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Periodic solutions and stability of eighth order rational difference equations

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Abstract

Some real life problems are modeled using difference equations. Extracting the exact solutions of such equations is an active topic for some scientists. This paper investigates the equilibrium points, stability, boundedness, periodicity, and some exact solutions for eighth order rational difference equations. The exact solutions are obtained using the iterations method. We also present some 2D figures to show the validity of the obtained results. The used methods can be applied for other nonlinear difference equations.

Keywords: Equilibrium points, stability, boundedness, exact solution, numerical solution.

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1. Introduction

Difference equations are widely considered as an important tool that plays a significant role in improving mathematics as a whole. Even though these equations have been analyzed, they have not received the attention that they deserve. Nowadays, several phenomena such as those occurring in finite mathematics, probability theory, economy, biology, queuing problems, physics, chemistry, electrical networks, control theory, are described using difference equations and systems of difference equations. Thus, difference equations can be used to solve various natural problems. Difference equations are also used to discretize the derivatives which appear in differential equations. For instance, some phenomena such as propagation of annual plants and trade models have been modeled by using difference equations [11]. Furthermore, Murray [21] used difference equations to describe a single species population growth.

Exact solutions and long term behaviors of difference equations play a significant role in interpreting the future pattern of the relevant model. However, the exact solutions of some nonlinear difference equations cannot be sometimes found. As a result, some researchers have discussed some qualitative behaviors of such equations. In other words, scientists have examined equilibrium points and their stability,

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periodicity and boundedness of the solutions. For instance, Gumus et al. [15] investigated the behavior of the following third order difference equation

$$x_{n+1} = \frac{\alpha x_n}{\beta + \gamma x_{n-1}^p x_{n-2}^q}.$$

Almatrafi and Alzubaidi [4] analyzed the stability, periodicity, and analytic solutions of the difference equation

$$x_{n+1} = c_1 x_{n-3} + \frac{c_2 x_{n-3}}{c_3 x_{n-3} - c_4 x_{n-7}}.$$

Elsayed [12] studied the stability of the rational difference equation

$$x_{n+1} = a + \frac{b x_{n-l} + c x_{n-k}}{d x_{n-l} + e x_{n-k}}.$$

Alayachi et al. [1] examined the qualitative properties of the difference equation

$$y_{n+1} = A y_{n-1} + \frac{B y_{n-1} y_{n-3}}{C y_{n-3} + D y_{n-5}}, \ n = 0, 1, \ldots.$$

In [5], the authors discussed the stability, periodicity and some solutions of the difference equation

$$u_{m+1} = a u_{m-1} + \frac{b u_{m-1} u_{m-4}}{c u_{m-4} - d u_{m-6}}, \quad m = 0, 1, \ldots.$$

Moreover, Amleh and Drymonis [10] explored the global stability of solution of the difference equation

$$x_{n+1} = \frac{(\alpha x_n + \beta x_n x_{n-1} + \gamma x_{n-1}) x_n}{A x_n + B x_n x_{n-1} + C x_{n-1}}.$$

Furthermore, Al-Shabi and Abo-Zeid [9] investigated the qualitative behaviors of the solutions of the difference equation

$$x_{n+1} = \frac{A x_{n-2r-1}}{B + C x_{n-2l} x_{n-2k}}.$$

The study in [22] discussed the stability and the periodicity of the following difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}.$$

More results about difference equations can be found in the refs. [2, 3, 6–8, 13, 14, 16, 17, 19, 20].

The main purpose of this paper is to explore the equilibrium points, stability and the exact solutions of the following difference equations

$$x_{m+1} = \frac{x_{m-1} x_{m-7}}{x_{m-5} (-A + B x_{m-1} x_{m-7})}, \quad m = 0, 1, \ldots,$$
$$x_{m+1} = \frac{x_{m-1} x_{m-7}}{x_{m-5} (A - B x_{m-1} x_{m-7})}, \quad m = 0, 1, \ldots,$$

where A and B are positive real numbers and the initial conditions x_i for all $i = -7, -6, \ldots, 0$, are arbitrary non-zero real numbers. We also plot some 2D graphs for the obtained results.

Next, we recall some definitions and theorems used in verifying the obtained results. The following concepts can be found in [18].

Definition 1.1. Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then, for every set of initial condition $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

2. Linearized stability theorem

Let F be a continuously differentiable function in some open neighborhood of an equilibrium point x^*. Let

$$p_i = \frac{\partial F}{\partial u_i}(x^*, x^*, \dots, x^*) \quad \text{for} \quad i = 0, 1, \dots, k,$$

denote the partial derivatives of $F(u_0, u_1, \dots, u_k)$ evaluated at the equilibrium x^* of Eq. (1.1). Then, the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k}\,, \quad n = 0, 1, \dots, \tag{2.1}$$

is called the linearized equation associated of Eq. (1.1) about the equilibrium point x^* and the equation

$$\lambda^{k+1} - p_0\lambda^k - \cdots - p_{k-1}\lambda - p_k = 0, \tag{2.2}$$

is called the characteristic equation of Eq. (2.1) about x^*.

Theorem 2.1 (Linear stability theorem, [18])**.** *Assume that $p_0, p_1, \dots, p_k$ are real numbers such that*

$$|p_0| + |p_1| + \cdots + |p_k| < 1, \quad \textit{or} \quad \sum_{i=1}^{k} |p_i| < 1.$$

Then, all roots of Eq. (2.2) lie inside the unit disk.

3. Qualitative behavior of $x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}(-A+Bx_{m-1}x_{m-7})}$

Here, we consider the equilibrium point, stability and some solutions for the following difference equation:

$$x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}\left(-A + Bx_{m-1}x_{m-7}\right)}, \tag{3.1}$$

where the initial conditions $x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$, and x_0 are arbitrary real numbers with $x_{-1}x_{-7} \neq -1$ and $x_0 x_{-6} \neq -1$.

3.1. Equilibrium points

This subsection analyzes the equilibrium points of Eq. (3.1) as shown in the following theorem.

Theorem 3.1. *The equilibrium points of Eq. (3.1) are $\bar{x} = 0$ and $\bar{x} = \sqrt{\frac{1+A}{B}}$.*

Proof. The equilibrium points of Eq. (3.1) are given by

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}\left(-A + B\bar{x}^2\right)},$$

which can be written as

$$\bar{x}^2\left(-1 - A + B\bar{x}^2\right) = 0.$$

Hence, $\bar{x} = 0$ and $\bar{x} = \sqrt{\frac{1+A}{B}}$. □

Theorem 3.2. *The equilibrium point $\bar{x} = \sqrt{\frac{1+A}{B}}$ is unstable.*

Proof. Let F be a function defined on I^3 by

$$F(u, v, w) = \frac{uw}{v\ (-A + Buw)}, \tag{3.2}$$

where $I \subseteq R$, such that $0 \in I$ and $F(I^3) \subseteq I$. Hence,

$$F_u(u, v, w) = \frac{-Aw}{v\ (-A + Buw)^2}, \quad F_v(u, v, w) = \frac{-uw}{v^2\ (-A + Buw)}, \quad F_w(u, v, w) = \frac{-Au}{v\ (-A + Buw)^2}.$$

Evaluating these derivatives at the equilibrium point $\bar{x} = \sqrt{\frac{1+A}{B}}$ gives

$$F_u(\bar{x},\bar{x},\bar{x}) = \frac{-A}{\left(-A + B\,\bar{x}^2\right)^2} = \frac{-A}{\left(-A + B\,\left(\frac{1+A}{B}\right)\right)^2} = -A = p_1,$$
$$F_v(\bar{x},\bar{x},\bar{x}) = \frac{-\,\bar{x}^2}{\bar{x}^2\left(-A + B\bar{x}^2\right)} = \frac{-\,1}{\left(-A + B\,\left(\frac{1+A}{B}\right)\right)} = -1 = p_2,$$
$$F_w(\bar{x},\bar{x},\bar{x}) = \frac{-A\bar{x}}{\bar{x}\left(-A + B\bar{x}^2\right)^2} = \frac{-A}{\left(-A + B\left(\frac{1+A}{B}\right)\right)^2} = -A = p_3.$$

Then, the linearized equation of Eq. (3.1) about $\bar{x} = \sqrt{\frac{1+A}{B}}$ is given by

$$y_{m+1} - p_1 y_{m-1} - p_2 y_{m-5} - p_3 y_{m-7} = 0.$$

Or,

$$y_{m+1} + A y_{m-1} + y_{m-5} + A y_{m-7} = 0.$$

According to Theorem 2.1, the stability occurs if

$$|p_1| + |p_2| + |p_3| < 1.$$

Therefore,

$$|A| + |-1| + |A| < 1.$$

This leads to $A < 0$, which contradicts the fact that $A > 0$. □

Theorem 3.3. *If* $A > 3$, *then the equilibrium point* $\bar{x} = 0$ *is asymptotically stable.*

Proof. From Eq. (3.2) and its derivatives, we have

$$F_u(\bar{x},\bar{x},\bar{x}) = -\frac{1}{A}, \quad F_v(\bar{x},\bar{x},\bar{x}) = \frac{1}{A}, \quad F_w(\bar{x},\bar{x},\bar{x}) = -\frac{1}{A}.$$

The linearized equation of Eq. (3.1) about $\bar{x} = 0$ is given by

$$y_{m+1} + \frac{1}{A} y_{m-1} - \frac{1}{A} y_{m-5} + \frac{1}{A} y_{m-7} = 0.$$

According to Theorem 2.1, the stability occurs if

$$\left|\frac{1}{A}\right| + \left|-\frac{1}{A}\right| + \left|\frac{1}{A}\right| < 1.$$

This leads to $A > 3$. □

3.2. *Qualitative behavior of solution of* $x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}(-1+x_{m-1}x_{m-7})}$

In this subsection, we discuss the solutions of the following difference equation:

$$x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}\left(-1 + x_{m-1}x_{m-7}\right)}, \tag{3.3}$$

where the initial conditions $x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$, and x_0 are arbitrary real numbers.

Theorem 3.4. *Let $\{x_m\}_{m=-7}^{\infty}$, be a solution of Eq. (3.3). Then, for $m = 0, 1, 2, \ldots,$ we have*

$$x_{12m-7} = \frac{\alpha}{(\alpha\mu-1)^m}, \qquad x_{12m-6} = \frac{\beta}{(\beta\tau-1)^m},$$
$$x_{12m-5} = \gamma(\alpha\mu-1)^m, \qquad x_{12m-4} = \delta(\beta\tau-1)^m,$$
$$x_{12m-3} = \frac{\eta}{(\alpha\mu-1)^m}, \qquad x_{12m-2} = \frac{\kappa}{(\beta\tau-1)^m},$$
$$x_{12m-1} = \mu(\alpha\mu-1)^m, \qquad x_{12m} = \tau(\beta\tau-1)^m,$$
$$x_{12m+1} = \frac{\alpha\mu}{\gamma(\alpha\mu-1)^{m+1}}, \qquad x_{12m+2} = \frac{\beta\tau}{\delta(\beta\tau-1)^{m+1}},$$
$$x_{12m+3} = \frac{\alpha\mu(\alpha\mu-1)^m}{\eta}, \qquad x_{12m+4} = \frac{\beta\tau(\beta\tau-1)^m}{\kappa},$$

where $x_{-7} = \alpha$, $x_{-6} = \beta$, $x_{-5} = \gamma$, $x_{-4} = \delta$, $x_{-3} = \eta$, $x_{-2} = \kappa$, $x_{-1} = \mu$, and $x_0 = \tau$.

Proof. For $m = 0$, the result holds. Now, we assume that $m > 0$ and that our assumption holds for $m-1$. That is,

$$x_{12m-19} = \frac{\alpha}{(\alpha\mu-1)^{m-1}}, \qquad x_{12m-18} = \frac{\beta}{(\beta\tau-1)^{m-1}},$$
$$x_{12m-17} = \gamma(\alpha\mu-1)^{m-1}, \qquad x_{12m-16} = \delta(\beta\tau-1)^{m-1},$$
$$x_{12m-15} = \frac{\eta}{(\alpha\mu-1)^{m-1}}, \qquad x_{12m-14} = \frac{\kappa}{(\beta\tau-1)^{m-1}},$$
$$x_{12m-13} = \mu(\alpha\mu-1)^{m-1}, \qquad x_{12m-12} = \tau(\beta\tau-1)^{m-1},$$
$$x_{12m-11} = \frac{\alpha\mu}{\gamma(\alpha\mu-1)^{m}}, \qquad x_{12m-10} = \frac{\beta\tau}{\delta(\beta\tau-1)^{m}},$$
$$x_{12m-9} = \frac{\alpha\mu(\alpha\mu-1)^{m-1}}{\eta}, \qquad x_{12m-8} = \frac{\beta\tau(\beta\tau-1)^{m-1}}{\kappa}.$$

Now, it follows from Eq. (3.3) that

$$x_{12m-7} = \frac{x_{12m-9}x_{12m-15}}{x_{12m-13}(-1+x_{12m-9}x_{12m-15})},$$
$$x_{12m-7} = \frac{\left(\frac{\alpha\mu(\alpha\mu-1)^{m-1}}{\eta}\right)\left(\frac{\eta}{(\alpha\mu-1)^{m-1}}\right)}{\mu(\alpha\mu-1)^{m-1}\left(-1+\left(\frac{\alpha\mu(\alpha\mu-1)^{m-1}}{\eta}\right)\left(\frac{\eta}{(\alpha\mu-1)^{m-1}}\right)\right)}$$
$$= \frac{\alpha}{(\alpha\mu-1)^{m-1}(-1+\alpha\mu)} = \frac{\alpha}{-(\alpha\mu+1)^{m-1}(\alpha\mu-1)} = \frac{\alpha}{(\alpha\mu-1)^m}.$$

Similarly, from Eq. (3.3), we have

$$x_{12m-5} = \frac{x_{12m-7}x_{12m-13}}{x_{12m-11}(-1-x_{12m-7}x_{12m-13})}$$
$$= \frac{\left(\frac{\alpha}{(\alpha\mu-1)^m}\right)\left(\mu(\alpha\mu-1)^{m-1}\right)}{\frac{\alpha\mu}{\gamma(\alpha\mu-1)^m}\left(-1+\left(\frac{\alpha}{(\alpha\mu-1)^m}\right)\left(\mu(\alpha\mu-1)^{m-1}\right)\right)}$$
$$= \frac{\frac{1}{\alpha\mu-1}}{\frac{1}{\gamma(\alpha\mu-1)^m}\left(-1+\frac{\alpha\mu}{\alpha\mu-1}\right)} = \frac{\frac{1}{\alpha\mu-1}}{\frac{1}{\gamma(\alpha\mu-1)^m}\left(\frac{-\alpha\mu+1+\alpha\mu}{\alpha\mu-1}\right)} = \frac{1}{\frac{1}{\gamma(\alpha\mu-1)^m}} = \gamma(\alpha\mu-1)^m.$$

□

Theorem 3.5. *Eq. (3.3) has a periodic solution of period twelve if and only if* $\alpha\mu = 2$ *and* $\beta\tau = 2$ *and it will take the form*

$$\left\{\alpha, \beta, \gamma, \delta, \eta, \kappa, \mu, \tau, \frac{\alpha\mu}{\gamma}, \frac{\beta\tau}{\delta}, \frac{\alpha\mu}{\eta}, \frac{\beta\tau}{\kappa}, \alpha, \beta, \gamma, \delta, \eta, \kappa, \mu, \tau, \frac{\alpha\mu}{\gamma}, \frac{\beta\tau}{\delta}, \frac{\alpha\mu}{\eta}, \frac{\beta\tau}{\kappa}, \ldots\right\}.$$

Proof. Assume that there exists a prime period twelve solution for Eq. (3.3) on the form

$$\alpha, \beta, \gamma, \delta, \eta, \kappa, \mu, \tau, \frac{\alpha\mu}{\gamma}, \frac{\beta\tau}{\delta}, \frac{\alpha\mu}{\eta}, \frac{\beta\tau}{\kappa}, \alpha, \beta, \gamma, \delta, \eta, \kappa, \mu, \tau, \frac{\alpha\mu}{\gamma}, \frac{\beta\tau}{\delta}, \frac{\alpha\mu}{\eta}, \frac{\beta\tau}{\kappa}, \ldots.$$

Then, from Eq. (3.3), we obtain

$$x_{12m-7} = \alpha = \frac{\alpha}{(\alpha\mu-1)^m}, \qquad x_{12m-6} = \beta = \frac{\beta}{(\beta\tau-1)^m},$$
$$x_{12m-5} = \gamma = \gamma(\alpha\mu-1)^m, \qquad x_{12m-4} = \delta = \delta(\beta\tau-1)^m,$$
$$x_{12m-3} = \eta = \frac{\eta}{(\alpha\mu-1)^m}, \qquad x_{12m-2} = \kappa = \frac{\kappa}{(\beta\tau-1)^m},$$
$$x_{12m-1} = \mu = \mu(\alpha\mu-1)^m, \qquad x_{12m} = \tau = \tau(\beta\tau-1)^m,$$
$$x_{12m+1} = \frac{\alpha\mu}{\gamma} = \frac{\alpha\mu}{\gamma(\alpha\mu-1)^{m+1}}, \qquad x_{12m+2} = \frac{\beta\tau}{\delta} = \frac{\beta\tau}{\delta(\beta\tau-1)^{m+1}},$$
$$x_{12m+3} = \frac{\alpha\mu}{\eta} = \frac{\alpha\mu(\alpha\mu-1)^m}{\eta}, \qquad x_{12m+4} = \frac{\beta\tau}{\kappa} = \frac{\beta\tau(\beta\tau-1)^m}{\kappa}.$$

Then, we can see that $\alpha\mu = 2$ and $\beta\tau = 2$. Conversely, suppose that $\alpha\mu = 2$ and $\beta\tau = 2$. Then we can see that,

$$x_{12m-7} = \frac{\alpha}{(\alpha\mu-1)^m} = \frac{\alpha}{(2-1)^m} = \alpha, \qquad x_{12m-6} = \frac{\beta}{(\beta\tau-1)^m} = \frac{\beta}{(2-1)^m} = \beta,$$
$$x_{12m-5} = \gamma(\alpha\mu-1)^m = \gamma(2-1)^m = \gamma, \qquad x_{12m-4} = \delta(\beta\tau-1)^m = \delta(2-1)^m = \delta,$$
$$x_{12m-3} = \frac{\eta}{(\alpha\mu-1)^m} = \frac{\eta}{(2-1)^m} = \eta, \qquad x_{12m-2} = \frac{\kappa}{(\beta\tau-1)^m} = \frac{\kappa}{(2-1)^m} = \kappa,$$
$$x_{12m-1} = \mu(\alpha\mu-1)^m = \mu(2-1)^m = \mu, \qquad x_{12m} = \tau(\beta\tau-1)^m = \tau(2-1)^m = \tau,$$
$$x_{12m+1} = \frac{\alpha\mu}{\gamma(\alpha\mu-1)^{m+1}} = \frac{\alpha\mu}{\gamma(2-1)^{m+1}} = \frac{\alpha\mu}{\gamma}, \qquad x_{12m+2} = \frac{\beta\tau}{\delta(\beta\tau-1)^{m+1}} = \frac{\beta\tau}{\delta(2-1)^{m+1}} = \frac{\beta\tau}{\delta},$$
$$x_{12m+3} = \frac{\alpha\mu(\alpha\mu-1)^m}{\eta} = \frac{\alpha\mu(2-1)^m}{\eta} = \frac{\alpha\mu}{\eta}, \qquad x_{12m+4} = \frac{\beta\tau(\beta\tau-1)^m}{\kappa} = \frac{\beta\tau(2-1)^m}{\kappa} = \frac{\beta\tau}{\kappa}.$$

□

4. Qualitative behaviors of $x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}(A-Bx_{m-1}x_{m-7})}$

In this section, we explore the qualitative properties of the following difference equation

$$x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}(A - Bx_{m-1}x_{m-7})}, \tag{4.1}$$

where the initial conditions $x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$, and x_0 are arbitrary real numbers with $x_{-1}x_{-7} \neq -1$ and $x_0x_{-6} \neq -1$.

4.1. Equilibrium points and local stability

This subsection is devoted to investigate the equilibrium points and the local stability of Eq. (4.1).

Theorem 4.1. *The equilibrium points of Eq.* (4.1) *are* $\overline{x} = 0$ *and* $\overline{x} = \sqrt{\frac{A-1}{B}}$.

Proof. The equilibrium points of Eq. (4.1) are given by

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(A - B\overline{x}^2\right)}.$$

Thus,

$$\overline{x}^2\left(-1 + A - B\overline{x}^2\right) = 0.$$

Hence, $\overline{x} = 0$ and $\overline{x} = \sqrt{\frac{A-1}{B}}, A > 1$. □

Theorem 4.2. *The equilibrium point* $\overline{x} = \sqrt{\frac{A-1}{B}}$ *is unstable.*

Proof. Suppose that G is a function defined on I^3 by

$$G(u, v, w) = \frac{uw}{v\ (A - Buw)},$$

where $I \subseteq R$, such that $0 \in I$ and $G(I^3) \subseteq I$. Then,

$$G_u(u, v, w) = \frac{Aw}{v\ (A - Buw)^2},\ G_v(u, v, w) = \frac{-uw}{v^2\ (A - Buw)},\ G_w(u, v, w) = \frac{Au}{v\ (A - Buw)^2}.$$

Calculating these derivatives at the equilibrium point $\overline{x} = \sqrt{\frac{A-1}{B}}$, leads to

$$\begin{aligned}
G_u(\overline{x}, \overline{x}, \overline{x}) &= \frac{A}{\left(A - B\ \overline{x}^2\right)^2} = \frac{A}{\left(A - B\ \left(\frac{A-1}{B}\right)\right)^2} = A = p_1,\\
G_v(\overline{x}, \overline{x}, \overline{x}) &= \frac{-\ \overline{x}^2}{\overline{x}^2\left(A - B\overline{x}^2\right)} = \frac{-\ 1}{\left(A - B\ \left(\frac{A-1}{B}\right)\right)} = -1 = p_2,\\
G_w(\overline{x}, \overline{x}, \overline{x}) &= \frac{A\overline{x}}{\overline{x}\left(A - B\overline{x}^2\right)^2} = \frac{A}{\left(A - B\left(\frac{A-1}{B}\right)\right)^2} = A = p_3.
\end{aligned}$$

The linearized equation of Eq. (4.1) about $\overline{x} = \sqrt{\frac{A-1}{B}}$ can be expressed as

$$y_{m+1} - p_1 y_{m-1} - p_2 y_{m-5} - p_3 y_{m-7} = 0.$$

Or,

$$y_{m+1} - A y_{m-1} + y_{m-5} - A y_{m-7} = 0.$$

According to Theorem 2.1, the stability occurs if

$$|p_1| + |p_2| + |p_3| < 1.$$

That is

$$|A| + |-1| + |A| < 1,$$

which gives $A < 0$. However, this result contradicts the fact that $A > 1$. □

Theorem 4.3. *If* $A > 3$, *then the equilibrium point* $\overline{x} = 0$ *is asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 3.3. □

4.2. Solutions of $x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}(1-x_{m-1}x_{m-7})}$

We now introduce the exact solutions of the following difference equation:

$$x_{m+1} = \frac{x_{m-1}x_{m-7}}{x_{m-5}\left(1-x_{m-1}x_{m-7}\right)}, \tag{4.2}$$

where the initial conditions $x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}$, and x_0 are arbitrary real numbers.

Theorem 4.4. *Let* $\{x_m\}_{m=-7}^{\infty}$, *be a solution to Eq. (4.2). Then, for* $m = 0, 1, 2, \ldots$, *we have*

$$x_{12m-7} = \alpha \prod_{j=0}^{m-1} \frac{(1-6j\alpha\mu)}{(1-(6j+3)\,\alpha\mu)}, \qquad x_{12m-6} = \beta \prod_{j=0}^{m-1} \frac{(1-6j\beta\tau)}{(1-(6j+3)\,\beta\tau)},$$

$$x_{12m-5} = \gamma \prod_{j=0}^{m-1} \frac{(1-(6j+1)\,\alpha\mu)}{(1-(6j+4)\,\alpha\mu)}, \qquad x_{12m-4} = \delta \prod_{j=0}^{m-1} \frac{(1-(6j+1)\,\beta\tau)}{(1-(6j+4)\,\beta\tau)},$$

$$x_{12m-3} = \eta \prod_{j=0}^{m-1} \frac{(1-(6j+2)\,\alpha\mu)}{(1-(6j+5)\,\alpha\mu)}, \qquad x_{12m-2} = \kappa \prod_{j=0}^{m-1} \frac{(1-(6j+2)\,\beta\tau)}{(1-(6j+5)\,\beta\tau)},$$

$$x_{12m-1} = \mu \prod_{j=0}^{m-1} \frac{(1-(6j+3)\,\alpha\mu)}{(1-(6j+6)\,\alpha\mu)}, \qquad x_{12m} = \tau \prod_{j=0}^{m-1} \frac{(1-(6j+3)\,\beta\tau)}{(1-(6j+6)\,\beta\tau)},$$

$$x_{12m+1} = \frac{\alpha\mu}{\gamma\,(1-\alpha\mu)} \prod_{j=0}^{m-1} \frac{(1-(6j+4)\,\alpha\mu)}{(1-(6j+7)\,\alpha\mu)}, \qquad x_{12m+2} = \frac{\beta\tau}{\delta\,(1-\beta\tau)} \prod_{j=0}^{m-1} \frac{(1-(6j+4)\,\beta\tau)}{(1-(6j+7)\,\beta\tau)},$$

$$x_{12m+3} = \frac{\alpha\mu}{\eta\,(1-2\alpha\mu)} \prod_{j=0}^{m-1} \frac{(1-(6j+5)\,\alpha\mu)}{(1-(6j+8)\,\alpha\mu)}, \qquad x_{12m+4} = \frac{\beta\tau}{\kappa\,(1-2\beta\tau)} \prod_{j=0}^{m-1} \frac{(1-(6j+5)\,\beta\tau)}{(1-(6j+8)\,\beta\tau)},$$

where $x_{-7} = \alpha$, $x_{-6} = \beta$, $x_{-5} = \gamma$, $x_{-4} = \delta$, $x_{-3} = \eta$, $x_{-2} = \kappa$, $x_{-1} = \mu$, *and* $x_0 = \tau$.

Proof. For $m = 0$, the solutions hold. Next, we suppose that $m > 0$ and that our assumption holds for $m-1$. That is,

$$x_{12m-19} = \alpha \prod_{j=0}^{m-2} \frac{(1-6j\alpha\mu)}{(1-(6j+3)\,\alpha\mu)}, \qquad x_{12m-18} = \beta \prod_{j=0}^{m-2} \frac{(1-6j\beta\tau)}{(1-(6j+3)\,\beta\tau)},$$

$$x_{12m-17} = \gamma \prod_{j=0}^{m-2} \frac{(1-(6j+1)\,\alpha\mu)}{(1-(6j+4)\,\alpha\mu)}, \qquad x_{12m-16} = \delta \prod_{j=0}^{m-2} \frac{(1-(6j+1)\,\beta\tau)}{(1-(6j+4)\,\beta\tau)},$$

$$x_{12m-15} = \eta \prod_{j=0}^{m-2} \frac{(1-(6j+2)\,\alpha\mu)}{(1-(6j+5)\,\alpha\mu)}, \qquad x_{12m-14} = \kappa \prod_{j=0}^{m-2} \frac{(1-(6j+2)\,\beta\tau)}{(1-(6j+5)\,\beta\tau)},$$

$$x_{12m-13} = \mu \prod_{j=0}^{m-2} \frac{(1-(6j+3)\,\alpha\mu)}{(1-(6j+6)\,\alpha\mu)}, \qquad x_{12m-12} = \tau \prod_{j=0}^{m-2} \frac{(1-(6j+3)\,\beta\tau)}{(1-(6j+6)\,\beta\tau)},$$

$$x_{12m-11} = \frac{\alpha\mu}{\gamma\,(1-\alpha\mu)} \prod_{j=0}^{m-2} \frac{(1-(6j+4)\,\alpha\mu)}{(1-(6j+7)\,\alpha\mu)}, \qquad x_{12m-10} = \frac{\beta\tau}{\delta\,(1-\beta\tau)} \prod_{j=0}^{m-2} \frac{(1-(6j+4)\,\beta\tau)}{(1-(6j+7)\,\beta\tau)},$$

$$x_{12m-9} = \frac{\alpha\mu}{\eta\,(1-2\alpha\mu)} \prod_{j=0}^{m-2} \frac{(1-(6j+5)\,\alpha\mu)}{(1-(6j+8)\,\alpha\mu)}, \qquad x_{12m-8} = \frac{\beta\tau}{\kappa\,(1-2\beta\tau)} \prod_{j=0}^{m-2} \frac{(1-(6j+5)\,\beta\tau)}{(1-(6j+8)\,\beta\tau)}.$$

Eq. (4.2) leads to

$$x_{12m-7} = \frac{x_{12m-9}x_{12m-15}}{x_{12m-13}(1-x_{12m-9}x_{12m-15})}$$

$$= \frac{\left(\frac{\alpha\mu}{\eta(1-2\alpha\mu)}\prod_{j=0}^{m-2}\frac{(1-(6j+5)\alpha\mu)}{(1-(6j+8)\alpha\mu)}\right)\left(\eta\prod_{j=0}^{m-2}\frac{(1-(6j+2)\alpha\mu)}{(1-(6j+5)\alpha\mu)}\right)}{\mu\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\left[1-\left(\frac{\alpha\mu}{\eta(1-2\alpha\mu)}\prod_{j=0}^{m-2}\frac{(1-(6j+5)\alpha\mu)}{(1-(6j+8)\alpha\mu)}\right)\left(\eta\prod_{j=0}^{m-2}\frac{(1-(6j+2)\alpha\mu)}{(1-(6j+5)\alpha\mu)}\right)\right]}$$

$$= \frac{\frac{\alpha}{(1-2\alpha\mu)}\prod_{j=0}^{m-2}\frac{(1-(6j+2)\alpha\mu)}{(1-(6j+8)\alpha\mu)}}{\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\left[1-\frac{\alpha\mu}{(1-2\alpha\mu)}\prod_{j=0}^{m-2}\frac{(1-(6j+2)\alpha\mu)}{(1-(6j+8)\alpha\mu)}\right]}$$

$$= \frac{\frac{\alpha(1-2\alpha\mu)(1-8\alpha\mu)\cdots(1-(6m-10)\alpha\mu)}{(1-2\alpha\mu)(1-8\alpha\mu)(1-14\alpha\mu)\cdots(1-(6m-4)\alpha\mu)}}{\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\left[1-\frac{\alpha\mu}{(1-2\alpha\mu)}\frac{(1-2\alpha\mu)(1-8\alpha\mu)\cdots(1-(6m-10)\alpha\mu)}{(1-8\alpha\mu)(1-14\alpha\mu)\cdots(1-(6m-4)\alpha\mu)}\right]}$$

$$= \frac{\frac{\alpha}{1-(6m-4)\alpha\mu}}{\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\left[1-\frac{\alpha\mu}{1-(6m-4)\alpha\mu}\right]}$$

$$= \frac{\frac{\alpha}{1-(6m-4)\alpha\mu}}{\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\left[\frac{1-(6m-4)\alpha\mu-\alpha\mu}{1-(6m-4)\alpha\mu}\right]} = \frac{\alpha}{\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\left[1-(6m-3)\,\alpha\mu\right]}$$

$$= \frac{\alpha}{1-(6m-3)\,\alpha\mu}\prod_{j=0}^{m-2}\frac{(1-(6j+6)\,\alpha\mu)}{(1-(6j+3)\,\alpha\mu)} = \alpha\prod_{j=0}^{m-1}\frac{(1-6j\alpha\mu)}{(1-(6j+3)\,\alpha\mu)}.$$

Moreover, from Eq. (4.2), we have

$$x_{12m-5} = \frac{x_{12m-7}x_{12m-13}}{x_{12m-11}\left(1-x_{12m-7}x_{12m-13}\right)}$$

$$= \frac{\left(\alpha\prod_{j=0}^{m-1}\frac{(1-6j\alpha\mu)}{(1-(6j+3)\alpha\mu)}\right)\left(\mu\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\right)}{\frac{\alpha\mu}{\gamma(1-\alpha\mu)}\prod_{j=0}^{m-2}\frac{(1-(6j+4)\alpha\mu)}{(1-(6j+7)\alpha\mu)}\left[1-\left(\alpha\prod_{j=0}^{m-1}\frac{(1-6j\alpha\mu)}{(1-(6j+3)\alpha\mu)}\right)\left(\mu\prod_{j=0}^{m-2}\frac{(1-(6j+3)\alpha\mu)}{(1-(6j+6)\alpha\mu)}\right)\right]}$$

$$= \frac{\prod_{j=0}^{m-1}\frac{1}{(1-(6j+3)\alpha\mu)}\prod_{j=0}^{m-2}\left(1-(6j+3)\,\alpha\mu\right)}{\frac{1}{\gamma(1-\alpha\mu)}\prod_{j=0}^{m-2}\frac{(1-(6j+4)\alpha\mu)}{(1-(6j+7)\alpha\mu)}\left[1-\alpha\mu\prod_{j=0}^{m-1}\frac{1}{(1-(6j+3)\alpha\mu)}\prod_{j=0}^{m-2}\left(1-(6j+3)\,\alpha\mu\right)\right]}$$

$$= \frac{\frac{1}{1-(6m-3)\alpha\mu}}{\frac{1}{\gamma(1-\alpha\mu)} \prod_{j=0}^{m-2} \frac{(1-(6j+4)\alpha\mu)}{(1-(6j+7)\alpha\mu)} \left[1 - \frac{\alpha\mu}{1-(6m-3)\alpha\mu}\right]}$$

$$= \frac{\frac{1}{1+(6m-3)\alpha\mu}}{\frac{1}{\gamma(1-\alpha\mu)} \prod_{j=0}^{m-2} \frac{(1-(6j+4)\alpha\mu)}{(1-(6j+7)\alpha\mu)} \left[\frac{1-(6m-3)\alpha\mu-\alpha\mu}{1-(6m-3)\alpha\mu}\right]}$$

$$= \frac{1}{\frac{1}{\gamma(1-\alpha\mu)} \prod_{j=0}^{m-2} \frac{(1-(6j+4)\alpha\mu)}{(1-(6j+7)\alpha\mu)} \left[1 - (6m-2)\,\alpha\mu\right]}$$

$$= \frac{\gamma\,(1-\alpha\mu)}{1-(6m-2)\,\alpha\mu} \prod_{j=0}^{m-2} \frac{(1-(6j+7)\,\alpha\mu)}{(1-(6j+4)\,\alpha\mu)} = \gamma \prod_{j=0}^{m-1} \frac{(1-(6j+1)\,\alpha\mu)}{(1-(6j+4)\,\alpha\mu)}.$$

□

5. Numerical examples

The obtained results are plotted in this section. We select the parameters according to the above-mentioned conditions.

Example 5.1. Figure 1 illustrates a stable solution for Eq. (3.1) about $\bar{x} = 0$ and under the conditions $A = 5$, $B = 1$, $x_{-7} = -2$, $x_{-6} = 2$, $x_{-5} = -1$, $x_{-4} = 1$, $x_{-3} = -1.5$, $x_{-2} = 1.5$, $x_{-1} = -2$, and $x_0 = 2$.

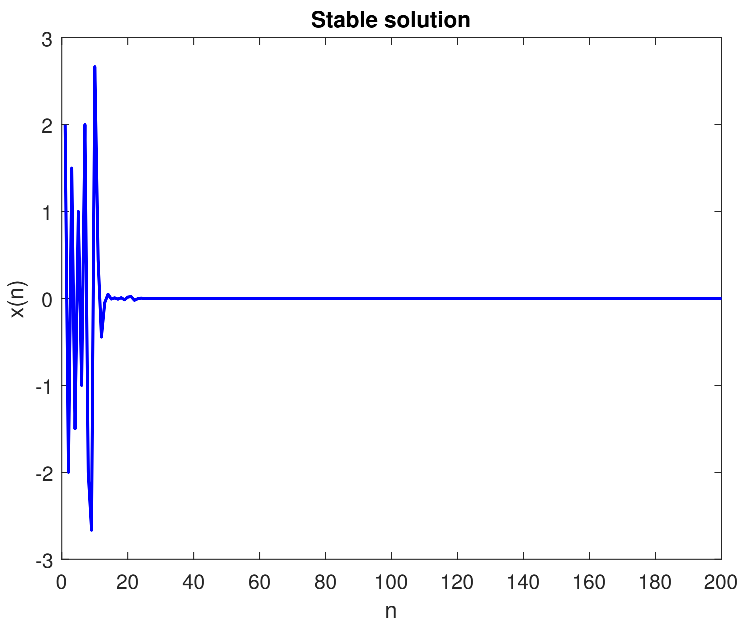

Figure 1: Stable solution for Eq. (3.1).

Example 5.2. Figure 2 shows an unstable solution for Eq. (3.1) under the conditions $A = 1$, $B = 1$, $x_{-7} = -2$, $x_{-6} = 2$, $x_{-5} = -1$, $x_{-4} = 1$, $x_{-3} = -1.5$, $x_{-2} = 1.5$, $x_{-1} = -2$, and $x_0 = 2$.

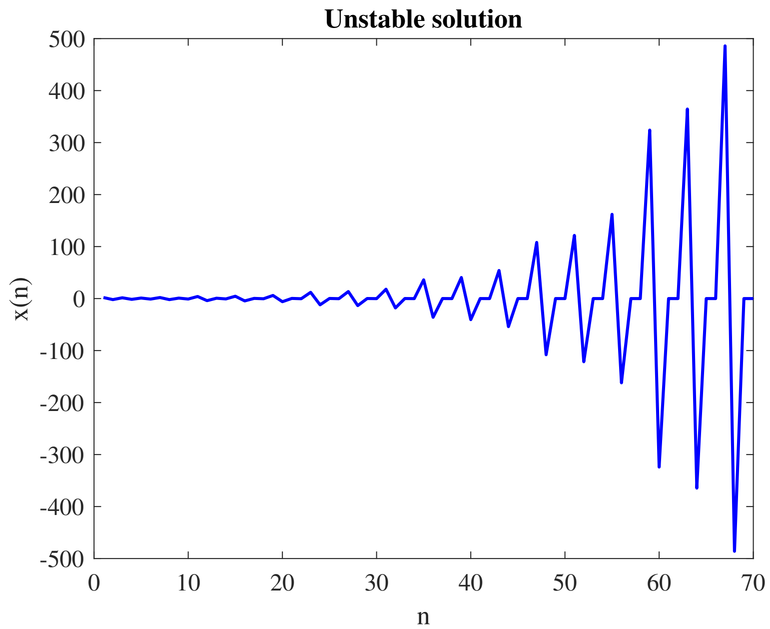

Figure 2: Unstable solution for Eq. (3.1).

Example 5.3. Figure 3 shows a periodic solution for Eq. (3.3) according to the assumptions $x_{-7} = 4$, $x_{-6} = 4$, $x_{-5} = 2$, $x_{-4} = 2$, $x_{-3} = 1$, $x_{-2} = 1$, $x_{-1} = 1$, and $x_0 = 1$.

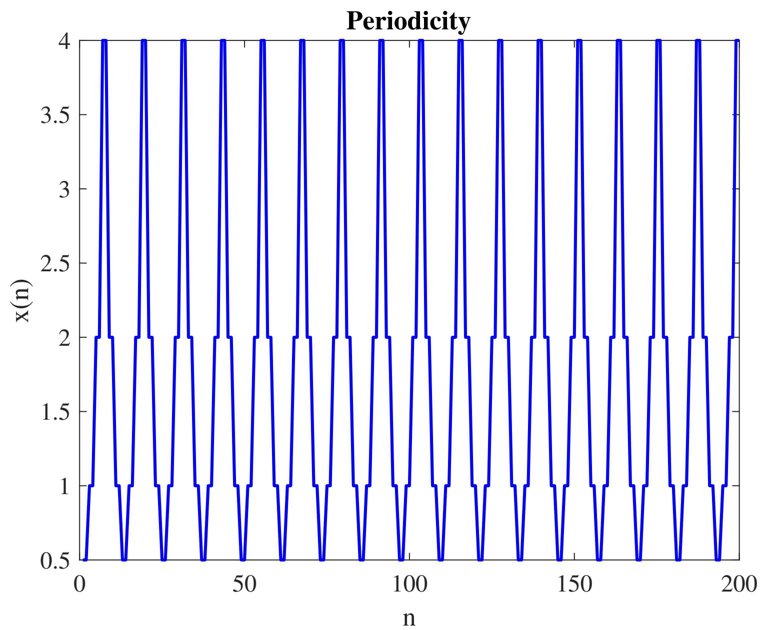

Figure 3: Periodic solution for Eq. (3.3).

Example 5.4. A stable solution for Eq. (4.1) about $\overline{x} = 0$ and under the parameters $A = 6$, $B = 1$, $x_{-7} = 2$, $x_{-6} = -2$, $x_{-5} = 1.5$, $x_{-4} = -1.5$, $x_{-3} = 1$, $x_{-2} = -1$, $x_{-1} = 2$, and $x_0 = -2$, is plotted in Figure 4.

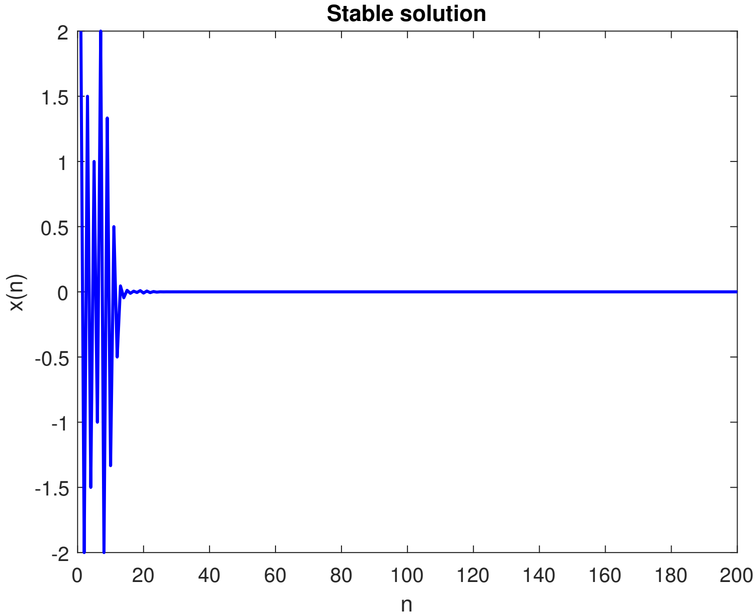

Figure 4: Stable solution for Eq. (4.1).

6. Conclusion

To sum up, we investigated the stability, periodicity and the solutions of Eqs. (3.1) and (4.1). In Theorem 3.2, we showed that the equilibrium point $\overline{x} = \sqrt{\frac{1+A}{B}}$ is unstable while Theorem 3.3 presented the asymptotic stability of the equilibrium point $\overline{x} = 0$ which occurs if $A > 3$. We also investigated the periodicity of Eq. (3.3) which has a periodic solution of period twelve if and only if $\alpha\mu = 2$ and $\beta\tau = 2$. Section 5 presented some 2D figures on the obtained results. For example, Figure 3 shows the periodic solution of Eq. (3.3).

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