Υ satisfies (Υ4). Thus, Υ is a c_s-distance on $\mathbb{X}$.

Example 2.4. Let $\mathbb{E} = \mathbb{R}$ and $\mathbb{P} = \{\upsilon \in \mathbb{E} : \upsilon \geqslant 0\}$. Let $\mathbb{X} = [0,1]$ and define a mapping $S : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{E}$ by $S(\upsilon,\nu,\omega) = |\upsilon-\omega|+|\nu-\omega|$ for all $\upsilon,\nu,\omega \in \mathbb{X}$. Then $(\mathbb{X}, S)$ is a complete cone S-metric space. Define a mapping $\Upsilon : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{E}$ by $\Upsilon(\upsilon,\nu,\omega) = (\omega+\nu)$ for all $\upsilon,\nu,\omega \in \mathbb{X}$. It is Clear that Υ satisfy (Υ1), (Υ2) and (Υ3). Let $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$ be given and put $\sigma = \frac{\epsilon}{4}$. Let $a \in \mathbb{X}$ and suppose that $\Upsilon(a,a,\upsilon) \ll \sigma$, $\Upsilon(a,a,\nu) \ll \sigma$ and $\Upsilon(a,a,\omega) \ll \sigma$. Then we have

$$\begin{aligned} S(\upsilon,\nu,\omega)(t) &= |\upsilon-\omega|+|\nu-\omega| \\ &\preceq 2\upsilon+\nu+\omega \\ &\preceq 2(\upsilon+a)+(\nu+a)+(\omega+a) \\ &= 2\Upsilon(a,a,\upsilon)+\Upsilon(a,a,\nu)+\Upsilon(a,a,\omega) \\ &\ll 2\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4} \\ &= \epsilon. \end{aligned}$$

Hence Υ satisfies (Υ4). Thus, Υ is a c_s-distance on $\mathbb{X}$.

Lemma 2.5. *The cone* S*-metric function is a* c_s*-distance on* X *where* $(\mathbb{X}, S)$ *is a cone* S*-metric space.*

Proof. Let $(\mathbb{X}, S)$ be a cone S-metric space and define the function $\Upsilon : \mathbb{X}\times\mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{E}$ by $\Upsilon(\upsilon,\nu,\omega) = S(\upsilon,\nu,\omega)$ for all $\upsilon,\nu,\omega \in \mathbb{X}$. It is Clear that Υ satisfy (Υ1), (Υ2) and (Υ3). Let $\epsilon \in \mathbb{E}$ with $\theta \ll \epsilon$ be given and put $\sigma = \frac{\epsilon}{3}$. Suppose that $\Upsilon(a,a,\upsilon) \ll \sigma$, $\Upsilon(a,a,\nu) \ll \sigma$ and $\Upsilon(a,a,\omega) \ll \sigma$. Then we have

$$\begin{aligned} S(\upsilon,\nu,\omega) &\preceq S(\upsilon,\upsilon,a)+S(\nu,\nu,a)+S(\omega,\omega,a) \\ &= S(a,a,\upsilon)+S(a,a,\nu)+S(a,a,\omega) \\ &= \Upsilon(a,a,\upsilon)+\Upsilon(a,a,\nu)+\Upsilon(a,a,\omega) \\ &\ll \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Hence Υ satisfies (Υ4). Thus, Υ is a c_s-distance on $\mathbb{X}$. □

Lemma 2.6. Let $(\mathbb{X}, S)$ be a cone S-metric space and Υ is a c_s-distance on $\mathbb{X}$. Let $\{\upsilon_n\}$ and $\{\nu_n\}$ be two sequences in X and $\upsilon,\nu,\omega \in \mathbb{X}$. Suppose that τ_n is a sequence in $\mathbb{P}$ converging to θ. Then the following hold:

1. If $\Upsilon(\upsilon_n, \upsilon_n, \nu) \preceq \tau_n$ and $\Upsilon(\upsilon_n, \upsilon_n, \omega) \preceq \tau_n$, then $\nu = \omega$.
2. If $\Upsilon(\upsilon_n, \upsilon_n, \nu_n) \preceq \tau_n$ and $\Upsilon(\upsilon_n, \upsilon_n, \omega) \preceq \tau_n$, then $\{\nu_n\}$ converges to ω.
3. If $\Upsilon(\upsilon_n, \upsilon_n, \upsilon_m) \preceq \tau_n$ for $m > n$, then $\{\upsilon_n\}$ is a Cauchy sequence in $\mathbb{X}$.
4. If $\Upsilon(\nu, \nu, \upsilon_n) \preceq \tau_n$, then $\{\upsilon_n\}$ is a Cauchy sequence in $\mathbb{X}$.

Proof.

1. Since τ_n is a sequence in $\mathbb{P}$ converging to θ, then there exist a positive integer N and

$$\theta \ll \epsilon \in \mathrm{int}\mathbb{P},$$

such that $\tau_n \ll \epsilon$ for all $n \geqslant N$. Hence, $\Upsilon(\upsilon_n, \upsilon_n, \nu) \ll \epsilon$ and $\Upsilon(\upsilon_n, \upsilon_n, \omega) \ll \epsilon$. By $\Upsilon 4$ with $\sigma = \epsilon$, it follows that $S(\nu, \nu, \omega) \ll \epsilon$. By Lemma 1.1 (4), it follows that $S(\nu, \nu, \omega) = \theta$. Hence $\nu = \omega$.
2. As in the proof of (1), there exist a positive integer N and $\theta \ll \epsilon \in \mathrm{int}\mathbb{P}$ such that $\tau_n \ll \epsilon$ for all $n \geqslant N$. Hence, $\Upsilon(\upsilon_n, \upsilon_n, \nu_n) \ll \epsilon$ and $\Upsilon(\upsilon_n, \upsilon_n, \omega) \ll \epsilon$. By $\Upsilon 4$ with $\sigma = \epsilon$, it follows that $S(\nu_n, \nu_n, \omega) \ll \epsilon$. Definition 1.3 (1) shows that $\{\nu_n\}$ converges to ω.
3. As in the proof of (1) and (2), there exist a positive integer n_0 and $\theta \ll \epsilon \in \mathrm{int}\mathbb{P}$ such that $\tau_n \ll \epsilon$ for all $n \geqslant n_0$. Hence, $\Upsilon(\upsilon_n, \upsilon_n, \upsilon_m) \ll \epsilon$ for all $m > n \geqslant n_0$. Clearly that $\Upsilon(\upsilon_n, \upsilon_n, \upsilon_{n+1}) \ll \epsilon$. Now, we have $\Upsilon(\upsilon_n, \upsilon_n, \upsilon_{n+1}) \ll \epsilon$ and $\Upsilon(\upsilon_n, \upsilon_n, \upsilon_m) \ll \epsilon$, By $\Upsilon 4$ with $\sigma = \epsilon$, it follows that $S(\upsilon_{n+1}, \upsilon_{n+1}, \upsilon_m) \ll \epsilon$. Definition 1.3 (2) shows that $\{\upsilon_n\}$ is a Cauchy sequence in $\mathbb{X}$.
4. The proof is similar to (3).

□

Remark 2.7.

1. $\Upsilon(\upsilon, \upsilon, \nu) = \Upsilon(\nu, \nu, \upsilon)$ does not necessarily for all $\upsilon, \nu \in \mathbb{X}$.
2. $\Upsilon(\upsilon, \nu, \omega) = \theta$ is not necessarily equivalent to $\upsilon = \nu = \omega$ for all $\upsilon, \nu, \omega \in \mathbb{X}$.

3. Common fixed point and fixed point results with c_s-distance in cone S-metric spaces

In this section, we will study the problems of the common fixed point and the fixed point for weakly compatible self mappings in cone *S*-metric spaces with a c_s-distance.

Theorem 3.1. *Suppose that $(\mathbb{X}, S)$ be a cone S-metric space and Υ is a c_s-distance on $\mathbb{X}$. Let $f, g : \mathbb{X} \longrightarrow \mathbb{X}$ be two self mappings satisfy the following contractive condition*

$$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1 \Upsilon(g\upsilon, g\upsilon, g\nu) + \alpha_2 \Upsilon(g\upsilon, g\upsilon, f\upsilon) + \alpha_3 \Upsilon(g\nu, g\nu, f\nu) + \alpha_4 \Upsilon(g\upsilon, g\upsilon, f\nu),$$

for all $\upsilon, \nu \in \mathbb{X}$ where $\alpha_i \in (0, 1), i = 1, 2, 3, 4$ such that $\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 < 1$. If $f(\mathbb{X})$ is a subset of $g(\mathbb{X})$ and $g(\mathbb{X})$ is a complete subspace of $\mathbb{X}$, then f and g have a coincidence point υ^* in $\mathbb{X}$. In addition, if $\omega = g\upsilon^* = f\upsilon^*$ then $\Upsilon(\omega, \omega, \omega) = \theta$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in $\mathbb{X}$. Opt a point υ_1 in X such that $g\upsilon_1 = f\upsilon_0$. This can be done for $f(\mathbb{X}) \subseteq g(\mathbb{X})$. Continuing this process we obtain a sequence $\{\upsilon_n\}$ in $\mathbb{X}$ such that $g\upsilon_{n+1} = f\upsilon_n$. Then we have

$$\begin{aligned}
\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) &= \Upsilon(f\upsilon_{n-1}, f\upsilon_{n-1}, f\upsilon_n) \\
&\preceq \alpha_1 \Upsilon(g\upsilon_{n-1}, g\upsilon x_{n-1}, g\upsilon_n) + \alpha_2 \Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, f\upsilon_{n-1}) \\
&\quad + \alpha_3 \Upsilon(g\upsilon_n, g\upsilon_n, f\upsilon_n) + \alpha_4 \Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, f\upsilon_n) \\
&= \alpha_1 \Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) + \alpha_2 \Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) \\
&\quad + \alpha_3 \Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) + \alpha_4 \Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_{n+1})
\end{aligned}$$

$$\begin{aligned}
&\preceq \alpha_1\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) + \alpha_2\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) \\
&\quad + \alpha_3\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) + \alpha_4[\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) \\
&\quad + \Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) + \Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1})] \\
&= (\alpha_1 + \alpha_2 + 2\alpha_4)\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) \\
&\quad + (\alpha_3 - \alpha_4)\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}).
\end{aligned}$$

So,

$$\begin{aligned}
\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) &\preceq \frac{\alpha_1 + \alpha_2 + 2\alpha_4}{1 - \alpha_3 - \alpha_4}\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) \\
&= h\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n) \\
&\preceq h^2\Upsilon(g\upsilon_{n-2}, g\upsilon_{n-2}, g\upsilon_{n-1}) \\
&\ \ \vdots \\
&\preceq h^n\Upsilon(g\upsilon_0, g\upsilon_0, g\upsilon_1),
\end{aligned}$$

where $h = \frac{\alpha_1+\alpha_2+2\alpha_4}{1-\alpha_3-\alpha_4} < 1$.

Let $m > n \geqslant 1$. Then we get

$$\begin{aligned}
\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_m) &\preceq 2\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) + 2\Upsilon(g\upsilon_{n+1}, g\upsilon_{n+1}, g\upsilon_{n+2}) \\
&\quad + \cdots + \Upsilon(g\upsilon_{m-1}, g\upsilon_{m-1}, g\upsilon_m) \\
&\preceq 2[\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) + \Upsilon(g\upsilon_{n+1}, g\upsilon_{n+1}, g\upsilon_{n+2}) \\
&\quad + \cdots + \Upsilon(g\upsilon_{m-1}, g\upsilon_{m-1}, g\upsilon_m)] \\
&\preceq 2(h^n + h^{n+1} + \cdots + h^{m-1})\Upsilon(g\upsilon_0, g\upsilon_0, g\upsilon_1) \\
&\preceq 2\frac{h^n}{1-h}\Upsilon(g\upsilon_0, g\upsilon_0, g\upsilon_1) \longrightarrow \theta \quad \text{as} \quad n \longrightarrow +\infty.
\end{aligned}$$

Consequently, Lemma 2.6 (3) explicates that $\{g\upsilon_n\}$ is a Cauchy sequence in $\mathbb{X}$. Since $g(\mathbb{X})$ is complete, there exists $\upsilon^* \in \mathbb{X}$ such that $g\upsilon_n \longrightarrow g\upsilon^*$ as $n \longrightarrow +\infty$. Therefore, we have

$$\Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon^*) \preceq 2\frac{h^n}{1-h}\Upsilon(g\upsilon_0, g\upsilon_0, g\upsilon_1) \longrightarrow \theta \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.1}$$

Note that

$$q(f\upsilon_{n-1}, f\upsilon_{n-1}, f\upsilon_n) = \Upsilon(g\upsilon_n, g\upsilon_n, g\upsilon_{n+1}) \preceq h\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon_n). \tag{3.2}$$

By utilization (3.2), we get

$$\begin{aligned}
\Upsilon(g\upsilon_n, g\upsilon_n, f\upsilon^*) &= \Upsilon(f\upsilon_{n-1}, f\upsilon_{n-1}, f\upsilon^*) \\
&\preceq h\Upsilon(g\upsilon_{n-1}, g\upsilon_{n-1}, g\upsilon^*) \\
&\preceq 2h\frac{h^{n-1}}{1-h}\Upsilon(g\upsilon_0, g\upsilon_0, g\upsilon_1) \\
&= 2\frac{h^n}{1-h}\Upsilon(g\upsilon_0, g\upsilon_0, g\upsilon_1) \longrightarrow \theta \quad \text{as} \quad n \longrightarrow +\infty.
\end{aligned} \tag{3.3}$$

Consequently, Lemma 2.6 (1), (3.1) and (3.3) explicate that $g\upsilon^* = f\upsilon^*$. Therefore, υ^* is a coincidence point of f and g, and ω is a point of coincidence of f and g where $w = g\upsilon^* = f\upsilon^*$ for some υ^* in $\mathbb{X}$.

Suppose that $\omega = g\upsilon^* = f\upsilon^*$. Then we have

$$\begin{aligned}
\Upsilon(\omega, \omega, \omega) &= \Upsilon(f\upsilon^*, f\upsilon^*, f\upsilon^*) \\
&\preceq \alpha_1\Upsilon(g\upsilon^*, g\upsilon^*, g\upsilon^*) + \alpha_2\Upsilon(g\upsilon^*, g\upsilon^*, f\upsilon^*)
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_3\Upsilon(g\upsilon^*, g\upsilon^*, f\upsilon^*) + \alpha_4\Upsilon(g\upsilon^*, g\upsilon^*, f\upsilon^*)\\
&= \alpha_1\Upsilon(\omega,\omega,\omega) + \alpha_2\Upsilon(\omega,\omega,\omega) + \alpha_3\Upsilon(\omega,\omega,\omega) + \alpha_4\Upsilon(\omega,\omega,\omega)\\
&= (\alpha_1 + a_2 + a_3 + a_4)\Upsilon(\omega,\omega,\omega).
\end{aligned}
$$

Since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) < 1$, Lemma 1.1 (1) explicates that $\Upsilon(\omega,\omega,\omega) = \theta$.

Finally, impose there is another point of coincidence μ of f and g such that $\mu = f\nu^* = g\nu^*$ for some ν^* in X. Then we have

$$
\begin{aligned}
\Upsilon(\omega,\omega,\mu) &= \Upsilon(f\upsilon^*, f\upsilon^*, f\nu^*)\\
&\preceq \alpha_1\Upsilon(g\upsilon^*, g\upsilon^*, g\nu^*) + \alpha_2\Upsilon(g\upsilon^*, g\upsilon^*, f\upsilon^*)\\
&\quad + \alpha_3\Upsilon(g\nu^*, g\nu^*, f\nu^*) + a_4\Upsilon(g\upsilon^*, g\upsilon^*, f\nu^*)\\
&= \alpha_1\Upsilon(\omega,\omega,\mu) + \alpha_2\Upsilon(\omega,\omega,\omega) + \alpha_3\Upsilon(\mu,\mu,\mu) + \alpha_4\Upsilon(\omega,\omega,\mu)\\
&= \alpha_1\Upsilon(\omega,\omega,\mu) + \alpha_4\Upsilon(\omega,\omega,\mu)\\
&= (a_1 + a_4)\Upsilon(\omega,\omega,\mu).
\end{aligned}
$$

Since $(\alpha_1 + \alpha_4) < 1$, Lemma 1.1 (1) explicates that $\Upsilon(\omega,\omega,\mu) = \theta$. Also, we have $\Upsilon(\omega,\omega,\omega) = \theta$. Thus, Lemma 2.6 (1) explicates that $\omega = \mu$. Therefore, ω is the unique point of coincidence. Now, let $\omega = g\upsilon^* = f\upsilon^*$. Since f and g are weakly compatible, we have

$$g\omega = gg\upsilon^* = gf\upsilon^* = fg\upsilon^* = f\omega.$$

Hence, gw is a point of coincidence. The uniqueness of the point of coincidence implies that $g\omega = g\upsilon^*$. Therefore, $\omega = g\omega = f\omega$. Hence, ω is the unique common fixed point of f and g. □

As a consequence of Theorem 3.1, we have the following common fixed point result under the concept of a c_s–distance in cone S-metric spaces.

Corollary 3.2. *Suppose that $(\mathbb{X}, S)$ be a cone S-metric space and Υ is a c_s-distance on $\mathbb{X}$. Let $f, g : \mathbb{X} \longrightarrow \mathbb{X}$ be two self mappings satisfy one of the following contractive conditions for all $\upsilon, \nu \in \mathbb{X}$:*

1. $$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha\Upsilon(g\upsilon, g\upsilon, g\nu),$$
   where $\alpha \in [0, 1)$ is a constant.
2. $$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1\Upsilon(g\upsilon, g\upsilon, f\upsilon) + \alpha_2\Upsilon(g\nu, g\nu, f\nu),$$
   where $\alpha_1, \alpha_2 \in [0, 1)$ are constants such that $\alpha_1 + \alpha_2 < 1$.
3. $$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1\Upsilon(g\upsilon, g\upsilon, g\nu) + \alpha_2\Upsilon(g\upsilon, g\upsilon, f\upsilon) + \alpha_3\Upsilon(g\nu, g\nu, f\nu),$$
   where $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$ are constants such that $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

If $f(\mathbb{X})$ is a subset of $g(\mathbb{X})$ and $g(\mathbb{X})$ is a complete subspace of $\mathbb{X}$, then f and g have a coincidence point υ^* in $\mathbb{X}$. In addition, if $\omega = g\upsilon^* = f\upsilon^*$ then $\Upsilon(\omega,\omega,\omega) = \theta$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.

Now, we give two examples to support our work.

Example 3.3 (The case of a normal cone)**.** Consider Example 2.2. Define the mappings $f : \mathbb{X} \longrightarrow \mathbb{X}$ by

$$f(x) = \begin{cases} \frac{4\upsilon}{3}, & \text{if } \upsilon \neq \frac{1}{2},\\ \frac{1}{3}, & \text{if } \upsilon = \frac{1}{2}, \end{cases}$$

and $g : \mathbb{X} \longrightarrow \mathbb{X}$ by $g\upsilon = 2\upsilon$ for all $\upsilon \in \mathbb{X}$. Clear that $f(\mathbb{X}) \subseteq g(\mathbb{X})$ and $g(\mathbb{X})$ is a complete subset of $\mathbb{X}$. Let $\upsilon, \nu \in \mathbb{X}$, we have the following cases:

1. If $\upsilon = \nu = \frac{1}{2}$, then we have

$$\begin{aligned}\Upsilon(f(\frac{1}{2}), f(\frac{1}{2}), f(\frac{1}{2})) &= \Upsilon\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\\ &= (\frac{2}{3}, \frac{2}{3})\\ &= \frac{1}{3}(2, 2)\\ &= \frac{1}{3}\Upsilon(1, 1, 1)\\ &= \frac{1}{3}\Upsilon\left(g(\frac{1}{2}), g(\frac{1}{2}), g(\frac{1}{2})\right)\\ &\preceq \frac{4}{5}\Upsilon\left(g(\frac{1}{2}), g(\frac{1}{2}), g(\frac{1}{2})\right),\end{aligned}$$

with $\alpha = \frac{4}{5}$.

2. If $\upsilon \neq \nu \neq \frac{1}{2}$, then we have

$$\begin{aligned}\Upsilon(f\upsilon, f\upsilon, f\nu) &= \left(\frac{4\nu}{3} + \frac{4\upsilon}{3}, \frac{4\nu}{3} + \frac{4\upsilon}{3}\right)\\ &= \frac{2}{3}(2\nu + 2\upsilon, 2\nu + 2\upsilon)\\ &= \frac{2}{3}\Upsilon(2\upsilon, 2\upsilon, 2\nu)\\ &= \frac{2}{3}\Upsilon(g\upsilon, g\upsilon, g\nu)\\ &\preceq \frac{4}{5}\Upsilon(g\upsilon, g\upsilon, g\nu),\end{aligned}$$

with $\alpha = \frac{4}{5}$.

3. If $\upsilon = \frac{1}{2}$, $\nu \neq \frac{1}{2}$, then we have

$$\begin{aligned}\Upsilon(f(\frac{1}{2}), f(\frac{1}{2}), f\nu) &= \left(\frac{4y}{3} + \frac{1}{3}, \frac{4\nu}{3} + \frac{1}{3}\right)\\ &= \frac{2}{3}\left(2\nu + \frac{1}{2}, 2\nu + \frac{1}{2}\right)\\ &\preceq \frac{2}{3}(2\nu + 1, 2\nu + 1)\\ &= \frac{2}{3}\Upsilon(1, 1, 2\nu)\\ &= \frac{2}{3}\Upsilon\left(g(\frac{1}{2}), g(\frac{1}{2}), g\nu\right)\\ &\preceq \frac{4}{5}\Upsilon\left(g(\frac{1}{2}), g(\frac{1}{2}), g\nu\right),\end{aligned}$$

with $\alpha = \frac{4}{5}$.

4. If $\upsilon \neq \frac{1}{2}$, $\nu = \frac{1}{2}$, then we have

$$\Upsilon(f\upsilon, f\upsilon, f(\frac{1}{2})) = \left(\frac{1}{3} + \frac{4\upsilon}{3}, \frac{1}{3} + \frac{4\upsilon}{3}\right)$$

$$
\begin{aligned}
&= \frac{2}{3}\left(\frac{1}{2}+2\upsilon, \frac{1}{2}+2\upsilon\right) \\
&\preceq \frac{2}{3}(1+2\upsilon, 1+2\upsilon) \\
&= \frac{2}{3}\Upsilon(2\upsilon, 2\upsilon, 1) \\
&= \frac{2}{3}\Upsilon\left(g\upsilon, g\upsilon, g(\frac{1}{2})\right) \\
&\preceq \frac{4}{5}\Upsilon\left(g\upsilon, g\upsilon, g(\frac{1}{2})\right),
\end{aligned}
$$

with $\alpha = \frac{4}{5}$.

Hence, $\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha q(g\upsilon, g\upsilon, g\nu)$ for all $\upsilon, \nu \in \mathbb{X}$ where $\alpha = \frac{4}{5} \in [0, 1)$. Also f and g are weakly compatible at $x = 0$. Therefore, all conditions of Corollary 3.2 are satisfied. Hence, f and g have a unique common fixed point $\upsilon = 0$ and $f(0) = g(0) = 0$ with $\Upsilon(0, 0, 0) = 0$.

Example 3.4 (The case of a nonnormal cone)**.** Consider Example 2.3. Define the mappings $f : \mathbb{X} \longrightarrow \mathbb{X}$ by $f\upsilon = \frac{\upsilon^2}{4}$ and $g : \mathbb{X} \longrightarrow \mathbb{X}$ by $g\upsilon = \frac{\upsilon}{2}$ for all $\upsilon \in \mathbb{X}$. Clear that $f(\mathbb{X}) \subseteq g(\mathbb{X})$ and $g(\mathbb{X})$ is a complete subset of $\mathbb{X}$. Let $\upsilon, \nu \in \mathbb{X}$, we have

$$
\begin{aligned}
\Upsilon(f\upsilon, f\upsilon, f\nu)(t) &= \Upsilon(\frac{\upsilon^2}{4}, \frac{\upsilon^2}{4}, \frac{\nu^2}{4})(t) \\
&= (\frac{\nu^2}{4} + \frac{\upsilon^2}{4}).e^t \\
&\preceq \frac{1}{2}(\frac{\nu}{2} + \frac{\upsilon}{2})e^t \\
&= \frac{1}{2}(g\nu + g\upsilon) \cdot e^t \\
&= \frac{1}{2}\Upsilon(g\upsilon, g\upsilon, g\nu)(t),
\end{aligned}
$$

with $\alpha = \frac{1}{2} < 1$. Also, f and g are weakly compatible at $\upsilon = 0$. Therefore, all conditions of Corollary 3.2 are satisfied. Hence, f and g have a unique common fixed point $\upsilon = 0$ and $f(0) = g(0) = 0$ with $\Upsilon(0, 0, 0) = 0$.

In the following theorem, we prove the fixed point theorem for self mappings in a complete cone S-metric space with a c_s-distance.

Theorem 3.5. *Let* $(\mathbb{X}, S)$ *be a complete cone S-metric space and* Υ *is a* c_s*-distance on* $\mathbb{X}$*. Let* $f : \mathbb{X} \longrightarrow \mathbb{X}$ *be a self mapping satisfies the following contractive condition*

$$
\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1\Upsilon(\upsilon, \upsilon, \nu) + \alpha_2\Upsilon(\upsilon, \upsilon, f\upsilon) + \alpha_3\Upsilon(\nu, \nu, f\nu) + \alpha_4\Upsilon(\upsilon, \upsilon, f\nu),
$$

for all $\upsilon, \nu \in \mathbb{X}$ *where* $\alpha_i \in (0, 1), i = 1, 2, 3, 4$ *such that* $\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 < 1$*. Then* f *has a fixed point* $\upsilon^* \in \mathbb{X}$ *and for any* $\upsilon \in \mathbb{X}$*, iterative sequence* $\{f^n\upsilon\}$ *converges to the fixed point. If* $\mu = f\mu$ *then* $\Upsilon(\mu, \mu, \mu) = \theta$*. The fixed point is unique.*

Proof. In Theorem 3.1, put $g\upsilon = \upsilon$. The proof is complete. □

As a consequence of Theorem 3.5, we get the fixed point theorem of Banach contraction type, Kannan contraction type and Reich contraction type under the concept of a c_s-distance in a cone S-metric space respectively.

Corollary 3.6. *Let $(\mathbb{X}, S)$ be a complete cone S-metric space and Υ is a c_s-distance on $\mathbb{X}$. Let $f : \mathbb{X} \longrightarrow \mathbb{X}$ be a self mapping satisfies one of the following contractive conditions for all $\upsilon, \nu \in \mathbb{X}$:*

1. $$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha\Upsilon(\upsilon, \upsilon, \nu),$$
   where $\alpha \in [0, 1)$ is a constant.
2. $$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1\Upsilon(\upsilon, \upsilon, f\upsilon) + \alpha_2\Upsilon(\nu, \nu, f\nu),$$
   where $\alpha_1, \alpha_2 \in [0, 1)$ are constants such that $\alpha_1 + \alpha_2 < 1$.
3. $$\Upsilon(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1\Upsilon(\upsilon, \upsilon, \nu) + \alpha_2\Upsilon(\upsilon, \upsilon, f\upsilon) + \alpha_3\Upsilon(\nu, \nu, f\nu),$$
   where $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$ are constants such that $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

Then f has a fixed point $\upsilon^* \in \mathbb{X}$ and for any $\upsilon \in \mathbb{X}$, iterative sequence $\{f^n\upsilon\}$ converges to the fixed point. If $\mu = f\mu$ then $\Upsilon(\mu, \mu, \mu) = \theta$. The fixed point is unique.

4. Some applications

Using Lemma 2.5, we prove some common fixed point and fixed point theorems in cone S-metric spaces with out assumption of normality for cones. Our results extend and generalize the fixed point results of Dhamodharan and Krishnakumar [4] and Saluja [10]

Theorem 4.1. *Suppose that $(\mathbb{X}, S)$ be a cone S-metric space and Υ. Let $f, g : \mathbb{X} \longrightarrow \mathbb{X}$ be two self mappings satisfy the following contractive condition*

$$S(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1 S(g\upsilon, g\upsilon, g\nu) + \alpha_2 S(g\upsilon, g\upsilon, f\upsilon) + \alpha_3 S(g\nu, g\nu, f\nu) + \alpha_4 S(g\upsilon, g\upsilon, f\nu),$$

for all $\upsilon, \nu \in \mathbb{X}$ where $\alpha_i \in (0, 1), i = l, 2, 3, 4$ such that $\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 < 1$. If $f(\mathbb{X})$ is a subset of $g(\mathbb{X})$ and $g(\mathbb{X})$ is a complete subspace of $\mathbb{X}$, then f and g have a coincidence point υ^* in $\mathbb{X}$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Since the cone S-metric function is a c_s–distance on X by Lemma 2.5. Put $\Upsilon(x, y, z) = S(x, y, z)$ in Theorem 3.1. The proof is complete. □

In the following theorem, we prove the fixed point theorem for self mappings in a complete cone S-metric space.

Theorem 4.2. *Let $(\mathbb{X}, S)$ be a complete cone S-metric space. Let $f : \mathbb{X} \longrightarrow \mathbb{X}$ be a self mapping satisfies the following contractive condition*

$$S(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1 S(\upsilon, \upsilon, \nu) + \alpha_2 S(\upsilon, \upsilon, f\upsilon) + \alpha_3 S(\nu, \nu, f\nu) + \alpha_4 S(\upsilon, \upsilon, f\nu),$$

for all $\upsilon, \nu \in \mathbb{X}$ where $\alpha_i \in (0, 1), i = l, 2, 3, 4$ such that $\alpha_1 + \alpha_2 + \alpha_3 + 3\alpha_4 < 1$. Then f has a fixed point $\upsilon^* \in \mathbb{X}$ and for any $\upsilon \in \mathbb{X}$, iterative sequence $\{f^n\upsilon\}$ converges to the fixed point. The fixed point is unique.

Proof. In Theorem 4.1, put $g\upsilon = \upsilon$. The proof is complete. □

As a consequence of Theorem 4.2, we get the fixed point theorem of Banach contraction type, Kannan contraction type and Reich contraction type in a cone S-metric space respectively.

Corollary 4.3. *Let $(\mathbb{X}, S)$ be a complete cone S-metric space. Let $f : \mathbb{X} \longrightarrow \mathbb{X}$ be a self mapping satisfies one of the following contractive conditions for all $\upsilon, \nu \in \mathbb{X}$:*

1.
$$S(f\upsilon, f\upsilon, f\nu) \preceq \alpha S(\upsilon, \upsilon, \nu),$$
where $\alpha \in [0, 1)$ *is a constant.*

2.
$$S(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1 S(\upsilon, \upsilon, f\upsilon) + \alpha_2 S(\nu, \nu, f\nu),$$
where $\alpha_1, \alpha_2 \in [0, 1)$ *are constants such that* $\alpha_1 + \alpha_2 < 1$.

3.
$$S(f\upsilon, f\upsilon, f\nu) \preceq \alpha_1 S(\upsilon, \upsilon, \nu) + \alpha_2 S(\upsilon, \upsilon, f\upsilon) + \alpha_3 S(\nu, \nu, f\nu),$$
where $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$ *are constants such that* $\alpha_1 + \alpha_2 + \alpha_3 < 1$.

Then f *has a fixed point* $\upsilon^* \in \mathbb{X}$ *and for any* $\upsilon \in \mathbb{X}$, *iterative sequence* $\{f^n\upsilon\}$ *converges to the fixed point. The fixed point is unique.*

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References

[1] S. Aleksić, Z. Kadelburg, Z. D. Mitrović, S. Radenović, *A new survey: Cone metric spaces*, J. Int. Math. Virtual Inst., **9** (2019), 93–121. 1
[2] K. Anthony Singh, M. R. Singh, *Some fixed point theorems of cone Sb-metric space*, J. Indian Acad. Math., **40** (2018), 255–272. 1
[3] K. Anthony Singh, M. R. Singh, *Some coupled fixed point theorems in cone Sb-metric space*, J. Math. Comput. Sci., **10** (2020), 891–905. 1
[4] D. Dhamodharan, R. Krishnakumar, *Cone S-Metric Space and Fixed Point Theorems of Contractive Mappings*, Ann. Pure Appl. Math., **14** (2017), 237–243. 1, 1.2, 1.3, 1.4, 4
[5] G. Jungck, S. Radenović, S. Radojević, V. Rakočević, *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory Appl., **2009** (2009), 13 pages. 1.1
[6] J. K. Kim, S. Sedghi, N. Shobkolaei, *Common fixed point theorems for the R-weakly commuting mappings in S-metric spaces*, J. Comput. Anal. Appl., **19** (2015), 751–759. 1
[7] N. Priyobarta, Y. Rohen, N. Mlaiki, *Complex valued Sb-metric spaces*, J. Math. Anal., **8** (2017), 13–24. 1
[8] M. U. Rahman, M. Sarwar, *Fixed point results of Altman integral type mappings in S-metric spaces*, Int. J. Anal. Appl., **10** (2016), 58–63. 1
[9] Y. Rohen, T. Došenović, S. Radenović, *A note on the paper "A fixed point theorem in S_b-metric spaces"*, Filomat, **31** (2017), 3335–3346. 1
[10] G. S. Saluja, *Fixed point theorems on cone S-metric spaces using implicit relation*, CUBO, **22** (2020), 273–288. 1, 4
[11] G. S. Saluja, *Some fixed point results under contractive type mappings in cone S_b-metric spaces*, Palest. J. Math., **10** (2021), 547–561. 1
[12] S. Sedghi, N. V. Dung, *Fixed point theorems on S-metric space*, Mat. Vesnik, **66** (2014), 113–124. 1
[13] S. Sedghi, A. Gholidahneh, K. P. R. Rao, *Common Fixed Point of Two R-Weakly Commuting Mappings in S_b-Metric Spaces*, Math. Sci. Lett., **6** (2017), 249–253. 1
[14] S. Sedghi, M. M. Rezaee, T. Došenović, S. Radenović, *Common fixed point theorems for contractive mappings satisfying Φ-maps in S-metric spaces*, Acta Univ. Sapientiae Math., **8** (2016), 298–311. 1
[15] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point in S-metric spaces*, Mat. Vesnik, **64** (2012), 258–266. 1
[16] S. Sedghi, N. Shobe, T. Došenović, *Fixed point results in S-metric spaces*, Nonlinear Funct. Anal. Appl., **20** (2015), 55–67. 1
[17] N. Souayah, N. Mlaiki, *A fixed point theorem in Sb-metric spaces*, J. Math. Computer Sci., **16** (2016), 131–139. 1
[18] N. Tas, N. özgür, *New generalized fixed point results on S_b-metric spaces*, Konuralp J. Math., **9** (2021), 24–32. 1
[19] N. Van Dung, N. T. Hieu, S. Radojević, *Fixed point theorems for g-monotone maps on partially ordered S-metric spaces*, Filomat, **28** (2014), 1885–1898. 1
[20] N. Yilmaz özgür, N. Tas, *Some fixed point theorems on S-metric spaces*, Mat. Vesnik, **69** (2017), 39–52. 1