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On solving variational inequality problems involving quasimonotone operators via modified Tseng's extragradient methods with convergence analysis

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Abstract

The main objective of this research is to find the numerical solution of variational inequalities involving quasimonotone operators in infinite-dimensional real Hilbert spaces. The main advantage of these iterative schemes is that they allow the uncomplicated calculation of step size rules that depend on the knowledge of an operator explanation instead of the Lipschitz constant or some other line search method. The proposed iterative schemes follow a monotone and non-monotone step size procedure based on mapping (operator) information as a replacement for its Lipschitz constant or some other line search method. The strong convergences are well proven, analogous to the proposed methods, and impose certain control specification conditions. Finally, to verify the effectiveness of the iterative methods, we present some numerical experiments.

Keywords: Variational inequality problem, Tseng's extragradient method, strong convergence theorems, quasimonotone operator, Lipschitz continuity.

2020 MSC: 47J25, 47H09, 47H06, 47J05.

1. Introduction

The main objective of this paper is to investigate the iterative methods used to estimate the solution to the variational inequality problem [26] in a real Hilbert space involving quasimonotone operators. Assume that Σ is a real Hilbert space and Δ is a nonempty, closed, and convex subset of Σ. Consider the operator $\Upsilon : \Sigma \to \Sigma$. The variational inequality problem for Υ on Δ is defined in the following manner:

$$\text{Find } \omega^* \in \Delta \text{ such that } \left\langle \Upsilon(\omega^*), y - \omega^* \right\rangle \geqslant 0, \ \forall y \in \Delta. \tag{VIP}$$

The mathematical model of the variational inequality problem is a key problem in nonlinear analysis. It is a significant mathematical model that unifies a number of crucial concepts in applied mathematics, such as a nonlinear system of equations, optimization conditions for problems with the optimization

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process, complementarity problems, network equilibrium problems, and finance (see for more details [13, 15–18]). As a consequence, this notion has various applications in the fields of mathematical programming, engineering, transport analysis, network economics, game theory, and computer science. The regularized method and the projection method are two prominent and general procedures for finding a solution to variational inequalities. It is also noted that the first approach is most commonly used to deal with the variational inequalities accompanied by the class of monotone operators. The regularized sub-problem in this method is strongly monotone, and its unique solution is found more conveniently than the initial problem. In this study, we discuss the projection methods that are well known for their simpler numerical computing. In addition, projection methods are useful for approximating the numerical solution of variational inequalities. Many researchers have developed various projection methods to solve such problems (see for more details [4, 6, 7, 10, 11, 14, 19, 22, 24, 28, 35, 36, 39]) and others in [3, 5, 8, 9, 12, 20, 23, 25, 29–33, 37, 38, 40]. Almost all methods for solving the problem (VIP) are based on the computation of a projection on the feasible set Δ. Korpelevich [19] and Antipin [1] introduced the following extragradient method. Their method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ y_n = P_\Delta[u_n - \varkappa\Upsilon(u_n)], \\ u_{n+1} = P_\Delta[u_n - \varkappa\Upsilon(y_n)], \end{cases} \tag{1.1}$$

where $0 < \varkappa < \frac{1}{L}$. Given the above method, we have used two projections on the underlying set Δ for each iteration. This, of course, can affect the computational effectiveness of the method if the feasible set Δ has a complicated structure. Here, we present some methods that can overcome this drawback. The first is the following subgradient extragradient method introduced by Censor et al. [10]. This method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ y_n = P_\Delta[u_n - \varkappa\Upsilon(u_n)], \\ u_{n+1} = P_{\Sigma_n}[u_n - \varkappa\Upsilon(y_n)], \end{cases}$$

where $0 < \varkappa < \frac{1}{L}$ and

$$\Sigma_n = \{z \in \Sigma : \langle u_n - \varkappa\Upsilon(u_n) - y_n, z - y_n\rangle \leqslant 0\}.$$

In this article, our main focus on the Tseng's extragradient method [28] that uses only one projection for each iteration. This method takes the following form:

$$\begin{cases} u_1 \in \Delta, \\ y_n = P_\Delta[u_n - \varkappa\Upsilon(u_n)], \\ u_{n+1} = y_n + \varkappa\big[\Upsilon(u_n) - \Upsilon(y_n)\big], \end{cases}$$

where $0 < \varkappa < \frac{1}{L}$. It is important to note that the above-mentioned methods have two major flaws: a fixed constant step size rule that is dependent on the Lipschitz constant of mapping and generates a weakly convergent iterative sequence. The Lipschitz constant is generally unknown or difficult to compute. From a computational point of view, it can be difficult to consider a fixed step size constraint that affects the method's efficiency and rate of convergence. In addition, the study of a strongly convergent iterative sequence is important in the context of an infinite-dimensional Hilbert space.

A natural question has been raised:

"Is it possible to introduce new strongly convergent Tseng's extragradient-type method by using a monotonic and non-monotonic variable step size rule to solve variational inequalities involving quasimonotone operator?

This research aims to explore variational inequalities involving quasimonotone operators in infinite-dimensional Hilbert spaces. Furthermore, to show that the iterative sequences generated by all four subgradient extragradient algorithms strongly converge to a solution. Both the monotone and non-monotone variable step size rules are used in subgradient and extragradient algorithms. The investigation of inertial algorithms is also presented, which usually improves the efficiency of the iterative sequence. The paper's key contribution is to investigate explicit monotone and non-monotone step size rules with inertial schemes and achieve strong convergence.

The paper is arranged in the following way. In Sect. 2, preliminary results were presented. Sect. 3 gives all new algorithms and their convergence analysis. Finally, Sect. 4 gives some numerical results to explain the practical efficiency of the proposed methods.

2. Preliminaries

This section contains a number of important identities and relevant lemmas. For any $u, y \in \Sigma$, we have

$$\|u+y\|^2 = \|u\|^2 + 2\langle u, y\rangle + \|y\|^2.$$

A *metric projection* $P_\Delta(y_1)$ of $y_1 \in \Sigma$ is defined by

$$P_\Delta(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \Delta\}.$$

First, we list some of the important identities of projection mapping and others.

Lemma 2.1 ([2]). *Let* $P_\Delta : \Sigma \to \Delta$ *be a metric projection. For any* $y_1, y_2, y_3 \in \Sigma$ *and* $\ell \in \mathbb{R}$, *then, the following inequalities are hold:*

(i) $y_3 = P_\Delta(y_1)$ *if and only if*
$$\langle y_1 - y_3, y_2 - y_3\rangle \leqslant 0, \ \forall y_2 \in \Delta;$$

(ii)
$$\|y_1 - P_\Delta(y_2)\|^2 + \|P_\Delta(y_2) - y_2\|^2 \leqslant \|y_1 - y_2\|^2, \ y_1 \in \Delta, y_2 \in \Sigma;$$

(iii)
$$\|y_1 - P_\Delta(y_1)\| \leqslant \|y_1 - y_2\|, \ y_2 \in \Delta, y_1 \in \Sigma;$$

(iv)
$$\|\ell y_1 + (1-\ell)y_2\|^2 = \ell\|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2;$$

(v)
$$\|y_1 + y_2\|^2 \leqslant \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle.$$

Lemma 2.2 ([34]). *Let* $\{e_n\} \subset [0, +\infty)$ *be a sequence satisfies the following condition*

$$e_{n+1} \leqslant (1 - f_n)e_n + f_n g_n, \ \forall n \in \mathbb{N}.$$

In addition, two sequences $\{f_n\} \subset (0,1)$ *and* $\{g_n\} \subset \mathbb{R}$ *satisfy the following conditions:*

$$\lim_{n\to+\infty} f_n = 0, \ \sum_{n=1}^{+\infty} f_n = +\infty \text{ and } \limsup_{n\to+\infty} g_n \leqslant 0.$$

Then, $\lim_{n\to+\infty} e_n = 0$.

Lemma 2.3 ([21]). *Let* $\{e_n\} \subset \mathbb{R}$ *be a sequence and there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that*

$$e_{n_i} < e_{n_{i+1}}, \ \forall i \in \mathbb{N}.$$

Then, there exists a nondecreasing sequence $m_k \subset \mathbb{N}$ *such that* $m_k \to +\infty$ *as* $k \to +\infty$, *with*

$$e_{m_k} \leqslant e_{m_{k+1}} \quad \text{and} \quad e_k \leqslant e_{m_{k+1}}, \ \forall k \in \mathbb{N}.$$

Indeed, $m_k = \max\{j \leqslant k : e_j \leqslant e_{j+1}\}$.

3. Main results

In this section, we propose a class of iterative algorithms based on Tseng's extragradient method for solving quasimonotone variational inequalities. The methods are all described in detail in the following text. The following conditions are assumed to be met in order to confirm the strong convergence.

(Υ1) The solution set for problem (VIP) is denoted by $VI(\Delta, \Upsilon)$ is nonempty.

(Υ2) An operator $\Upsilon : \Sigma \to \Sigma$ is said to be quasimonotone if

$$\langle \Upsilon(u), y - u \rangle > 0 \Longrightarrow \langle \Upsilon(y), y - u \rangle \geqslant 0, \ \forall u, y \in \Delta.$$

(Υ3) An operator $\Upsilon : \Sigma \to \Sigma$ is said to be *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|\Upsilon(u) - \Upsilon(y)\| \leqslant L\|u - y\|, \ \forall u, y \in \Delta.$$

(Υ4) An operator $\Upsilon : \Sigma \to \Sigma$ is *sequentially weakly continuous* if $\{\Upsilon(u_n)\}$ weakly converges to $\Upsilon(u)$ for every sequence $\{u_n\}$ weakly converges to u.

Now, we are in a position to propose a new variant of the extragradient method to solve quasimonotone variational inequalities in real Hilbert spaces and prove a strong convergence result for the proposed method.

Algorithm 1 (Halpern extragradient method with fixed step size rule)

STEP 0: Let $u_1 \in \Delta$, $0 < \varkappa < \frac{1}{L}$ and $\{\vartheta_n\} \subset (0, 1)$ meet the following conditions:

$$\lim_{n \to +\infty} \vartheta_n = 0 \text{ and } \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

STEP 1: Compute

$$y_n = P_\Delta(u_n - \varkappa\Upsilon(u_n)).$$

If $u_n = y_n$, STOP. Otherwise, go to **STEP 2**.

STEP 2: Compute

$$z_n = y_n + \varkappa\big[\Upsilon(u_n) - \Upsilon(y_n)\big].$$

STEP 3: Compute

$$u_{n+1} = \vartheta_n u_1 + (1 - \vartheta_n) z_n.$$

Set $n := n + 1$ and go back to **STEP 1**.

Lemma 3.1. *Suppose that $\Upsilon : \Sigma \to \Sigma$ satisfies the conditions (Υ1)-(Υ4) and sequence $\{u_n\}$ generated by Algorithm 1. Then, we have*

$$\|u_{n+1} - \omega^*\|^2 \leqslant \|u_n - \omega^*\|^2 - (1 - \varkappa^2 L^2)\|u_n - y_n\|^2.$$

Proof. Since $\omega^* \in VI(\Delta, \Upsilon)$, we have

$$\begin{aligned}
\|u_{n+1} - \omega^*\|^2 &= \|y_n + \varkappa[\Upsilon(u_n) - \Upsilon(y_n)] - \omega^*\|^2 \\
&= \|y_n - \omega^*\|^2 + \varkappa^2\|\Upsilon(u_n) - \Upsilon(y_n)\|^2 + 2\varkappa\langle y_n - \omega^*, \Upsilon(u_n) - \Upsilon(y_n)\rangle \\
&= \|y_n + u_n - u_n - \omega^*\|^2 + \varkappa^2\|\Upsilon(u_n) - \Upsilon(y_n)\|^2 + 2\varkappa\langle y_n - \omega^*, \Upsilon(u_n) - \Upsilon(y_n)\rangle \\
&= \|y_n - u_n\|^2 + \|u_n - \omega^*\|^2 + 2\langle y_n - u_n, u_n - \omega^*\rangle \\
&\quad + \varkappa^2\|\Upsilon(u_n) - \Upsilon(y_n)\|^2 + 2\varkappa\langle y_n - \omega^*, \Upsilon(u_n) - \Upsilon(y_n)\rangle \\
&= \|u_n - \omega^*\|^2 + \|y_n - u_n\|^2 + 2\langle y_n - u_n, y_n - \omega^*\rangle + 2\langle y_n - u_n, u_n - y_n\rangle \\
&\quad + \varkappa^2\|\Upsilon(u_n) - \Upsilon(y_n)\|^2 + 2\varkappa\langle y_n - \omega^*, \Upsilon(u_n) - \Upsilon(y_n)\rangle.
\end{aligned} \tag{3.1}$$

It is given that $y_n = P_\Delta[u_n - \varkappa\Upsilon(u_n)]$ and it gives that

$$\langle u_n - \varkappa \Upsilon(u_n) - y_n, y - y_n \rangle \leqslant 0, \ \forall y \in \Delta.$$

Thus, we have

$$\langle u_n - y_n, \omega^* - y_n \rangle \leqslant \varkappa \langle \Upsilon(u_n), \omega^* - y_n \rangle. \tag{3.2}$$

Combining expressions (3.1) and (3.2), we have

$$\begin{aligned} \|u_{n+1} - \omega^*\|^2 &\leqslant \|u_n - \omega^*\|^2 + \|y_n - u_n\|^2 + 2\varkappa \langle \Upsilon(u_n), \omega^* - y_n \rangle - 2\langle u_n - y_n, u_n - y_n \rangle \\ &\quad + \varkappa^2 \|\Upsilon(u_n) - \Upsilon(y_n)\|^2 - 2\varkappa \langle \Upsilon(u_n) - \Upsilon(y_n), \omega^* - y_n \rangle \\ &= \|u_n - \omega^*\|^2 - \|u_n - y_n\|^2 + \varkappa^2 \|\Upsilon(u_n) - \Upsilon(y_n)\|^2 - 2\varkappa \langle \Upsilon(y_n), y_n - \omega^* \rangle. \end{aligned} \tag{3.3}$$

It is given that ω^* is the solution of the problem (VIP) implies that

$$\langle \Upsilon(\omega^*), y - \omega^* \rangle \geqslant 0, \ \forall y \in \Delta.$$

It implies that

$$\langle \Upsilon(y), y - \omega^* \rangle \geqslant 0, \ \forall y \in \Delta.$$

By substituting $y = y_n \in \Delta$, we have

$$\langle \Upsilon(y_n), y_n - \omega^* \rangle \geqslant 0. \tag{3.4}$$

From expressions (3.3) and (3.4), we obtain

$$\begin{aligned} \|u_{n+1} - \omega^*\|^2 &\leqslant \|u_n - \omega^*\|^2 - \|u_n - y_n\|^2 + \varkappa^2 L^2 \|u_n - y_n\|^2 \\ &= \|u_n - \omega^*\|^2 - (1 - \varkappa^2 L^2) \|u_n - y_n\|^2. \end{aligned}$$

□

Next, we introduce a variant of Algorithm 1 in which the constant step size $\varkappa$ is chosen adaptively and thus produced a sequence $\varkappa_n$ that does not require the knowledge of the Lipschitz-type constants L.

Algorithm 2 (Monotonic explicit Halpern extragradient method with variable step size rule)

STEP 0: Let $u_1 \in \Delta$, $\varkappa_1 > 0$, $\chi \in (0,1)$ and $\{\vartheta_n\} \subset (0,1)$ meet the following conditions:

$$\lim_{n \to +\infty} \vartheta_n = 0 \text{ and } \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

STEP 1: Compute

$$y_n = P_\Delta(u_n - \varkappa_n \Upsilon(u_n)).$$

If $u_n = y_n$, STOP. Otherwise, go to **STEP 2**.

STEP 2: Compute

$$z_n = y_n + \varkappa_n \left[\Upsilon(u_n) - \Upsilon(y_n) \right].$$

STEP 3: Compute $u_{n+1} = \vartheta_n u_1 + (1 - \vartheta_n) z_n$.

STEP 4: Compute

$$\varkappa_{n+1} = \begin{cases} \min\left\{ \varkappa_n, \frac{\chi \|u_n - y_n\|}{\|\Upsilon(u_n) - \Upsilon(y_n)\|} \right\}, & \text{if } \Upsilon(u_n) - \Upsilon(y_n) \neq 0, \\ \varkappa_n, & \text{otherwise.} \end{cases} \tag{3.5}$$

Set $n := n + 1$ and go back to **STEP 1**.

Lemma 3.2. *The sequence $\{\varkappa_n\}$ generated by (3.5) is decreasing monotonically and converges to $\varkappa > 0$.*

Proof. It is given that Υ is Lipschitz-continuous with constant $L > 0$. Let $\Upsilon(u_n) \neq \Upsilon(y_n)$ such that

$$\frac{\chi\|u_n - y_n\|}{\|\Upsilon(u_n) - \Upsilon(y_n)\|} \geqslant \frac{\chi\|u_n - y_n\|}{L\|u_n - y_n\|} \geqslant \frac{\chi}{L}.$$

The above expression implies that $\lim_{n\to+\infty} \varkappa_n = \varkappa$. □

Now, we propose a second variant of the first method to solve quasimonotone variational inequalities in real Hilbert spaces and prove a strong convergence result for the proposed method. The second method involves a non-monotonic self-adaptive step rule to make the method independent of the Lipschitz constant. The second method is written as follows.

Algorithm 3 (Non-monotonic explicit Halpern extragradient method with variable step size rule)

STEP 0: Let $u_1 \in \Delta$, $\varkappa_1 > 0$, $\chi \in (0,1)$ and sequence $\{\varphi_n\}$ satisfying $\sum_{n=1}^{+\infty} \varphi_n < +\infty$. Moreover, $\{\vartheta_n\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{n\to+\infty} \vartheta_n = 0 \text{ and } \sum_{n=1}^{+\infty} \vartheta_n = +\infty.$$

STEP 1: Compute

$$y_n = P_\Delta(u_n - \varkappa_n \Upsilon(u_n)).$$

If $u_n = y_n$, STOP. Otherwise, go to **STEP 2**.
STEP 2: Compute $z_n = y_n + \varkappa_n\left[\Upsilon(u_n) - \Upsilon(y_n)\right]$.
STEP 3: Compute $u_{n+1} = \vartheta_n u_1 + (1 - \vartheta_n) z_n$.
STEP 4: Compute

$$\varkappa_{n+1} = \begin{cases} \min\left\{\varkappa_n + \varphi_n, \frac{\chi\|u_n - y_n\|}{\|\Upsilon(u_n) - \Upsilon(y_n)\|}\right\}, & \text{if } \Upsilon(u_n) - \Upsilon(y_n) \neq 0, \\ \varkappa_n + \varphi_n, & \text{otherwise.} \end{cases} \tag{3.6}$$

Set $n := n + 1$ and go back to **STEP 1**.

Lemma 3.3. *A sequence $\{\varkappa_n\}$ generated by (3.6) is convergent to $\varkappa$ and satisfying the following inequality*

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leqslant \varkappa_n \leqslant \varkappa_1 + P \quad \textit{where} \quad P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof. It is given that Υ is Lipschitz-continuous with constant $L > 0$. Let $\Upsilon(u_n) \neq \Upsilon(y_n)$ such that

$$\frac{\chi\|u_n - y_n\|}{\|\Upsilon(u_n) - \Upsilon(y_n)\|} \geqslant \frac{\chi\|u_n - y_n\|}{L\|u_n - y_n\|} \geqslant \frac{\chi}{L}.$$

By using mathematical induction on the definition of $\varkappa_{n+1}$, we have

$$\min\left\{\frac{\chi}{L}, \varkappa_1\right\} \leqslant \varkappa_n \leqslant \varkappa_1 + P.$$

Let $[\varkappa_{n+1} - \varkappa_n]^+ = \max\left\{0, \varkappa_{n+1} - \varkappa_n\right\}$ and $[\varkappa_{n+1} - \varkappa_n]^- = \max\left\{0, -(\varkappa_{n+1} - \varkappa_n)\right\}$. From the definition of $\{\varkappa_n\}$, we have

$$\sum_{n=1}^{+\infty} (\varkappa_{n+1} - \varkappa_n)^+ = \sum_{n=1}^{+\infty} \max\left\{0, \varkappa_{n+1} - \varkappa_n\right\} \leqslant P < +\infty.$$

That is, the series $\sum_{n=1}^{+\infty}(\varkappa_{n+1}-\varkappa_n)^+$ is convergent. Next, we need to prove the convergence of $\sum_{n=1}^{+\infty}(\varkappa_{n+1}-\varkappa_n)^-$. Let $\sum_{n=1}^{+\infty}(\varkappa_{n+1}-\varkappa_n)^- = +\infty$. Due to the reason that $\varkappa_{n+1}-\varkappa_n = (\varkappa_{n+1}-\varkappa_n)^+ - (\varkappa_{n+1}-\varkappa_n)^-$, thus, we have

$$\varkappa_{k+1}-\varkappa_1 = \sum_{n=0}^{k}(\varkappa_{n+1}-\varkappa_n) = \sum_{n=0}^{k}(\varkappa_{n+1}-\varkappa_n)^+ - \sum_{n=0}^{k}(\varkappa_{n+1}-\varkappa_n)^-. \tag{3.7}$$

By allowing $k \to +\infty$ in (3.7), we have $\varkappa_k \to -\infty$ as $k \to +\infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\varkappa_{n+1}-\varkappa_n)^+$ and $\sum_{n=0}^{k}(\varkappa_{n+1}-\varkappa_n)^-$ taking $k \to +\infty$ in (3.7), we obtain $\lim_{n\to+\infty}\varkappa_n = \varkappa$. This completes the proof. □

Lemma 3.4. *Assume that $\Upsilon : \Sigma \to \Sigma$ satisfies the conditions* (Υ1)-(Υ4). *Let $\{u_n\}$ be a sequence generated by Algorithms 2 and 3. For each $\omega^* \in VI(\Delta,\Upsilon)$, we have*

$$\|u_{n+1}-\omega^*\|^2 \leqslant \|u_n-\omega^*\|^2 - \left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n-y_n\|^2.$$

Proof. Let $\omega^* \in VI(\Delta,\Upsilon)$ and by definition of u_{n+1}, we have

$$\begin{aligned}\|u_{n+1}-\omega^*\|^2 &= \|y_n+\varkappa_n[\Upsilon(u_n)-\Upsilon(y_n)]-\omega^*\|^2\\ &= \|y_n-\omega^*\|^2+\varkappa_n^2\|\Upsilon(u_n)-\Upsilon(y_n)\|^2+2\varkappa_n\langle y_n-\omega^*,\Upsilon(u_n)-\Upsilon(y_n)\rangle\\ &= \|y_n+u_n-u_n-\omega^*\|^2+\varkappa_n^2\|\Upsilon(u_n)-\Upsilon(y_n)\|^2+2\varkappa_n\langle y_n-\omega^*,\Upsilon(u_n)-\Upsilon(y_n)\rangle\\ &= \|y_n-u_n\|^2+\|u_n-\omega^*\|^2+2\langle y_n-u_n,u_n-\omega^*\rangle\\ &\quad+\varkappa_n^2\|\Upsilon(u_n)-\Upsilon(y_n)\|^2+2\varkappa_n\langle y_n-\omega^*,\Upsilon(u_n)-\Upsilon(y_n)\rangle\\ &= \|u_n-\omega^*\|^2+\|y_n-u_n\|^2+2\langle y_n-u_n,y_n-\omega^*\rangle+2\langle y_n-u_n,u_n-y_n\rangle\\ &\quad+\varkappa_n^2\|\Upsilon(u_n)-\Upsilon(y_n)\|^2+2\varkappa_n\langle y_n-\omega^*,\Upsilon(u_n)-\Upsilon(y_n)\rangle.\end{aligned} \tag{3.8}$$

It is given that $y_n = P_\Delta[u_n-\varkappa_n\Upsilon(u_n)]$ and it further implies that

$$\langle u_n-\varkappa_n\Upsilon(u_n)-y_n, y-y_n\rangle \leqslant 0,\ \forall y \in \Delta.$$

Moreover, equivalently for some $\omega^* \in VI(\Delta,\Upsilon)$, we can write

$$\langle u_n-y_n,\omega^*-y_n\rangle \leqslant \varkappa_n\langle\Upsilon(u_n),\omega^*-y_n\rangle. \tag{3.9}$$

Combining expressions (3.8) and (3.9), we have

$$\begin{aligned}\|u_{n+1}-\omega^*\|^2 &\leqslant \|u_n-\omega^*\|^2+\|y_n-u_n\|^2+2\varkappa_n\langle\Upsilon(u_n),\omega^*-y_n\rangle-2\langle u_n-y_n,u_n-y_n\rangle\\ &\quad+\varkappa_n^2\|\Upsilon(u_n)-\Upsilon(y_n)\|^2-2\varkappa_n\langle\Upsilon(u_n)-\Upsilon(y_n),\omega^*-y_n\rangle\\ &= \|u_n-\omega^*\|^2-\|u_n-y_n\|^2+\varkappa_n^2\|\Upsilon(u_n)-\Upsilon(y_n)\|^2-2\varkappa_n\langle\Upsilon(y_n),y_n-\omega^*\rangle.\end{aligned} \tag{3.10}$$

It is given that ω^* is the solution of the problem (VIP) implying that

$$\langle\Upsilon(\omega^*),y-\omega^*\rangle > 0,\ \forall y \in \Delta.$$

Due to the property of Υ on Δ, we obtain

$$\langle \Upsilon(y), y-\omega^*\rangle \geqslant 0, \ \forall y \in \Delta.$$

Substituting $y = y_n \in \Delta$, we have

$$\langle \Upsilon(y_n), y_n-\omega^*\rangle \geqslant 0. \tag{3.11}$$

Combining expressions (3.10) and (3.11) we obtain

$$\begin{aligned}\|u_{n+1}-\omega^*\|^2 &\leqslant \|u_n-\omega^*\|^2 - \|u_n-y_n\|^2 + \chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\|u_n-y_n\|^2\\ &= \|u_n-\omega^*\|^2 - \left(1-\chi^2\frac{\varkappa_n^2}{\varkappa_{n+1}^2}\right)\|u_n-y_n\|^2.\end{aligned}$$

□

Lemma 3.5. *Let $\Upsilon : \Sigma \to \Sigma$ be an operator satisfying the conditions (Υ1)–(Υ4). If there exists a subsequence $\{u_{n_k}\}$ weakly convergent to $\hat{u}$ and $\lim_{k\to+\infty}\|u_{n_k}-y_{n_k}\| = 0$, then, $\hat{u} \in VI(\Delta, \Upsilon)$.*

Proof. Since $\{u_{n_k}\}$ is weakly convergent to $\hat{u}$ and due to $\lim_{k\to+\infty}\|u_{n_k}-y_{n_k}\| = 0$, the sequence $\{y_{n_k}\}$ also is weakly convergent to $\hat{u}$. Next, we need to prove that $\hat{u} \in VI(\Delta, \Upsilon)$. By value of y_n, we have

$$y_{n_k} = P_\Delta[u_{n_k} - \varkappa_{n_k}\Upsilon(u_{n_k})]$$

that is equivalent to

$$\langle u_{n_k} - \varkappa_{n_k}\Upsilon(u_{n_k}) - y_{n_k}, y-y_{n_k}\rangle \leqslant 0, \ \forall y \in \Delta.$$

The above inequality implies that

$$\langle u_{n_k}-y_{n_k}, y-y_{n_k}\rangle \leqslant \varkappa_{n_k}\langle \Upsilon(u_{n_k}), y-y_{n_k}\rangle, \ \forall y \in \Delta.$$

Thus, we obtain

$$\frac{1}{\varkappa_{n_k}}\langle u_{n_k}-y_{n_k}, y-y_{n_k}\rangle + \langle \Upsilon(u_{n_k}), y_{n_k}-u_{n_k}\rangle \leqslant \langle \Upsilon(u_{n_k}), y-u_{n_k}\rangle, \ \forall y \in \Delta. \tag{3.12}$$

By the use of $\lim_{k\to+\infty}\|u_{n_k}-y_{n_k}\| = 0$ and $k \to +\infty$ in expression (3.12), we have

$$\liminf_{k\to+\infty}\langle \Upsilon(u_{n_k}), y-u_{n_k}\rangle \geqslant 0, \ \forall y \in \Delta.$$

Furthermore, it implies that

$$\langle \Upsilon(y_{n_k}), y-y_{n_k}\rangle = \langle \Upsilon(y_{n_k}) - \Upsilon(u_{n_k}), y-u_{n_k}\rangle + \langle \Upsilon(u_{n_k}), y-u_{n_k}\rangle + \langle \Upsilon(y_{n_k}), u_{n_k}-y_{n_k}\rangle. \tag{3.13}$$

Since $\lim_{k\to+\infty}\|u_{n_k}-y_{n_k}\| = 0$, thus, we have

$$\lim_{k\to+\infty}\|\Upsilon(u_{n_k}) - \Upsilon(y_{n_k})\| = 0, \tag{3.14}$$

which together with expressions (3.13) and (3.14), we obtain

$$\liminf_{k\to+\infty}\langle \Upsilon(y_{n_k}), y-y_{n_k}\rangle \geqslant 0, \ \forall y \in \Delta.$$

Moreover, let us take a positive sequence $\{\epsilon_k\}$ that is decreasing and convergent to zero. For each $\{\epsilon_k\}$ there exists a least positive integer denoted by m_k such that

$$\langle \Upsilon(u_{n_i}), y-u_{n_i}\rangle + \epsilon_k > 0, \ \forall i \geqslant m_k. \tag{3.15}$$

Since $\{\epsilon_k\}$ is decreasing sequence and it is easy to see that the sequence $\{m_k\}$ is increasing. If there exists a natural number $N_0 \in \mathbb{N}$ such that for all $\Upsilon(u_{n_{m_k}}) \neq 0$, $n_{m_k} \geqslant N_0$. Consider that

$$\Gamma_{n_{m_k}} = \frac{\Upsilon(u_{n_{m_k}})}{\|\Upsilon(u_{n_{m_k}})\|^2}, \ \forall n_{m_k} \geqslant N_0.$$

Due to the above definition, we have

$$\langle \Upsilon(u_{n_{m_k}}), \Gamma_{n_{m_k}} \rangle = 1, \ \forall n_{m_k} \geqslant N_0. \tag{3.16}$$

Moreover, from expressions (3.15) and (3.16) for all $n_{m_k} \geqslant N_0$, we have

$$\langle \Upsilon(u_{n_{m_k}}), \ y + \epsilon_k \Gamma_{n_{m_k}} - u_{n_{m_k}} \rangle > 0.$$

By the definition of quasimonotone, we have

$$\langle \Upsilon(y + \epsilon_k \Gamma_{n_{m_k}}), \ y + \epsilon_k \Gamma_{n_{m_k}} - u_{n_{m_k}} \rangle > 0.$$

For all $n_{m_k} \geqslant N_0$, we have

$$\langle \Upsilon(y), y - u_{n_{m_k}} \rangle \geqslant \langle \Upsilon(y) - \Upsilon(y + \epsilon_k \Gamma_{n_{m_k}}), \ y + \epsilon_k \Gamma_{n_{m_k}} - u_{n_{m_k}} \rangle - \epsilon_k \langle \Upsilon(y), \Gamma_{n_{m_k}} \rangle. \tag{3.17}$$

Due to $\{u_{n_k}\}$ converges weakly to $\hat{u} \in \Delta$ with Υ is weakly sequentially continuous on the set Δ we obtain $\{\Upsilon(u_{n_k})\}$ converges weakly to $\Upsilon(\hat{u})$. Let $\Upsilon(\hat{u}) \neq 0$ implies that

$$\|\Upsilon(\hat{u})\| \leqslant \liminf_{k \to +\infty} \|\Upsilon(u_{n_k})\|.$$

Since $\{u_{n_{m_k}}\} \subset \{u_{n_k}\}$ and $\lim_{k \to +\infty} \epsilon_k = 0$, we have

$$0 \leqslant \lim_{k \to +\infty} \|\epsilon_k \Gamma_{n_{m_k}}\| = \lim_{k \to +\infty} \frac{\epsilon_k}{\|\Upsilon(u_{n_{m_k}})\|} \leqslant \frac{0}{\|\Upsilon(\hat{u})\|} = 0.$$

By letting $k \to +\infty$ in (3.17), we obtain

$$\langle \Upsilon(y), y - \hat{u} \rangle \geqslant 0, \ \forall y \in \Delta. \tag{3.18}$$

Let $u \in \Delta$ be an arbitrary element and for $0 < \varkappa \leqslant 1$, let

$$\hat{u}_\varkappa = \varkappa u + (1 - \varkappa)\hat{u}.$$

Then $\hat{u}_\varkappa \in \Delta$ and from (3.18) we have

$$\varkappa \langle \Upsilon(\hat{u}_\varkappa), u - \hat{u} \rangle \geqslant 0.$$

Hence

$$\langle \Upsilon(\hat{u}_\varkappa), u - \hat{u} \rangle \geqslant 0. \tag{3.19}$$

Let $\varkappa \to 0$. Then $\hat{u}_\varkappa \to \hat{u}$ along a line segment. By the continuity of an operator, $\Upsilon(\hat{u}_\varkappa)$ converges to $\Upsilon(\hat{u})$ as $\varkappa \to 0$. It follows from (3.19) that

$$\langle \Upsilon(\hat{u}), u - \hat{u} \rangle \geqslant 0.$$

Therefore $\hat{u}$ is a solution of problem (VIP). □

Theorem 3.6. *Assume that a sequence $\{u_n\}$ generated by Algorithm 3 and $\omega^* \in VI(\Delta, \Upsilon)$. Then, $\{u_n\}$ converges strongly to $\omega^* = P_{VI(\Delta,\Upsilon)}(u_1)$.*

Proof. Since $\varkappa_n \to \varkappa$ there exists a positive number $\epsilon \in (0, 1-\chi^2)$ such that

$$\lim_{n\to+\infty}\Big(1-\frac{\chi^2\varkappa_n^2}{\varkappa_{n+1}^2}\Big) = 1-\chi^2 > \epsilon > 0.$$

Thus, there exists a finite number $n_1 \in \mathbb{N}$ such that

$$\Big(1-\frac{\chi^2\varkappa_n^2}{\varkappa_{n+1}^2}\Big) > \epsilon > 0, \ \forall n \geqslant n_1.$$

Hence, by Lemma 3.4, we obtain

$$\|z_n - \omega^*\|^2 \leqslant \|u_n - \omega^*\|^2, \ \forall n \geqslant n_1. \tag{3.20}$$

Since $\omega^* \in VI(\Delta, \Upsilon)$ and by the use of definition of $\{u_{n+1}\}$, we have

$$\begin{aligned}\|u_{n+1}-\omega^*\| &= \|\vartheta_n u_1 + (1-\vartheta_n)z_n - \omega^*\| \\ &= \|\vartheta_n[u_1-\omega^*] + (1-\vartheta_n)[z_n-\omega^*]\| \leqslant \vartheta_n\|u_1-\omega^*\| + (1-\vartheta_n)\|z_n-\omega^*\|.\end{aligned} \tag{3.21}$$

Combining expressions (3.20) with (3.21) and $\vartheta_n \subset (0,1)$, we have

$$\begin{aligned}\|u_{n+1}-\omega^*\| &\leqslant \vartheta_n\|u_1-\omega^*\| + (1-\vartheta_n)\|u_n-\omega^*\|. \\ &\leqslant \max\Big\{\|u_1-\omega^*\|, \|u_n-\omega^*\|\Big\} \leqslant \max\Big\{\|u_1-\omega^*\|, \|u_{n_1}-\omega^*\|\Big\}.\end{aligned}$$

Thus, we conclude that the $\{u_n\}$ is a bounded sequence. Next, we need to prove the strong convergence of the iterative sequence $\{u_n\}$ generated by Algorithm 3. The Lipschitz continuity property indicates that the solution set $VI(\Delta, \Upsilon)$ is a convex and closed set (see for details [27]). Let $\omega^* = P_{VI(\Delta,\Upsilon)}(u_1)$ and by Lemma 2.1 (i) we have

$$\langle u_1-\omega^*, y-\omega^*\rangle \leqslant 0, \ \forall y \in VI(\Delta, \Upsilon).$$

By using Lemma 2.1 (iv), we obtain

$$\begin{aligned}\|u_{n+1}-\omega^*\|^2 &= \|\vartheta_n u_1 + (1-\vartheta_n)z_n - \omega^*\|^2 \\ &= \|\vartheta_n[u_1-\omega^*] + (1-\vartheta_n)[z_n-\omega^*]\|^2 \\ &= \vartheta_n\|u_1-\omega^*\|^2 + (1-\vartheta_n)\|z_n-\omega^*\|^2 - \vartheta_n(1-\vartheta_n)\|u_1-z_n\|^2 \\ &\leqslant \vartheta_n\|u_1-\omega^*\|^2 + (1-\vartheta_n)\Big[\|u_n-\omega^*\|^2 - \Big(1-\frac{\chi^2\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\|u_n-y_n\|^2\Big] \\ &\quad - \vartheta_n(1-\vartheta_n)\|u_1-z_n\|^2 \\ &\leqslant \vartheta_n\|u_1-\omega^*\|^2 + \|u_n-\omega^*\|^2 - (1-\vartheta_n)\Big(1-\frac{\chi^2\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\|u_n-y_n\|^2.\end{aligned}$$

The above expression implies that

$$(1-\vartheta_n)\Big(1-\frac{\chi^2\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\|u_n-y_n\|^2 \leqslant \vartheta_n\|u_1-\omega^*\|^2 + \|u_n-\omega^*\|^2 - \|u_{n+1}-\omega^*\|^2. \tag{3.22}$$

The remainder of the proof shall be splited into the following two parts.

Case 1: Assume that there exists a fixed number $n_2 \in \mathbb{N}$ such that

$$\|u_{n+1}-\omega^*\| \leqslant \|u_n-\omega^*\|, \ \forall n \geqslant n_2.$$

Thus, above implies that $\lim_{n\to+\infty}\|u_n-\omega^*\|$ exists and let $\lim_{n\to+\infty}\|u_n-\omega^*\|=l$. From expression (3.22) we have

$$(1-\vartheta_n)\Big(1-\frac{\chi^2\varkappa_n^2}{\varkappa_{n+1}^2}\Big)\|u_n-y_n\|^2\leqslant\vartheta_n\|u_1-\omega^*\|^2+\|u_n-\omega^*\|^2-\|u_{n+1}-\omega^*\|^2. \tag{3.23}$$

By existence of $\lim_{n\to+\infty}\|u_n-\omega^*\|=l$ and $\vartheta_n\to 0$, we can deduce that

$$\lim_{n\to+\infty}\|u_n-y_n\|=0.$$

Furthermore, we have

$$\|z_n-y_n\|=\|y_n+\varkappa[\Upsilon(u_n)-\Upsilon(y_n)]-y_n\|\leqslant\varkappa L\|u_n-y_n\|.$$

It follows that

$$\lim_{n\to+\infty}\|u_n-z_n\|\leqslant\lim_{n\to+\infty}\|u_n-y_n\|+\lim_{n\to+\infty}\|y_n-z_n\|=0. \tag{3.24}$$

Furthermore, we obtain

$$\begin{aligned}\big\|u_{n+1}-u_n\big\|&=\big\|\vartheta_n u_1+(1-\vartheta_n)z_n-u_n\big\|\\&=\big\|\vartheta_n[u_1-u_n]+(1-\vartheta_n)[z_n-u_n]\big\|\leqslant\vartheta_n\big\|u_1-u_n\big\|+(1-\vartheta_n)\big\|z_n-u_n\big\|.\end{aligned}$$

It follows that

$$\lim_{n\to+\infty}\|u_{n+1}-u_n\|=0.$$

Thus, the implies that the sequences $\{y_n\}$ and $\{z_n\}$ are bounded. Due to the reflexivity of Σ and the boundedness of $\{u_n\}$ guarantees that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\}\rightharpoonup\hat{u}\in\Sigma$ as $k\to+\infty$. Next, we need to prove that $\hat{u}\in VI(\Delta,\Upsilon)$. By Lemma 3.4, it follows that $\hat{u}\in VI(\Delta,\Upsilon)$. Next, consider that

$$\limsup_{n\to+\infty}\langle u_1-\omega^*,u_n-\omega^*\rangle=\limsup_{k\to+\infty}\langle u_1-\omega^*,u_{n_k}-\omega^*\rangle=\langle u_1-\omega^*,\hat{u}-\omega^*\rangle\leqslant 0.$$

By the use of $\lim_{n\to+\infty}\big\|u_{n+1}-u_n\big\|=0$, we may deduce that

$$\limsup_{n\to+\infty}\langle u_1-\omega^*,u_{n+1}-\omega^*\rangle\leqslant\limsup_{n\to+\infty}\langle u_1-\omega^*,u_{n+1}-u_n\rangle+\limsup_{n\to+\infty}\langle u_1-\omega^*,u_n-\omega^*\rangle\leqslant 0. \tag{3.25}$$

By using Lemma 2.1 (v), we have

$$\begin{aligned}\big\|u_{n+1}-\omega^*\big\|^2&=\big\|\vartheta_n u_1+(1-\vartheta_n)z_n-\omega^*\big\|^2\\&=\big\|\vartheta_n[u_1-\omega^*]+(1-\vartheta_n)[z_n-\omega^*]\big\|^2\\&\leqslant(1-\vartheta_n)^2\big\|z_n-\omega^*\big\|^2+2\vartheta_n\langle u_1-\omega^*,(1-\vartheta_n)[z_n-\omega^*]+\vartheta_n[u_1-\omega^*]\rangle\\&=(1-\vartheta_n)^2\big\|z_n-\omega^*\big\|^2+2\vartheta_n\langle u_1-\omega^*,u_{n+1}-\omega^*\rangle\\&\leqslant(1-\vartheta_n)\big\|u_n-\omega^*\big\|^2+2\vartheta_n\langle u_1-\omega^*,u_{n+1}-\omega^*\rangle.\end{aligned} \tag{3.26}$$

From expressions (3.25), (3.26), and using Lemma 2.2, we may deduce that $\big\|u_n-\omega^*\big\|\to 0$ as $n\to+\infty$.

Case 2: Assume that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i}-\omega^*\|\leqslant\|u_{n_{i+1}}-\omega^*\|,\ \forall i\in\mathbb{N}.$$

Thus, by Lemma 2.3, there exists a sequence $\{m_k\}\subset\mathbb{N}$ as $\{m_k\}\to+\infty$, such that

$$\|u_{m_k}-\omega^*\|\leqslant\|u_{m_{k+1}}-\omega^*\|\quad\text{and}\quad\|u_k-\omega^*\|\leqslant\|u_{m_{k+1}}-\omega^*\|,\ \text{for all } k\in\mathbb{N}. \tag{3.27}$$

As similar to the Case 1, expression (3.23) provides that

$$(1-\vartheta_{m_k})\left(1-\frac{\chi^2\varkappa_{m_k}^2}{\varkappa_{m_k+1}^2}\right)\|u_{m_k}-y_{m_k}\|^2 \leqslant \vartheta_{m_k}\|u_1-\omega^*\|^2+\|u_{m_k}-\omega^*\|^2-\|u_{m_k+1}-\omega^*\|^2.$$

Due to $\vartheta_{m_k}\to 0$, we deduce the following:

$$\lim_{k\to+\infty}\|u_{m_k}-y_{m_k}\|=0.$$

It follows that

$$\|z_{m_k}-y_{m_k}\|=\|y_{m_k}+\varkappa[\Upsilon(u_{m_k})-\Upsilon(y_{m_k})]-y_{m_k}\|\leqslant \varkappa L\|u_{m_k}-y_{m_k}\|.$$

Furthermore, we have

$$\lim_{k\to+\infty}\|u_{m_k}-z_{m_k}\|\leqslant \lim_{k\to+\infty}\|u_{m_k}-y_{m_k}\|+\lim_{k\to+\infty}\|y_{m_k}-z_{m_k}\|=0.$$

Also, we can obtain

$$\begin{aligned}\left\|u_{m_k+1}-u_{m_k}\right\| &= \left\|\vartheta_{m_k}u_1+(1-\vartheta_{m_k})z_{m_k}-u_{m_k}\right\|\\ &= \left\|\vartheta_{m_k}[u_1-u_{m_k}]+(1-\vartheta_{m_k})[z_{m_k}-u_{m_k}]\right\|\\ &\leqslant \vartheta_{m_k}\left\|u_1-u_{m_k}\right\|+(1-\vartheta_{m_k})\left\|z_{m_k}-u_{m_k}\right\| \longrightarrow 0.\end{aligned}$$

We use the same argument as in Case 1, which is as follows:

$$\limsup_{k\to+\infty}\langle u_1-\omega^*, u_{m_k+1}-\omega^*\rangle \leqslant 0. \tag{3.28}$$

Now, using expressions (3.26) and (3.27), we have

$$\begin{aligned}\left\|u_{m_k+1}-\omega^*\right\|^2 &\leqslant (1-\vartheta_{m_k})\left\|u_{m_k}-\omega^*\right\|^2+2\vartheta_{m_k}\langle u_1-\omega^*, u_{m_k+1}-\omega^*\rangle\\ &\leqslant (1-\vartheta_{m_k})\left\|u_{m_k+1}-\omega^*\right\|^2+2\vartheta_{m_k}\langle u_1-\omega^*, u_{m_k+1}-\omega^*\rangle.\end{aligned}$$

It continues from that

$$\left\|u_{m_k+1}-\omega^*\right\|^2 \leqslant 2\langle u_1-\omega^*, u_{m_k+1}-\omega^*\rangle. \tag{3.29}$$

Since $\vartheta_{m_k}\to 0$ and $\left\|u_{m_k}-\omega^*\right\|$ is bounded, thus, by expressions (3.28) and (3.29) we obtain

$$\|u_{m_k+1}-\omega^*\|^2\to 0, \text{ as } k\to+\infty.$$

It implies that

$$\lim_{n\to+\infty}\|u_k-\omega^*\|^2 \leqslant \lim_{n\to+\infty}\|u_{m_k+1}-\omega^*\|^2 \leqslant 0.$$

Consequently, $u_n\to\omega^*$. This completes the proof of the theorem. □

4. Numerical illustration

This section describes the numerical performance of the proposed algorithms, in contrast to some related work in the literature, as well as the analysis of how variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on HP i-5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Let $\Sigma = l_2$ be a real Hilbert space with with the sequences of real numbers satisfying the following condition

$$\|u_1\|^2 + \|u_2\|^2 + \cdots + \|u_n\|^2 + \cdots < +\infty.$$

Assume that a mapping $\Upsilon : \Delta \to \Delta$ is defined by

$$G(u) = (5 - \|u\|)u, \ \forall u \in \Sigma,$$

where $\Delta = \{u \in \Sigma : \|u\| \leqslant 3\}$. We can easily see that Υ is weakly sequentially continuous on Σ and the solution set is $VI(\Delta, \Upsilon) = \{0\}$. For any $u, y \in \Sigma$, we have

$$\begin{aligned} \big\|\Upsilon(u) - \Upsilon(y)\big\| &= \big\|(5 - \|u\|)u - (5 - \|y\|)y\big\| \\ &= \big\|5(u-y) - \|u\|(u-y) - (\|u\| - \|y\|)y\big\| \\ &\leqslant 5\|u-y\| + \|u\|\|u-y\| + \big|\|u\| - \|y\|\big|\|y\| \\ &\leqslant 5\|u-y\| + 3\|u-y\| + 3\|u-y\| \\ &\leqslant 11\|u-y\|. \end{aligned}$$

Hence Υ is L-Lipschitz continuous with $L = 11$. For any $u, y \in \Sigma$, let $\langle \Upsilon(u), y - u \rangle > 0$, such that

$$(5 - \|u\|)\langle u, y - u \rangle > 0.$$

Since $\|u\| \leqslant 3$, it implies that

$$\langle u, y - u \rangle > 0.$$

Consider that

$$\begin{aligned} \langle \Upsilon(y), y - u \rangle &= (5 - \|y\|)\langle y, y - u \rangle \\ &\geqslant (5 - \|y\|)\langle y, y - u \rangle - (5 - \|y\|)\langle u, y - u \rangle \geqslant 2\|u - y\|^2 \geqslant 0. \end{aligned}$$

Hence a mapping Υ is quasimonotone on Δ. Let $u = (\frac{5}{2}, 0, 0, \ldots, 0, \ldots)$ and $y = (3, 0, 0, \ldots, 0, \ldots)$ such that

$$\langle \Upsilon(u) - \Upsilon(y), u - y \rangle = \left(\frac{5}{2} - 3\right)^3 < 0.$$

Consider the following projection formula:

$$P_\Delta(u) = \begin{cases} u, & \text{if } \|u\| \leqslant 3, \\ \frac{3u}{\|u\|}, & \text{otherwise.} \end{cases}$$

Figures 1-6 and Table 1 show numerical results. The control conditions are taken in the following way: (i) Algorithm 1 (shortly, **Alg01**): $\varkappa = \frac{0.7}{L}, \vartheta_n = \frac{1}{(n+2)}, D_n = \|u_n - u_y\|$; (ii) Algorithm 2 (shortly, **Alg02**): $\varkappa_1 = 0.22, \chi = 0.44, \vartheta_n = \frac{1}{(n+2)}, D_n = \|u_n - y_n\|$; (iii) Algorithm 3 (shortly, **Alg03**): $\varkappa_1 = 0.22, \chi = 0.44, \varphi_n = \frac{100}{(n+1)^2}, \vartheta_n = \frac{1}{(n+2)}, D_n = \|u_n - y_n\|$.

Table 1: Numerical values for Example 4.1.

	Number of iterations			Execution time in seconds		
u_1	**Alg01**	**Alg02**	**Alg03**	**Alg01**	**Alg02**	**Alg03**
$(1, 1, \cdots, 1_{10000}, 0, 0, \cdots)$	111	105	91	4.3553636	6.7229686	7.3973612
$(1, 2, \cdots, 10000, 0, 0, \cdots)$	124	100	87	6.2130909	9.4322004	6.7588432
$(8, 8, \cdots, 8_{10000}, 0, 0, \cdots)$	133	116	101	5.4634891	7.8256418	8.8856192

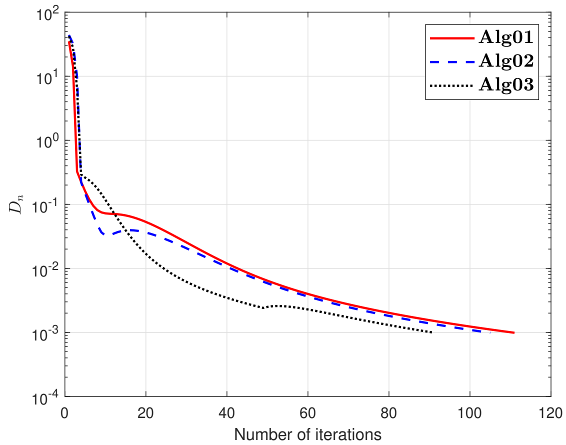

Figure 1: Numerical illustration of Algorithms 1, 2, and 3 while $u_1 = (1, 1, \ldots, 1_{10000}, 0, 0, \ldots)$.

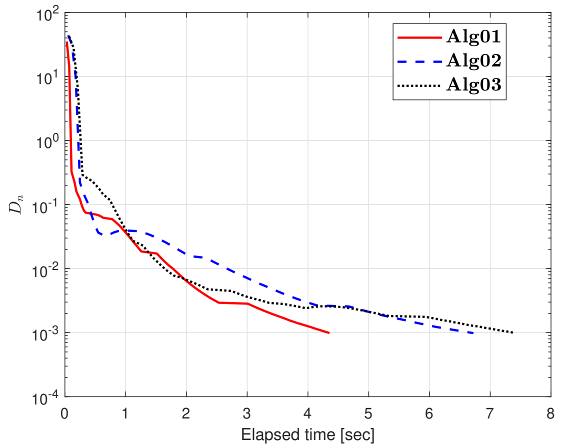

Figure 2: Numerical illustration of Algorithms 1, 2, and 3 while $u_1 = (1, 1, \ldots, 1_{10000}, 0, 0, \ldots)$.

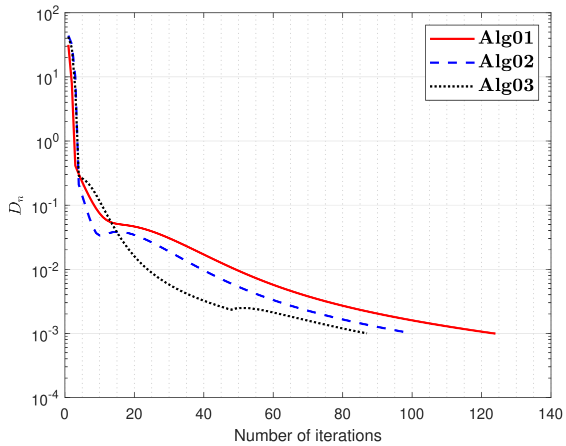

Figure 3: Numerical illustration of Algorithms 1, 2, and 3 while $u_1 = (1, 2, \ldots, 10000, 0, 0, \ldots)$.

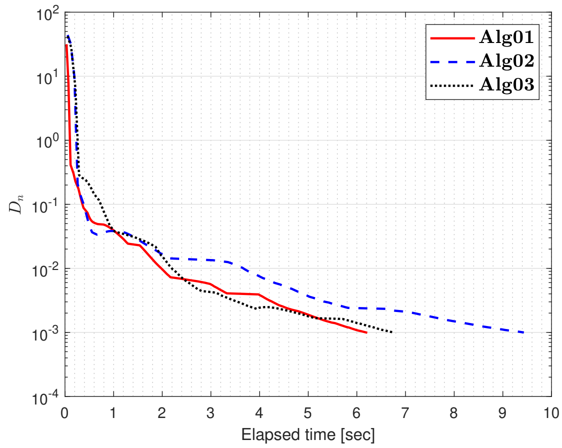

Figure 4: Numerical illustration of Algorithms 1, 2, and 3 while $u_1 = (1, 2, \ldots, 10000, 0, 0, \ldots)$.

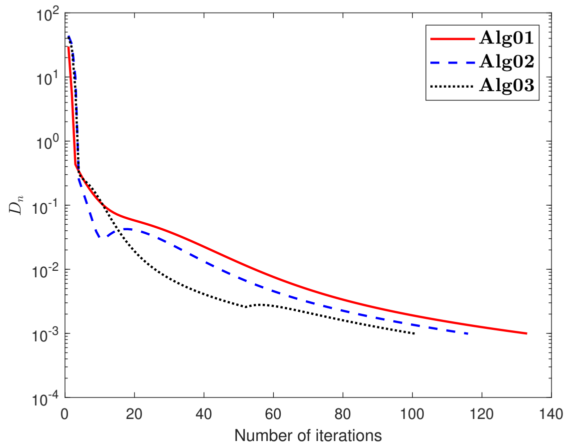

Figure 5: Numerical illustration of Algorithms 1, 2, and 3 while $u_1 = (8, 8, \ldots, 8_{10000}, 0, 0, \ldots)$.

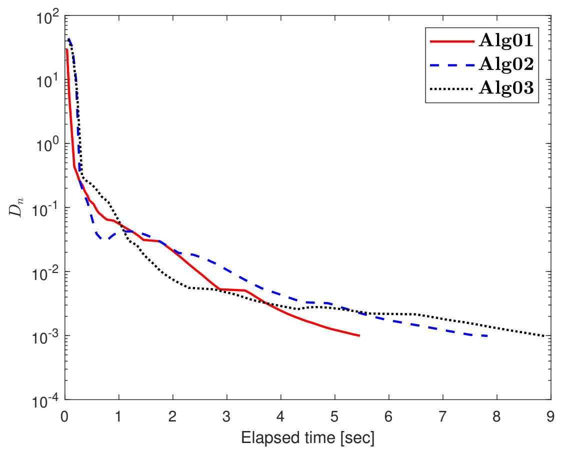

Figure 6: Numerical illustration of Algorithms 1, 2, and 3 while $u_1 = (8, 8, \ldots, 8_{10000}, 0, 0, \ldots)$.

5. Conclusion

To find a numerical solution to the quasimonotone variational inequality problems in real Hilbert space, we have developed different modified extragradient-type methods. While following a different step size rule, all sequences generated by the proposed method are strongly convergent to the solution. Numerical findings are summarized to demonstrate the numerical effectiveness of our algorithm in comparison to other proposed methods. These numerical studies have indicated that the variable step size rule outperforms the fixed step size rule in most situations.

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