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Bifurcations of resonant double homoclinic loops for higher dimensional systems

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Abstract

In this work, we study the bifurcation problems of double homoclinic loops with resonant condition for higher dimensional systems. The Poincaré maps are constructed by using the foundational solutions of the linear variational systems as the local coordinate systems in the small tubular neighborhoods of the homoclinic orbits. We obtain the existence, number and existence regions of the small homoclinic loops, small periodic orbits, and the large homoclinic loops, large periodic orbits, respectively. Moreover, the complete bifurcation diagrams are given.

Keywords: Double homoclinic loops, resonance, bifurcation, higher dimensional system.
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1. Introduction and Hypotheses

In the study of the complex dynamic behaviors of high-dimensional nonlinear dynamical systems, the bifurcation problems of homoclinic orbits and heteroclinic loops have been becoming an important research field. By using the traditional Poincaré map construction method, [1, 8, 11] discussed the bifurcations of non-degenerated homoclinic loops. In [13], Zhu discussed the bifurcation problems of non-degenerated homoclinic loop by using the generalized Floquet theory. In [3, 4, 5, 6], by using the foundational solutions of the linear variational systems of the unperturbed systems along the homoclinic orbits as the local coordinate systems to construct the Poincaré maps, the authors studied the bifurcations and stability of homoclinic loops for higher dimensional systems. In [2, 10], the stability of double homoclinic loops was studied. In [7, 12], Lu and Zhang studied the double homoclinic loops bifurcations under the non-resonant condition. In this paper, we study the bifurcations of double homoclinic loops under the resonant condition for higher

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dimensional systems. In this case, we get the complete bifurcation diagrams. Specially, we obtain the more complex bifurcation phenomena than that of [12].

Suppose the following C^r system

$$\dot{z} = f(z) \ , \tag{1.1}$$

where $r \geq 5$, $z \in \mathbf{R}^{m+n}$, satisfies the following hypotheses.

(H1) (Hyperbolicity) $z = 0$ is the hyperbolic equilibrium of system (1.1), the stable manifold W_0^s and the unstable manifold W_0^u of $z = 0$ are m-dimensional and n-dimensional respectively. λ_1 and $-\rho_1$ are simple eigenvalues of $Df(0)$, such that any other eigenvalue σ of $Df(0)$ satisfies either $\mathrm{Re}\sigma < -\rho_0 < -\rho_1 < 0$ or $\mathrm{Re}\sigma > \lambda_0 > \lambda_1 > 0$, where λ_0 and ρ_0 are some positive numbers.

(H2) (Non-degeneration) System (1.1) has a double homoclinic loops $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_i = \{z = r_i(t) : t \in \mathbf{R}, r_i(\pm\infty) = 0\}$, $i = 1, 2$. For any $P \in \Gamma$, $\mathrm{codim}(T_P W_0^u + T_P W_0^s) = 1$, where, $T_P W_0^s$ and $T_P W_0^u$ are the tangent spaces of W_0^s and W_0^u at P respectively.

(H3) (Strong inclination) $\lim\limits_{t\to+\infty}(T_{r_i(t)}W_0^s + T_{r_i(t)}W_0^u) = T_0 W_0^s \oplus T_0 W_0^{uu}$, $\lim\limits_{t\to-\infty}(T_{r_i(t)}W_0^s + T_{r_i(t)}W_0^u) = T_0 W_0^{ss} \oplus T_0 W_0^u$, where $i = 1, 2$, W_0^{ss} and W_0^{uu} are the strong stable manifold and the strong unstable manifold of $z = 0$ respectively, $T_0 W_0^{ss}$ is the generalized eigenspace corresponding to those eigenvalues with smaller real part than $-\rho_0$, $T_0 W_0^{uu}$ is the generalized eigenspace corresponding to those eigenvalues with larger real part than λ_0. Let $e_i^\pm = \lim\limits_{t\to\mp\infty} \dot{r}_i(t)/|\dot{r}_i(t)|$, $e_i^+ \in T_0 W_0^u$ and $e_i^- \in T_0 W_0^s$ are the unit eigenvectors corresponding to λ_1 and $-\rho_1$ respectively. $e_1^+ = -e_2^+$, $e_1^- = -e_2^-$. span $(T_0 W_0^{uu}, e_i^+) = T_0 W_0^u$, span $(T_0 W_0^{ss},\ e_i^-) = T_0 W_0^s$.

(H4) (Resonance condition) $\rho_1 = \lambda_1$.

Now, we consider the bifurcation problems of the following C^r system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.2}$$

where $\mu \in \mathbf{R}^l$, $l \geq 3$, $0 \leq |\mu| \ll 1$, $g(0, \mu) = g(z, 0) = 0$.

2. Local coordinate systems

Suppose that **(H1)–(H3)** are established. Then, for $|\mu| \ll 1$, in the small enough neighborhood U of $z = 0$, we introduce a C^r transformation such that system (1.2) has the following form

$$\begin{cases} \dot{x} = [\lambda_1(\mu) + \cdots\cdots]x + u[O(y) + O(v)], \\ \dot{y} = [-\rho_1(\mu) + \cdots\cdots]y + v[O(x) + O(u)], \\ \dot{u} = [B_1(\mu) + \cdots\cdots]u + x[O(x) + O(y) + O(v)], \\ \dot{v} = [-B_2(\mu) + \cdots\cdots]v + y[O(x) + O(y) + O(u)], \end{cases} \tag{2.1}$$

where $z = (x, y, u^*, v^*)^*$, $x \in R^1$, $y \in R^1$, $u \in \mathbf{R}^{n-1}$, $v \in \mathbf{R}^{m-1}$, $*$ means transposition, $\lambda_1(0) = \rho_1(0)$, $\mathrm{Re}\sigma(B_1(\mu)) > \lambda_0$, $\mathrm{Re}\sigma(-B_2(\mu)) < -\rho_0$. Moreover, the unstable manifold, stable manifold, strong unstable manifold, strong stable manifold and local homoclinic orbits have the following forms, respectively

$$\begin{array}{ll} W^u_{loc} = \{y = 0, v = 0\}, & W^s_{loc} = \{x = 0, u = 0, \}, \\ W^{uu}_{loc} = \{x = 0, y = 0, v = 0\}, & W^{ss}_{loc} = \{x = 0, u = 0, y = 0\}, \\ \Gamma_i \cap W^u_{loc} = \{y = 0, v = 0, u = u_i(x)\}, & \Gamma_i \cap W^s_{loc} = \{x = 0, u = 0, v = v_i(y)\}, \end{array}$$

where $i = 1, 2$, $u_i(0) = \dot{u}_i(0) = 0$, $v_i(0) = \dot{v}_i(0) = 0$.

Denote $r_i(t) = (r_i^x(t), r_i^y(t), (r_i^u(t))^*, (r_i^v(t))^*)^*$, $i = 1, 2$. Suppose that $r_1(-T_1) = (\delta, 0, \delta_{1,u}^*, 0^*)^*$, $r_1(T_1) = (0, \delta, 0^*,\ \delta_{1,v}^*)^*$, $r_2(-T_2) = (-\delta, 0, \delta_{2,u}^*, 0^*)^*$, $r_2(T_2) = (0, -\delta, 0^*,\ \delta_{2,v}^*)^*$, where, $T_i > 0$, $i = 1, 2$, δ is small enough, such that $\{(x, y, u, v) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$. Obviously, we have $|\delta_{i,u}| = O(\delta^{\omega_1})$, $|\delta_{i,v}| = O(\delta^{\omega_2})$, $\omega_1 = \mathrm{Re}\frac{\sigma(B_1(\mu))}{\lambda_1(\mu)} > 1$, $\omega_2 = \mathrm{Re}\frac{\sigma(B_2(\mu))}{\rho_1(\mu)} > 1$.

Consider the linear system

$$\dot{z} = (Df(r_i(t)))z. \tag{2.2}$$

Similar to [3, 5, 6, 7, 12], system (2.2) has a fundamental solution matrix $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$ satisfying

$$\begin{aligned} &z_i^1(t) \in (T_{r_i(t)}W^s)^c \cap (T_{r_i(t)}W^u)^c, \\ &z_i^2(t) = (-1)^i \dot{r}_i(t)/|\dot{r}_i^y(T_i)| \in T_{r_i(t)}W^s \cap T_{r_i(t)}W^u, \\ &z_i^3(t) = (z_i^{3,1}(t), \cdots, z_i^{3,n-1}(t)) \in (T_{r_i(t)}W^s)^c \cap (T_{r_i(t)}W^u) = T_{r_i(t)}W^{uu}, \\ &z_i^4(t) = (z_i^{4,1}(t), \cdots, z_i^{4,m-1}(t)) \in (T_{r_i(t)}W^s) \cap (T_{r_i(t)}W^u)^c = T_{r_i(t)}W^{ss}, \end{aligned}$$

and

$$Z_i(T_i) = \begin{pmatrix} 1 & 0 & w_i^{31} & 0 \\ 0 & 1 & w_i^{32} & 0 \\ 0 & 0 & w_i^{33} & 0 \\ w_i^{14}, & w_i^{24} & w_i^{34} & I \end{pmatrix}, \quad Z_i(-T_i) = \begin{pmatrix} w_i^{11} & w_i^{21} & 0 & w_i^{41} \\ w_i^{12} & 0 & 0 & w_i^{42} \\ w_i^{13} & w_i^{23} & I & w_i^{43} \\ 0, & 0 & 0 & w_i^{44} \end{pmatrix},$$

where $i = 1, 2$, $w_i^{21} < 0$, $w_i^{12} \neq 0$, $\det w_i^{33} \neq 0$, $\det w_i^{44} \neq 0$, and $|w_i^{1j}(w_i^{12})^{-1}| \ll 1$, $j \neq 2$; $|w_i^{2j}(w_i^{21})^{-1}| \ll 1$, $j = 3, 4$; $|w_i^{3j}(w_i^{33})^{-1}| \ll 1$, $j \neq 3$; $|w_i^{4j}(w_i^{44})^{-1}| \ll 1$, $j \neq 4$.

Denote $\Phi_i(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t)) = (Z_i^{-1}(t))^*$, $i = 1, 2$, so, $\Phi_i(t)$ is a fundamental solution matrix of the adjoint system $\dot{\phi} = -(Df(r_i(t)))^*\phi$ of (2.2), and $\phi_i^1(t) \in (T_{r_i(t)}W^s)^c \cap (T_{r_i(t)}W^u)^c$ is bounded and tends to zero exponentially as $t \to \pm\infty$[4, 7, 9, 12, 13].

We select $z_i^1(t)$, $z_i^2(t)$, $z_i^3(t)$, $z_i^4(t)$ as the local coordinate systems along Γ_i, $i = 1, 2$.

Let $\Delta_i = w_i^{12}/|w_i^{12}|$, $i = 1, 2$. We say that Γ_i is non-twisted as $\Delta_i = 1$, and twisted as $\Delta_i = -1$. In this paper, we consider the case $\Delta_1 = \Delta_2 = 1$.

3. Poincaré Maps and the bifurcation equations

Denote $h_i(t) = r_i(t) + Z_i(t)N_i(t)$, $N_i(t) = (n_i^1, 0, (n_i^3)^*, (n_i^4)^*)^*$, $i = 1, 2$, let $S_i^- = \{z = h_i(-T_i) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$, $S_i^+ = \{z = h_i(T_i) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$ be the cross sections of Γ_i at $t = -T_i$ and $t = T_i$, respectively.

Now, we set up Poincaré maps. (Figure 1)

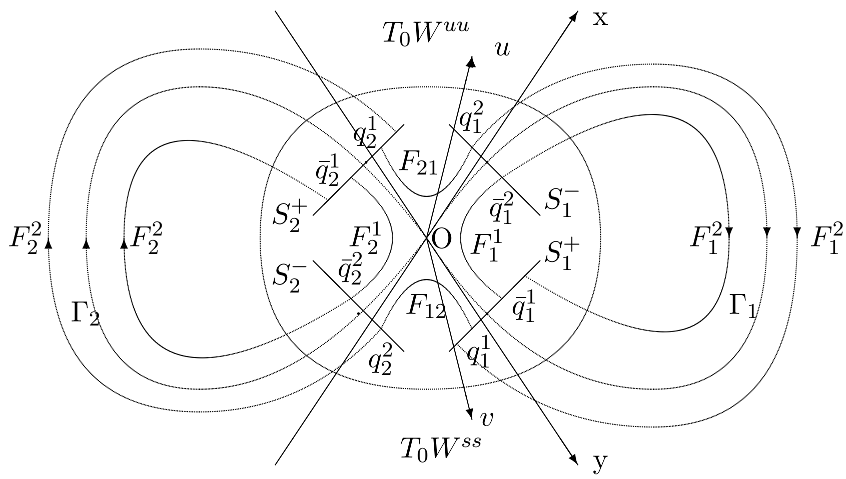


Figure 1

In U, denote $F_{21}: S_2^+ \to S_1^-$, $F_{21}(q_2^{2j+1}) = q_1^{2j+2}$; $F_{12}: S_1^+ \to S_2^-$, $F_{12}(q_1^{2j+1}) = q_2^{2j+2}$; $F_1^1: S_1^+ \to S_1^-$, $F_1^1(\bar{q}_1^{2j+1}) = \bar{q}_1^{2j+2}$; $F_2^1: S_2^+ \to S_2^-$, $F_2^1(\bar{q}_2^{2j+1}) = \bar{q}_2^{2j+2}$; where $i = 1,2$, $j = 0,1,\cdots$. In the tubular neighborhood of Γ_i, let F_i^2 be the map from S_i^- to S_i^+, $F_i^2(q_i^{2j}) = q_i^{2j+1}$, $F_i^2(\bar{q}_i^{2j}) = \bar{q}_i^{2j+1}$, where $i = 1,2$, $j = 1,2,\cdots$.

At first, we set up the relationship between the Cartesian coordinates and the normal coordinates of the points in the neighborhood of homoclinic loop. Let

$$\begin{aligned}
q_i^{2j+2}(x_i^{2j+2}, y_i^{2j+2}, (u_i^{2j+2})^*, (v_i^{2j+2})^*)^* &= r_i(-T_i) + Z_i(-T_i)N_i^{2j+2},\\
\bar{q}_i^{2j+2}(\bar{x}_i^{2j+2}, \bar{y}_i^{2j+2}, (\bar{u}_i^{2j+2})^*, (\bar{v}_i^{2j+2})^*)^* &= r_i(-T_i) + Z_i(-T_i)\bar{N}_i^{2j+2},\\
q_i^{2j+1}(x_i^{2j+1}, y_i^{2j+1}, (u_i^{2j+1})^*, (v_i^{2j+1})^*)^* &= r_i(T_i) + Z(T_i)N_i^{2j+1},\\
\bar{q}_i^{2j+1}(\bar{x}_i^{2j+1}, \bar{y}_i^{2j+1}, (\bar{u}_i^{2j+1})^*, (\bar{v}_i^{2j+1})^*)^* &= r_i(T_i) + Z(T_i)\bar{N}_i^{2j+1},
\end{aligned}$$

$$\begin{aligned}
N_i^{2j+2} &= (n_i^{2j+2,1}, 0, (n_i^{2j+2,3})^*, (n_i^{2j+2,4})^*)^*,\\
N_i^{2j+1} &= (n_i^{2j+1,1}, 0, (n_i^{2j+1,3})^*, (n_i^{2j+1,4})^*)^*,\\
\bar{N}_i^{2j+2} &= (\bar{n}_i^{2j+2,1}, 0, (\bar{n}_i^{2j+2,3})^*, (\bar{n}_i^{2j+2,4})^*)^*,\\
\bar{N}_i^{2j+1} &= (\bar{n}_i^{2j+1,1}, 0, (\bar{n}_i^{2j+1,3})^*, (\bar{n}_i^{2j+1,4})^*)^*.
\end{aligned}$$

By $Z_i^{-1}(T_i)$, $Z_i^{-1}(-T_i)$, we get

$$y_1^{2j+1} \approx \delta,\ \ x_1^{2j+2} \approx \delta,\ \ y_2^{2j+1} \approx -\delta,\ \ x_2^{2j+2} \approx -\delta \tag{3.1}$$

and

$$\begin{cases}
n_i^{2j+2,1} = (w_i^{12})^{-1}[y_i^{2j+2} - w_i^{42}(w_i^{44})^{-1}v_i^{2j+2}],\\
n_i^{2j+2,3} = u_i^{2j+2} - \delta_{iu} + b_i(w_i^{12})^{-1}y_i^{2j+2} + a_i(w_i^{44})^{-1}v_i^{2j+2},\\
n_i^{2j+2,4} = (w_i^{44})^{-1}v_i^{2j+2},
\end{cases} \tag{3.2}$$

$$\begin{cases}
n_i^{2j+1,1} = x_i^{2j+1} - w_i^{31}(w_i^{33})^{-1}u_i^{2j+1},\\
n_i^{2j+1,3} = (w_i^{33})^{-1}u_i^{2j+1},\\
n_i^{2j+1,4} = -w_i^{14}x_i^{2j+1} + c_i(w_i^{33})^{-1}u_i^{2j+1} + v_i^{2j+1} - \delta_{iv},
\end{cases} \tag{3.3}$$

where, $b_i = w_i^{11}w_i^{23}(w_i^{21})^{-1} - w_i^{13}$, $a_i = -w_i^{43} + w_i^{13}(w_i^{12})^{-1}w_i^{42} - w_i^{23}(w_i^{21})^{-1}[-w_i^{41} + w_i^{11}(w_i^{12})^{-1}w_i^{42}]$, $c_i = (w_i^{14}w_i^{31} + w_i^{24}w_i^{32} - w_i^{34})$.

As well, the relationship between the two kinds coordinates of $\bar{q}_i^{2j+2}$, $\bar{q}_i^{2j+1}$ also satisfies (3.1), (3.2) and (3.3).

Now, we consider the map F_i^2. Substituting transformation $z = h_i(t)$ into (1.2), and using $\dot{r}_i(t) = f(r_i(t))$, $\dot{Z}_i(t) = Df(r_i(t))Z_i(t)$, we get

$$Z_i(t)(\dot{n_i^1}, 0, \dot{(n_i^3)}^*, \dot{(n_i^4)}^*)^* = g_\mu(r_i(t), 0)\mu + h.o.t.\ .$$

Multiplying the both sides of the above equation by $\Phi_i^*(t)$ and using $\Phi_i^*(t)Z_i(t) = I$, we have

$$(\dot{n_i^1}, 0, \dot{(n_i^3)}^*, \dot{(n_i^4)}^*)^* = \Phi_i^*(t)g_\mu(r_i(t), 0)\mu + h.o.t.\ .$$

Integrating it, we have F_i^2 defined by the following

$$\begin{cases}
n_i^{2j+3,k} = n_i^{2j+2,k} + M_i^k\mu + h.o.t.\ ,\\
\bar{n}_i^{2j+3,k} = \bar{n}_i^{2j+2,k} + M_i^k\mu + h.o.t.\ ,
\end{cases} \quad k = 1,3,4, \tag{3.4}$$

where, $M_i^k = \int_{-\infty}^{+\infty}(\phi_i^k(t))^* g_\mu(r_i(t), 0)dt$, $k = 1,3,4$, $i = 1,2$.

Next, we consider the map in U. For convenience, we may assume $\rho_1(\mu) = (1 + \alpha(\mu))\lambda_1(\mu)$, where, $\alpha(\mu) \in R^1$, $|\alpha(\mu)| \ll 1$, $\alpha(0) = 0$.

Assume that τ_{21} is the flying time from q_2^1 to q_1^2, τ_{12} is the time from q_1^1 to q_2^2, τ_1 is the time from $\bar{q}_1^1$ to $\bar{q}_1^2$, τ_2 is the time from $\bar{q}_2^1$ to $\bar{q}_2^2$. Set $s_j = e^{-\lambda_1(\mu)\tau_j}$, $j = 21, 12, 1, 2$, which are called the Silnikov times. By (2.1), we have

$$x = e^{\lambda_1(\mu)(t-T-\tau)}x^2 + h.o.t.\ , \quad y = e^{-(1+\alpha(\mu))\lambda_1(\mu)(t-T)}y^1 + h.o.t.\ ,$$
$$u = e^{B_1(\mu)(t-T-\tau)}u^2 + h.o.t.\ , \quad v = e^{-B_2(\mu)(t-T)}v^1 + h.o.t.\ .$$

Neglecting the higher order terms, the above formulas defined the following maps:

$$F_1^1:\ \bar{x}_1^1 \approx \delta s_1,\ \bar{y}_1^2 \approx \delta s_1^{(1+\alpha(\mu))},\ \bar{u}_1^1 \approx s_1^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_1^2,\ \bar{v}_1^2 \approx s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^1. \tag{3.5}$$

$$F_2^1:\ \bar{x}_2^1 \approx -\delta s_2,\ \bar{y}_2^2 \approx -\delta s_2^{(1+\alpha(\mu))},\ \bar{u}_2^1 \approx s_2^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_2^2,\ \bar{v}_2^2 \approx s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^1. \tag{3.6}$$

$$F_{21}:\ x_2^1 \approx \delta s_{21},\ y_1^2 \approx -\delta s_{21}^{(1+\alpha(\mu))},\ u_2^1 \approx s_{21}^{B_1(\mu)/\lambda_1(\mu)}u_1^2,\ v_1^2 \approx s_{21}^{B_2(\mu)/\lambda_1(\mu)}v_2^1. \tag{3.7}$$

$$F_{12}:\ x_1^1 \approx -\delta s_{12},\ y_2^2 \approx \delta s_{12}^{(1+\alpha(\mu))},\ u_1^1 \approx s_{12}^{B_1(\mu)/\lambda_1(\mu)}u_2^2,\ v_2^2 \approx s_{12}^{B_2(\mu)/\lambda_1(\mu)}v_1^1. \tag{3.8}$$

At last, by (3.2)–(3.4) and (3.5)–(3.8), we can get Poincaré maps as follows:
$\bar{F}_1 = F_1^2 \circ F_1^1$ is

$$\begin{cases} \bar{n}_1^{3,1} = (w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} + M_1^1\mu + h.o.t.\ , \\ \bar{n}_1^{3,3} = \bar{u}_1^2 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} + M_1^3\mu + h.o.t.\ , \\ \bar{n}_1^{3,4} = (w_1^{44})^{-1}s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^1 + M_1^4\mu + h.o.t.\ . \end{cases} \tag{3.9}$$

$\bar{F}_2 = F_2^2 \circ F_2^1$ is

$$\begin{cases} \bar{n}_2^{3,1} = -(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t.\ , \\ \bar{n}_2^{3,3} = \bar{u}_2^2 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} + M_2^3\mu + h.o.t.\ , \\ \bar{n}_2^{3,4} = (w_2^{44})^{-1}s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^1 + M_2^4\mu + h.o.t.\ . \end{cases} \tag{3.10}$$

$F_1 = F_1^2 \circ F_{21}$ is

$$\begin{cases} n_1^{3,1} = -(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} + M_1^1\mu + h.o.t.\ , \\ n_1^{3,3} = u_1^2 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} + M_1^3\mu + h.o.t.\ , \\ n_1^{3,4} = (w_1^{44})^{-1}s_{21}^{B_2(\mu)/\lambda_1(\mu)}v_2^1 + M_1^4\mu + h.o.t.\ . \end{cases} \tag{3.11}$$

$F_2 = F_2^2 \circ F_{12}$ is

$$\begin{cases} n_2^{3,1} = (w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t.\ , \\ n_2^{3,3} = u_2^2 - \delta_{2u} + b_2(w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} + M_2^3\mu + h.o.t.\ , \\ n_2^{3,4} = (w_2^{44})^{-1}s_{12}^{B_2(\mu)/\lambda_1(\mu)}v_1^1 + M_2^4\mu + h.o.t.\ . \end{cases} \tag{3.12}$$

Meanwhile, we get the successor functions as follows:
$\bar{G}_1(s_1, \bar{u}_1^2, \bar{v}_1^1) = (\bar{G}_1^1, \bar{G}_1^3, \bar{G}_1^4) = (\bar{F}_1(\bar{q}_1^1) - \bar{q}_1^1)$ is

$$\begin{cases} \bar{G}_1^1 = & \delta[(w_1^{12})^{-1}s_1^{(1+\alpha(\mu))} - s_1] + M_1^1\mu + h.o.t.\ , \\ \bar{G}_1^3 = & \bar{u}_1^2 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_1^{(1+\alpha(\mu))} - (w_1^{33})^{-1}s_1^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_1^2 \\ & +M_1^3\mu + h.o.t.\ , \\ \bar{G}_1^4 = & -\bar{v}_1^1 + \delta_{1v} + w_1^{14}\delta s_1 + (w_1^{44})^{-1}s_1^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_1^1 + M_1^4\mu + h.o.t.\ . \end{cases} \tag{3.13}$$

$\bar{G}_2(s_2, \bar{u}_2^2, \bar{v}_2^1) = (\bar{G}_2^1, \bar{G}_2^3, \bar{G}_2^4) = (\bar{F}_2(\bar{q}_2^1) - \bar{q}_2^1)$ is

$$\begin{cases} \bar{G}_2^1 = & \delta[-(w_2^{12})^{-1}s_2^{(1+\alpha(\mu))} + s_2] + M_2^1\mu + h.o.t.\ , \\ \bar{G}_2^3 = & \bar{u}_2^2 - \delta_{2u} - b_2(w_2^{12})^{-1}\delta s_2^{(1+\alpha(\mu))} - (w_2^{33})^{-1}s_2^{B_1(\mu)/\lambda_1(\mu)}\bar{u}_2^2 \\ & +M_2^3\mu + h.o.t.\ , \\ \bar{G}_2^4 = & -\bar{v}_2^1 + \delta_{2v} - w_2^{14}\delta s_2 + (w_2^{44})^{-1}s_2^{B_2(\mu)/\lambda_1(\mu)}\bar{v}_2^1 + M_2^4\mu + h.o.t.\ . \end{cases} \tag{3.14}$$

$G(s_{12}, s_{21}, u_1^2, u_2^2, v_1^1, v_2^1) = (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = (F_1(q_2^1) - q_1^1, F_2(q_1^1) - q_2^1)$ is

$$\begin{cases} G_1^1 = & \delta[-(w_1^{12})^{-1} s_{21}^{(1+\alpha(\mu))} + s_{12}] + M_1^1\mu + h.o.t.\ , \\ G_1^3 = & u_1^2 - \delta_{1u} - b_1(w_1^{12})^{-1}\delta s_{21}^{(1+\alpha(\mu))} - (w_1^{33})^{-1} s_{12}^{B_1(\mu)/\lambda_1(\mu)} u_2^2 \\ & +M_1^3\mu + h.o.t.\ , \\ G_1^4 = & -v_1^1 + \delta_{1v} - w_1^{14}\delta s_{12} + (w_1^{44})^{-1} s_{21}^{B_2(\mu)/\lambda_1(\mu)} v_2^1 + M_1^4\mu + h.o.t.\ , \\ G_2^1 = & \delta[(w_2^{12})^{-1} s_{12}^{(1+\alpha(\mu))} - s_{21}] + M_2^1\mu + h.o.t.\ , \\ G_2^3 = & u_2^2 - \delta_{2u} + b_2(w_2^{12})^{-1}\delta s_{12}^{(1+\alpha(\mu))} - (w_2^{33})^{-1} s_{21}^{B_1(\mu)/\lambda_1(\mu)} u_1^2 \\ & +M_2^3\mu + h.o.t.\ , \\ G_2^4 = & -v_2^1 + \delta_{2v} + w_2^{14}\delta s_{21} + (w_2^{44})^{-1} s_{12}^{B_2(\mu)/\lambda_1(\mu)} v_1^1 + M_2^4\mu + h.o.t.\ . \end{cases} \tag{3.15}$$

Thus, we get the three bifurcation equations as follows:

$$\bar{G}_1(s_1, \bar{u}_1^2, \bar{v}_1^1) = (\bar{G}_1^1, \bar{G}_1^3, \bar{G}_1^4) = 0. \tag{3.16}$$

$$\bar{G}_2(s_2, \bar{u}_2^2, \bar{v}_2^1) = (\bar{G}_2^1, \bar{G}_2^3, \bar{G}_2^4) = 0. \tag{3.17}$$

$$G(s_{12}, s_{21}, u_1^2, u_2^2, v_1^1, v_2^1) = (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0. \tag{3.18}$$

Obviously, for system (1.2), there is an one to one correspondence between the 1-homoclinic loops and 1-periodic orbits bifurcated from Γ and the solutions of the bifurcation equations satisfy $s_j \geq 0$, $j = 1, 2, 21, 12$.

We call the 1-homoclinic loop and 1-periodic orbit bifurcated from single homoclinic loop Γ_i as small homoclic loop and small period orbit, respectively; call the 1-homoclinic loop and 1-periodic orbit bifurcated from $\Gamma = \Gamma_1 \cup \Gamma_2$ as large homoclic loop and large period orbit, respectively.

4. Resonant bifurcations

At first, we consider the bifurcations of the single homoclinic loop Γ_i, $i = 1, 2$.

Theorem 4.1. *Suppose* **(H1)–(H4)** *are fulfilled,* $|\mu| \ll 1$, $\alpha(\mu)(1 - w_i^{12}) > 0$. *If* $M_i^1\mu \neq 0$, *then, there exist two* $(l-1)$*-dimensional surfaces* $\Sigma_i \in \{\mu : \alpha(\mu)(-1)^{i+1} M_1^1\mu > 0\}$, *and* L_i, *which have the same normal vector* M_i^1, *such that*

(1) System (1.2) *has a unique 2-multiple 1-periodic orbit near* Γ_i *if and only if* $\mu \in \Sigma_i$.

(2) System (1.2) *has no 1-homoclinic and 1-periodic orbit near* Γ_i *if and only if*

$$\mu \in \{\alpha(\mu) > 0, (-1)^{i+1} M_i^1\mu > (-1)^{i+1}\beta_i(\mu)\} \text{ or } \mu \in \{\alpha(\mu) < 0, (-1)^{i+1} M_i^1\mu < (-1)^{i+1}\beta_i(\mu)\}.$$

(3) System (1.2) *has exactly two 1-periodic orbits near* Γ_i *if and only if*

$$\mu \in \{\alpha(\mu) > 0, (-1)^{i+1}\beta_i^0(\mu) < (-1)^{i+1} M_i^1\mu < (-1)^{i+1}\beta_i(\mu)\}$$

or

$$\mu \in \{\alpha(\mu) < 0, (-1)^{i+1}\beta_i(\mu) < (-1)^{i+1} M_i^1\mu < (-1)^{i+1}\beta_i^0(\mu)\}.$$

(4) System (1.2) *has exactly one 1-homoclinic orbit and one 1-periodic orbit near* Γ_i *if and only if* $\mu \in L_i$.

(5) System (1.2) *has exactly one 1-periodic orbit near* Γ_i *if and only if*

$$\mu \in \{\alpha(\mu) > 0, (-1)^{i+1} M_i^1\mu < (-1)^{i+1}\beta_i^0(\mu)\} \text{ or } \mu \in \{\alpha(\mu) < 0, (-1)^{i+1} M_i^1\mu > (-1)^{i+1}\beta_i^0(\mu)\}.$$

Where, for $i = 1, 2$, $L_i := \{\mu :\ M_i^1\mu = \beta_i^0(\mu)\}$ *is a surface defined by*

$$s_i{}^{1+\alpha(\mu)} = w_i^{12}(s_i + (-1)^i\delta^{-1} M_i^1\mu) + h.o.t.\ , \tag{4.1}$$

with $s_i = 0$, Σ_i *is a surface defined by*

$$M_i^1\mu = \beta_i(\mu) := (-1)^{i+1}\delta(w_i^{12})^{\frac{1}{\alpha(\mu)}}\alpha(\mu)(1 + \alpha(\mu))^{-1-\frac{1}{\alpha(\mu)}} + h.o.t.\ . \tag{4.2}$$

Proof. For Γ_1, it is easy to see that, for $0 \le s_1, |\mu| \ll 1$, equation $(\bar{G}_1^3, \bar{G}_1^4) = 0$ of (3.16) always has a unique solution $\bar{u}_1^2 = \bar{u}_1^2(s_1, \mu)$, $\bar{v}_1^1 = \bar{v}_1^1(s_1, \mu)$. Substituting it into $\bar{G}_1^1 = 0$, we get the bifurcation equation as

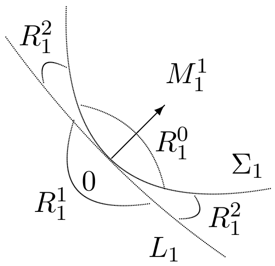
$$\delta[(w_1^{12})^{-1} s_1^{(1+\alpha(\mu))} - s_1] + M_1^1 \mu + h.o.t. = 0. \tag{4.3}$$

Similarly, about the bifurcation of Γ_2, we have the bifurcation equation as

$$\delta[-(w_2^{12})^{-1} s_2^{(1+\alpha(\mu))} + s_2] + M_2^1 \mu + h.o.t. = 0. \tag{4.4}$$

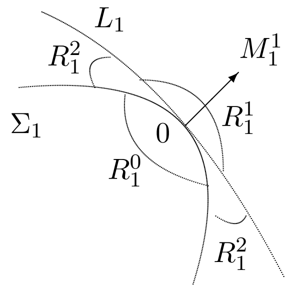
By the analysis of the existence of solutions of the equations (4.3) and (4.4) which satisfy $s_j \ge 0$, we get the results of the theorem. The method of the analysis is similar to that of [3], we don't state in detail here. $\square$

Σ_i is called 2-multiple 1-periodic orbit bifurcation surface, L_i is called 1-homoclinic orbit bifurcation surface. The bifurcations diagrams of Theorem 4.1 are the Figures 2, 3, 4 and 5.



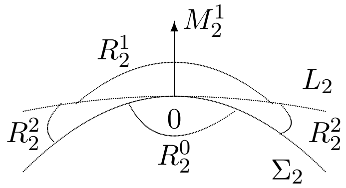
$\alpha(\mu) > 0, 0 < w_1^{12} < 1$

Figure 2



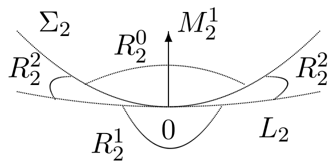
$\alpha(\mu) < 0, w_1^{12} > 1$

Figure 3



$\alpha(\mu) > 0, 0 < w_2^{12} < 1$

Figure 4



$\alpha(\mu) < 0, w_2^{12} > 1$

Figure 5

Theorem 4.2. *Suppose* **(H1)–(H4)** *are fulfilled,* $|\mu| \ll 1$, $\alpha(\mu)(1 - w_i^{12}) < 0$, *then, we have*

(1) If $(-1)^{i+1}\alpha(\mu)(M_i^1\mu - \beta_i^0(\mu)) > 0$, *then, system* (1.2) *has a unique 1-periodic orbit near* Γ_i.

(2) If $(-1)^{i+1}\alpha(\mu)(M_i^1\mu - \beta_i^0(\mu)) < 0$, *then, system* (1.2) *has no 1-periodic orbit near* Γ_i.

Proof. By the definition of L_i and some simple analysis for the intersection points of the curve $Y = {s_i}^{1+\alpha(\mu)}$ and the line $Y = w_i^{12}(s_i + (-1)^i \delta^{-1} M_i^1 \mu) + h.o.t.$, we get the conclusions of this theorem. $\square$

Theorem 4.3. *Suppose* **(H1)–(H4)** *are fulfilled,* $|\mu| \ll 1$, $\alpha(\mu) = 0$, $w_i^{12} \ne 1$. *If* $M_i^1\mu \ne 0$, *then, we have the following.*

(1) System (1.2) *has a unique 1-periodic orbit near* Γ_i *if and only if* $[(w_i^{12})^{-1} - 1]^{-1}(-1)^{i+1} M_i^1 \mu < 0$.

(2) *System* (1.2) *has no 1-homoclinic orbit and 1-periodic orbit near* Γ_i *if and only if*

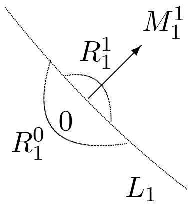
$$[(w_i^{12})^{-1} - 1]^{-1}(-1)^{i+1}M_i^1\mu > 0.$$

(3) *System* (1.2) *has exactly a unique 1-homoclinic loop near* Γ_i *if and only if* $\mu \in L_i$*, where,* L_i *is defined by the following equation with* $s_i = 0$*.*

$$\delta[(w_i^{12})^{-1} - 1]s_i + (-1)^{i+1}M_i^1\mu + h.o.t. = 0. \tag{4.5}$$

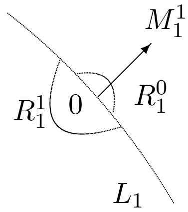
Proof. If $\alpha(\mu) = 0$, then, by (4.1), we only need to consider the solution of bifurcation equation (4.5). So, if $(w_i^{12})^{-1} \neq 1$, we have the unique solution $s_i = (-1)^i\delta^{-1}[(w_i^{12})^{-1} - 1]^{-1}M_i^1\mu + h.o.t.$. □

For the bifurcations diagrams of Theorem 4.2 and Theorem 4.3, see the Figures 6, 7, 8 and 9.



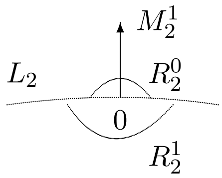
$\alpha(\mu) \geq 0, w_1^{12} > 1$

Figure 6



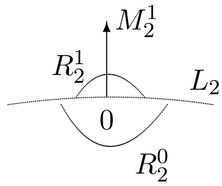
$\alpha(\mu) \leq 0, 0 < w_1^{12} < 1$

Figure 7



$\alpha(\mu) \geq 0, w_2^{12} > 1$

Figure 8



$\alpha(\mu) \leq 0, 0 < w_2^{12} < 1$

Figure 9

Next we discuss the large 1-homoclinic loop and large 1-periodic orbit bifurcated by $\Gamma = \Gamma_1 \cup \Gamma_2$, that is, discuss the solutions $Q(s_{12}, s_{21}, u_1^2, u_2^2, v_1^1, v_2^1)$ of the bifurcation equation (3.18) which satisfy $s_{12} \geq 0$, $s_{21} \geq 0$. By (3.15), for $0 \leq s_{12}, s_{21}, |\mu| \ll 1$, the equation $(G_1^3, G_1^4, G_2^3, G_2^4) = 0$ always has unique solution $u_1^2 = u_1^2(s_{21}, s_{12}, \mu)$, $u_2^2 = u_2^2(s_{21}, s_{12}, \mu)$, $v_1^1 = v_1^1(s_{21}, s_{12}, \mu)$, $v_2^1 = v_2^1(s_{21}, s_{12}, \mu)$. Substituting it into $(G_1^1, G_2^1) = 0$, we have

$$\begin{cases} \delta[-(w_1^{12})^{-1}s_{21}^{(1+\alpha(\mu))} + s_{12}] + M_1^1\mu + h.o.t. = 0, \\ \delta[(w_2^{12})^{-1}s_{12}^{(1+\alpha(\mu))} - s_{21}] + M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.6}$$

Case 1. $\alpha(\mu) > 0$

In this case, we have the following conclusions.

Theorem 4.4. *If* **(H1)–(H4)** *are satisfied,* $|\mu| \ll 1$*,* $\alpha(\mu) > 0$*,* $rank(M_1^1, M_2^1) = 2$*, then, we have the following (see* Figure 10*).*

(1) *In* $\{M_1^1\mu < 0,\ M_2^1\mu < 0\}$*, there exists a* $(l-1)$*-dimensional surface* L_{21}^{12} *which is tangent to* L_2 *at* $\mu = 0$*, and, in* $\{M_1^1\mu > 0,\ M_2^1\mu > 0\}$*, there exists a* $(l-1)$*-dimensional surface* L_{12}^{21} *which is tangent to* L_1 *at* $\mu = 0$*, such that, the necessary and sufficient condition that system* (1.2) *has a large homoclinic loop near* Γ *is* $\mu \in L_{12}^{21} \cup L_{21}^{12}$*.*

(2) *There exists an area* R_1 *that bounds with* L_{12}^{21} *and* L_{21}^{12}*, vector* M_1^1 *directs to the outside of* R_1 *from* L_{12}^{21}*, and* M_2^1 *directs to the inner of* R_1 *from* L_{21}^{12}*, such that, for* $\mu \in R_1$*, system* (1.2) *has a large periodic orbit near* Γ*.*

(3) *For* $\mu \in L = L_1 \cap L_2$*, double homoclinic loops* Γ *is preserved.*

Where, L_i*,* $i = 1, 2$ *are defined by* $M_i^1\mu + h.o.t. = 0$ *which are expressed by the two equations of* (4.6) *satisfying* $s_{12} = s_{21} = 0$*.* L_{21}^{12} *defined by*

$$(w_2^{12})^{-1}(-\delta^{-1}M_1^1\mu + h.o.t)^{(1+\alpha(\mu))} + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$

and L_{12}^{21} *defined by*

$$(w_1^{12})^{-1}(\delta^{-1}M_2^1\mu + h.o.t.)^{(1+\alpha(\mu))} - \delta^{-1}M_1^1\mu + h.o.t. = 0.$$

Proof. By (3.15), we have $\partial(G_1^1, G_2^1, G_1^3, G_2^3, G_1^4, G_2^4)/\partial Q|_{(Q,\mu)=0} = \mathrm{diag}(\delta, -\delta, 1, 1, -1, -1) + (g_{ij})$, where, except $g_{51} = w_1^{14}\delta$, $g_{62} = -w_2^{14}\delta$, other elements of (g_{ij}) are all zero. So, $\|\partial G/\partial Q|_{(Q,\mu)=0}\| \neq 0$. According to the implicit function theorem, we have, near $(Q, \mu) = (0, 0)$, the equation (3.18) has a unique solution

$$s_{21} = s_{21}(\mu), u_1^2 = u_1^2(\mu), v_1^1 = v_1^1(\mu), s_{12} = s_{12}(\mu), u_2^2 = u_2^2(\mu), v_2^1 = v_2^1(\mu),$$

satisfies $s_{21}(0) = 0$, $s_{12}(0) = 0$, $u_1^2(0) = 0$, $u_2^2(0) = 0$, $v_1^1(0) = 0$, $v_2^1(0) = 0$.

If (4.6) has a solution $s_{12} = s_{21} = 0$, then (4.6) is turned to $M_i^1\mu + h.o.t. = 0$, $i = 1, 2$. So, if $\mathrm{rank}(M_1^1, M_2^1) = 2$, then, when $\mu \in L = L_1 \cap L_2$ and $|\mu| \ll 1$, double homoclinic loop Γ are preserved, where, L_1, L_2 are expressed by the two equations of (4.6) satisfying $s_{12} = s_{21} = 0$.

If (4.6) has a solution $s_{21} = 0$, $s_{12} > 0$, then (4.6) is turned to

$$s_{12} = -\delta^{-1} M_1^1 \mu + h.o.t. = 0, \tag{4.7}$$

$$\delta(w_2^{12})^{-1}(-\delta^{-1} M_1^1 \mu + h.o.t)^{(1+\alpha(\mu))} + M_2^1\mu + h.o.t. = 0. \tag{4.8}$$

If $\mathrm{rank}(M_1^1, M_2^1) = 2$, then, in $\{M_1^1\mu < 0,\ M_2^1\mu < 0\}$, (4.8) defines a $(l-1)$-dimensional surface L_{21}^{12} which is tangent to L_2 at $\mu = 0$, such that, system (1.2) has a unique large homoclinic loop in the neighborhood of Γ if $\mu \in L_{21}^{12}$ and $\mu \ll 1$.

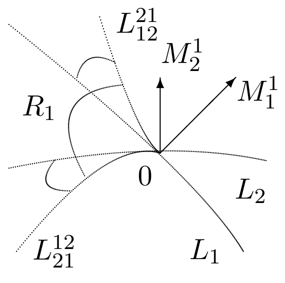
Similarly, in $\{M_1^1\mu > 0,\ M_2^1\mu > 0\}$, we can get the $(l-1)$-dimensional surface L_{12}^{21} which is tangent to L_1 at $\mu = 0$, such that, system (1.2) has a unique large homoclinic loop in the neighborhood of Γ if $\mu \in L_{12}^{21}$ and $\mu \ll 1$.

If (4.6) has a solution $s_{21} > 0$, $s_{12} > 0$, then, making the derivative of (4.6) about μ, we get

$$(s_{12})_\mu M_1^1 = -\delta^{-1}|M_1^1|^2 + O(|\mu|) + O(s_{12}^{\alpha(\mu)}),$$

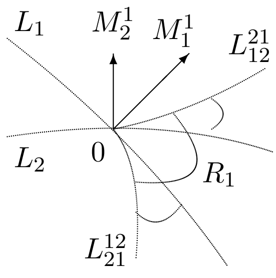
$$(s_{21})_\mu M_2^1 = \delta^{-1}|M_2^1|^2 + O(|\mu|) + O(s_{21}^{\alpha(\mu)}).$$

The above expressions show when $|\mu| \ll 1$ and $M_i^1 \neq 0$, $i = 1, 2$, in L_{12}^{21}, directional derivative of s_{12} along M_1^1 is negative; in L_{21}^{12}, directional derivative of s_{21} along M_2^1 is positive. Notice that $\{\mu : s_{12}(\mu) = 0, s_{21}(\mu) > 0\} \subset L_{12}^{21}$, $\{\mu : s_{21}(\mu) = 0, s_{12}(\mu) > 0\} \subset L_{21}^{12}$ and $\{\mu : s_{12}(\mu) = s_{21}(\mu) = 0\} \subset L_{12}^{21} \cap L_{21}^{12}$, then, (4.6) has a solution satisfying $s_{12} > 0$, $s_{21} > 0$ if and only if $\mu \in R_1$, where, R_1 is a area which have the boundaries L_{12}^{21} and L_{21}^{12}, and vector M_1^1 directs to the outside of R_1 from the boundary L_{12}^{21}, M_2^1 directs to the inner of R_1 from the boundary L_{21}^{12}. So, if $\mu \in R_1$ and $\mu \ll 1$, the system (1.2) has a large periodic orbit in the neighborhood of $\Gamma = \Gamma_1 \cap \Gamma_2$. □



$\alpha(\mu) > 0$

Figure 10



$\alpha(\mu) < 0$

Figure 11

Case 2. $\alpha(\mu) < 0$

In this case, $1 + \alpha(\mu) < 1$, by times scale transformations $s_{12} \to (s_{12})^{\frac{1}{1+\alpha(\mu)}}$, $s_{21} \to (s_{21})^{\frac{1}{1+\alpha(\mu)}}$, (4.6) becomes

$$\begin{cases} -(w_1^{12})^{-1}s_{21} + (s_{12})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (w_2^{12})^{-1}s_{12} - (s_{21})^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.9}$$

Thus, similar to that of Theorem 4.4, we have,

Theorem 4.5. *If* **(H1)–(H4)** *are satisfied,* $|\mu| \ll 1$*,* $\alpha(\mu) < 0$*,* $rank(M_1^1, M_2^1) = 2$*, then, we have the following (see* Figure 11*).*

(1) In $\{M_1^1\mu < 0,\ M_2^1\mu < 0\}$*, there exists a* $(l-1)$*-dimensional surface* L_{21}^{12} *which is tangent to* L_1 *at* $\mu = 0$*, and, in* $\{M_1^1\mu > 0,\ M_2^1\mu > 0\}$*, there exists a* $(l-1)$*-dimensional surface* L_{12}^{21} *which is tangent to* L_2 *at* $\mu = 0$*, such that, the necessary and sufficient condition that system* (1.2) *has a large homoclinic loop near* Γ *is* $\mu \in L_{12}^{21} \cup L_{21}^{12}$*.*

(2) There exists an area R_1 *that bounds with* L_{12}^{21} *and* L_{21}^{12}*, vector* M_2^1 *directs to the outside of* R_1 *from* L_{12}^{21}*, and* M_1^1 *directs to the inner of* R_1 *from* L_{21}^{12}*, such that, for* $\mu \in R_1$*, system* (1.2) *has a large periodic orbit near* Γ*.*

(3) For $\mu \in L = L_1 \cap L_2$*, double homoclinic loops* Γ *is preserved.*

Where, L_i*,* $i = 1, 2$ *are defined by* $M_i^1\mu + h.o.t. = 0$ *which are expressed by the two equations of* (4.9) *satisfying* $s_{12} = s_{21} = 0$*.* L_{21}^{12} *defined by*

$$\left(-\delta^{-1}w_2^{12}M_2^1\mu + h.o.t.\right)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_1^1\mu + h.o.t. = 0,$$

and L_{12}^{21} *defined by*

$$-(\delta^{-1}w_1^{12}M_1^1\mu + h.o.t.)^{\frac{1}{1+\alpha(\mu)}} + \delta^{-1}M_2^1\mu + h.o.t. = 0.$$

Case 3. $\alpha(\mu) = 0$

In this case, (4.6) becomes

$$\begin{cases} -(w_1^{12})^{-1}s_{21} + s_{12} + \delta^{-1}M_1^1\mu + h.o.t. = 0, \\ (w_2^{12})^{-1}s_{12} - s_{21} + \delta^{-1}M_2^1\mu + h.o.t. = 0. \end{cases} \tag{4.10}$$

So,

$$\begin{pmatrix} s_{12} \\ s_{21} \end{pmatrix} = \delta^{-1}D^{-1}\begin{pmatrix} \left(M_1^1 - (w_1^{12})^{-1}M_2^1\right)\mu \\ \left((w_2^{12})^{-1}M_1^1 - M_2^1\right)\mu \end{pmatrix} + h.o.t.\ , \tag{4.11}$$

where, $D = (w_1^{12}w_2^{12})^{-1} - 1$.

Denote $M_0^1 := (w_2^{12})^{-1}M_1^1 - M_2^1$, $M_0^2 := M_1^1 - (w_1^{12})^{-1}M_2^1$. Thus, we get the following theorem.

Theorem 4.6. *Suppose that* **(H1)–(H4)** *hold,* $|\mu| \ll 1$*. If* $\alpha(\mu) = 0$*,* $(w_1^{12}w_2^{12})^{-1} \neq 1$*,* $rank\{M_1^1, M_2^1\} = 2$*, then,* (4.11) *has a unique solution* $0 \leq s_{12}(\mu), s_{21}(\mu) \ll 1$ *satisfying* $s_{12}(0) = s_{21}(0) = 0$*. Moreover*

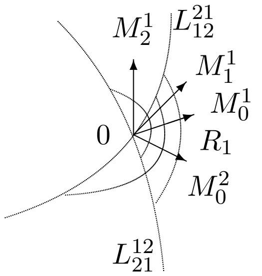
(1) In the region $\{D^{-1}M_0^2\mu > 0\}$*, there is a* $(l-1)$*-dimensional surface* L_{21}^{12} *which has normal vector* M_0^1 *at* $\mu = 0$*, such that for* $\mu \in L_{21}^{12}$*,* (4.11) *has a solution* $s_{21} = 0$*,* $s_{12} > 0$*, that is, system* (1.2) *has a large homoclinic loop.*

(2) In the region $\{D^{-1}M_0^1\mu > 0\}$*, there is a* $(l-1)$*-dimensional surface* L_{12}^{21} *which has normal vector* M_0^2 *at* $\mu = 0$*, such that for* $\mu \in L_{12}^{21}$*,* (4.11) *has a solution* $s_{12} = 0$*,* $s_{21} > 0$*, that is, system* (1.2) *has a large homoclinic loop.*

(3) If $\mu \in L_{12}^{21} \cap L_{21}^{12}$*, then,* (4.11) *has a solution* $s_{12} = 0$*,* $s_{21} = 0$*, that is, system* (1.2) *has a double homoclinic loops.*

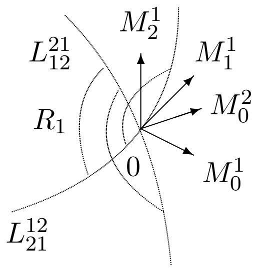
(4) If $\mu \in \{D^{-1}M_0^2\mu > 0\} \cap \{D^{-1}M_0^1\mu > 0\}$*, then,* (4.11) *has a solution* $s_{12} > 0$*,* $s_{21} > 0$*, that is, system* (1.2) *has a large periodic orbit.*

Figures 12 and 13 are the bifurcation diagrams of Theorem 4.6.



$\alpha(\mu) = 0, D > 0$

Figure 12



$\alpha(\mu) = 0, D < 0$

Figure 13

5. Conclusion

In this paper, we have discussed the bifurcation problems of double homoclinic loops with resonant condition for higher dimensional system. In the parameter space, for the different values of $\alpha(\mu)$, w_1^{12}, and w_2^{12}, we obtain the existence, number and existence regions of the small homoclinic loops, periodic orbits, and large homoclinic loops, periodic orbits, respectively.

Finally, combining the related results of Theorems 4.1–4.6, we can get the complete bifurcations figures in the parameter space for the different values of $\alpha(\mu)$, w_1^{12}, and w_2^{12}.

Theorem 5.1. *Suppose that* **(H1)–(H4)** *hold,* $|\mu| \ll 1$, *rank*$\{M_1^1, M_2^1\} = 2$, *then, we have the following conclusions.*

(1) *For the case* $\alpha(\mu) > 0$, $0 < w_1^{12} < 1$, $0 < w_2^{12} < 1$, *the bifurcations figure is the combination of* Figures 2, 4 *and* 10.

(2) *For the case* $\alpha(\mu) < 0$, $w_1^{12} > 1$, $w_2^{12} > 1$, *the bifurcations figure is the combination of* Figures 3, 5 *and* 11.

(3) *For the case* $\alpha(\mu) > 0$, $0 < w_1^{12} < 1$, $w_2^{12} > 1$, *the bifurcations figure is the combination of* Figures 2, 8 *and* 10.

(4) *For the case* $\alpha(\mu) > 0$, $w_1^{12} > 1$, $0 < w_2^{12} < 1$, *the bifurcations figure is the combination of* Figures 6, 4 *and* 10.

(5) *For the case* $\alpha(\mu) > 0$, $w_1^{12} > 1$, $w_2^{12} > 1$, *the bifurcations figure is the combination of* Figures 6, 8 *and* 10.

(6) *For the case* $\alpha(\mu) < 0$, $w_1^{12} > 1$, $0 < w_2^{12} < 1$, *the bifurcations figure is the combination of* Figures 3, 9 *and* 11.

(7) *For the case* $\alpha(\mu) < 0$, $0 < w_1^{12} < 1$, $w_2^{12} > 1$, *the bifurcations figure is the combination of* Figures 7, 5 *and* 11.

(8) *For the case* $\alpha(\mu) < 0$, $0 < w_1^{12} < 1$, $0 < w_2^{12} < 1$, *the bifurcations figure is the combination of* Figures 7, 9 *and* 11.

(9) *For the case* $\alpha(\mu) = 0$, $w_1^{12} > 1$, $w_2^{12} > 1$, $D = (w_1^{12} w_2^{12})^{-1} - 1 < 0$, *the bifurcations figure is the combination of* Figures 6, 8 *and* 13.

(10) *For the case* $\alpha(\mu) = 0$, $0 < w_1^{12} < 1$, $0 < w_2^{12} < 1$, $D = (w_1^{12} w_2^{12})^{-1} - 1 > 0$, *the bifurcations figure is the combination of* Figures 7, 9 *and* 12.

(11) For the case $\alpha(\mu) = 0$, $w_1^{12} > 1$, $0 < w_2^{12} < 1$, $D = (w_1^{12}w_2^{12})^{-1} - 1 > 0$, *the bifurcations figure is the combination of* Figures 6, 9 *and* 12.

(12) For the case $\alpha(\mu) = 0$, $w_1^{12} > 1$, $0 < w_2^{12} < 1$, $D = (w_1^{12}w_2^{12})^{-1} - 1 < 0$, *the bifurcations figure is the combination of* Figures 6, 9 *and* 13.

(13) For the case $\alpha(\mu) = 0$, $0 < w_1^{12} < 1$, $w_2^{12} > 1$, $D = (w_1^{12}w_2^{12})^{-1} - 1 > 0$, *the bifurcations figure is the combination of* Figures 7, 8 *and* 12.

(14) For the case $\alpha(\mu) = 0$, $0 < w_1^{12} < 1$, $w_2^{12} > 1$, $D = (w_1^{12}w_2^{12})^{-1} - 1 < 0$, *the bifurcations figure is the combination of* Figures 7, 8 *and* 13.

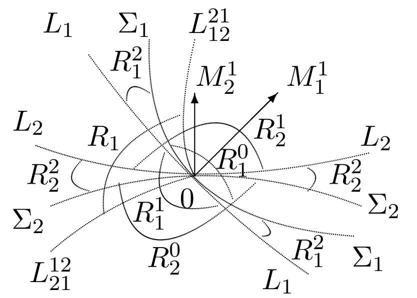
For example, the bifurcations figure for the case **(1)** ($\alpha(\mu) > 0$, $0 < w_1^{12} < 1$, $0 < w_2^{12} < 1$) is the following Figure 14, the bifurcations figure for the case **(2)** ($\alpha(\mu) < 0$, $w_1^{12} > 1$, $w_2^{12} > 1$) is the following Figure 15. Here, the relative positions of Σ_1, Σ_2 and L_{12}^{21}, L_{21}^{12} are determined by their expressions as follows:

(i) For the case $\alpha(\mu) > 0$, $0 < w_1^{12} < 1$, $0 < w_2^{12} < 1$,

$$M_1^1\mu|_{\mu\in L_{12}^{21}} > M_1^1\mu|_{\mu\in\Sigma_1} > 0, \quad M_2^1\mu|_{\mu\in L_{21}^{12}} < M_2^1\mu|_{\mu\in\Sigma_2} < 0.$$

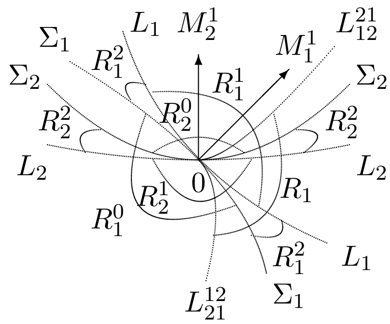
(ii) For the case $\alpha(\mu) < 0$, $w_1^{12} > 1$, $w_2^{12} > 1$,

$$M_1^1\mu|_{\mu\in L_{21}^{12}} < M_1^1\mu|_{\mu\in\Sigma_1} < 0, \quad M_2^1\mu|_{\mu\in L_{12}^{21}} > M_2^1\mu|_{\mu\in\Sigma_2} > 0.$$



$\alpha(\mu) > 0, 0 < w_1^{12} < 1, 0 < w_2^{12} < 1$

Figure 14



$\alpha(\mu) < 0, w_1^{12} > 1, w_2^{12} > 1$

Figure 15

Acknowledgments

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