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Maximal elements for Kakutani maps

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Abstract

We present some new general existence theorems for maximal type elements for upper semicontinuous maps with convex compact values.

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1. Introduction

In this paper we present a variety of new collectively fixed point and coincidence point results and from these existence theorems we establish some new maximal type element results for majorized type maps (see [12]). The maps considered in this paper are Kakutani maps (or more generally, admissible maps with respect to Gorniewicz) and multivalued maps with continuous selections (see [3, 4, 10, 11] and the references therein).

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geqslant 1$, and $H_0(X) \approx K$.

Let X, Y, and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a perfect map, i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

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(i) p is a Vietoris map; and
(ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [7]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to K(Y)$ is said to Kakutani (and we write $\phi \in Kak(X, Y)$); here $K(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

The following classes of maps will play a major role in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in DKT(Z, W)$ [4] if W is convex and there exists a map $S : Z \to W$ with $co(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$. We say $G \in HLPY(Z, W)$ [11] if W is convex and there exists a map $S : Z \to W$ with $co(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\text{int}\, S^{-1}(w) : w \in W\}$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \text{ Fix}\, F \neq \emptyset \text{ for all } F \in \mathcal{X}(Z, Z)\},$$

where $\text{Fix}\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;
(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii) $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, \ldots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leqslant 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α-close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha : K \to V$ such that

(i) π_α and $i : K \to V$ are α-close;
(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

X is said to be q-Schauder admissible if any nonempty compact convex subset Ω of X is Schauder admissible.

Theorem 1.1 ([1, 8])**.** *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists an $x \in X$ with $x \in \Psi(x)$.*

Remark 1.2. Other variations of Theorem 1.1 can be found in [9].

We now list two well known results from the literature [12] (see also [2]).

Theorem 1.3. *Let X and Y be two topological spaces and A an open subset of X. Suppose $F_1 : X \to 2^Y$, $F_2 : A \to 2^Y$ are upper semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the map $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & x \notin A, \\ F_2(x), & x \in A, \end{cases}$$

is upper semicontinuous.

Theorem 1.4. *Let X and Y be topological spaces. If $F, G : X \to 2^Y$ have compact values and are upper semicontinuous then $F \cap G$ is also upper semicontinuous.*

We recall that a point $x \in X$ is a maximal element of a set valued map F from a topological space X to another topological space Y if $F(x) = \emptyset$.

2. Maximal element type results

In this section we begin by presenting collectively fixed point results in a variety of settings. Our goal later is to consider collectively coincidence results.

Theorem 2.1. *Let $\{X_i\}_{i=1}^N$ be a family of compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \dots, N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ is upper semicontinuous with nonempty convex compact values (i.e., $F_i \in Kak(X, X_i)$). Also assume X is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Then there exists an $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, \dots, N\}$ (here x_i is the projection of x on X_i).*

Proof. Let $F(x) = \prod_{i=1}^N F_i(x)$ for $x \in X$ and note $F \in Kak(X, X)$ (see [2]). Now Theorem 1.1 guarantees the result. □

Remark 2.2.

(i). Note we could replace $\{X_i\}_{i=1}^N$ with $\{X_i\}_{i \in I}$ (where I is an index set) in Theorem 1.1.

(ii). In Theorem 1.1 we could replace $F_i \in Kak(X, X_i)$ with $F_i \in Ad(X, X_i)$ (recall a finite product of admissible maps is admissible). This remark could also be applied to other results in this paper.

Theorem 2.3. *Let $\{X_i\}_{i=1}^N$ be a family of sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \dots, N\}$ let K_i be a nonempty compact subset of X_i and suppose $F_i : X \equiv \prod_{i=1}^N X_i \to K_i$ is upper semicontinuous with nonempty convex compact values (i.e., $F_i \in Kak(X, K_i)$). Also assume $K \equiv \prod_{i=1}^N K_i$ is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Then there exists an $x \in K$ with $x_i \in F_i(x)$ for $i \in \{1, \dots, N\}$.*

Proof. Let $F(x) = \prod_{i=1}^N F_i(x)$ for $x \in K$ and note $F \in Kak(K, K)$. Now apply Theorem 1.1. □

Theorem 2.4. *Let $\{X_i\}_{i=1}^N$ be a family of sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \dots, N\}$ let K_i be a nonempty subset of X_i and suppose $F_i : X \equiv \prod_{i=1}^N X_i \to K_i$. Let $F : X \to K$ be given by $F(x) = \prod_{i=1}^N F_i(x)$ with $K \equiv \prod_{i=1}^N K_i$ and assume $co(K) \subseteq X$ is compact and $F : X \to co(K)$ is upper semicontinuous with nonempty convex compact values (i.e., $F \in Kak(X, co(K))$). Also suppose $co(K)$ is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Then there exists an $x \in co(K)$ with $x_i \in F_i(x)$ for $i \in \{1, \dots, N\}$.*

Proof. Note $F \in Kak(co(K), co(K))$ and apply Theorem 1.1. □

The conclusion in the above results is the existence of a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1, \dots, N\}$. One can adjust so that the conclusion is the existence of a $x \in X$ and a $i \in \{1, \dots, N\}$ with $x_i \in F_i(x)$. To see this we will consider Theorem 2.1 in this setting.

Theorem 2.5. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,\dots,N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ is upper semicontinuous with convex compact values. Assume X is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Also suppose for each $x \in X$ there exists an $i \in \{1,\dots,N\}$ with $F_i(x) \neq \emptyset$. Finally assume for each $i \in \{1,\dots,N\}$ that $U_i = \{x \in X : F_i(x) \neq \emptyset\}$ is open in X. Then there exists an $x \in X$ and a $i_0 \in \{1,\dots,N\}$ with $x_{i_0} \in F_{i_0}(x)$.*

Proof. Fix $i \in \{1,\dots,N\}$. Define a map $G_i : X \to X_i$ by

$$G_i(x)) = \begin{cases} F_i(x), & x \in U_i, \\ X_i, & x \in X \backslash U_i. \end{cases}$$

Note G_i has nonempty convex compact values and $G_i : X \to X_i$ is upper semicontinuous from Theorem 1.3 (i.e., $G_i \in Kak(X, X_i)$). Let $G : X \to X$ be given by

$$G(x) = \prod_{j=1}^N G_j(x) \quad \text{for } x \in X.$$

Note $G \in Kak(X, X)$. Now Theorem 1.1 guarantees a $y \in X$ with $y \in G(y)) = \prod_{j=1}^N G_j(y)$, i.e., $y_i \in G_i(y)$ for $i \in \{1,\dots,N\}$. Now by assumption there exists an $i_0 \in \{1,\dots,N\}$ with $F_{i_0}(y) \neq \emptyset$. Thus $y_{i_0} \in G_{i_0}(y) = F_{i_0}(y)$. □

Collectively fixed point theory can be rewritten as a maximal element type result. To illustrate this we will consider Theorems 2.1 and 2.5.

Theorem 2.6. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,\dots,N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ is upper semicontinuous with convex compact values. Also assume X is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Now suppose for each $x \in X$ there exists a $j \in \{1,\dots,N\}$ with $x_j \notin F_j(x)$. Then there exists an $x \in X$ and a $i_0 \in \{1,\dots,N\}$ with $F_{i_0}(x) = \emptyset$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ we have $F_i(x) \neq \emptyset$ for all $i \in \{1,\dots,N\}$. Thus $F_i \in Kak(X, X_i)$ for all $i \in \{1,\dots,N\}$. Now Theorem 2.1 guarantees a $x \in X$ with $x_i \in F_i(x)$ for $i \in \{1,\dots,N\}$, a contradiction. □

Theorem 2.7. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,\dots,N\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to X_i$ is upper semicontinuous with convex compact values. Assume X is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Also suppose for all $i \in \{1,\dots,N\}$ that $x_i \notin F_i(x)$ for each $x \in X$. Finally suppose for each $i \in \{1,\dots,N\}$ that $U_i = \{x \in X : F_i(x) \neq \emptyset\}$ is open in X. Then there exists an $x \in X$ with $F_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists an $i \in \{1,\dots,N\}$ with $F_i(x) \neq \emptyset$. Now Theorem 2.5 guarantees an $x \in X$ and a $i \in \{1,\dots,N\}$ with $x_i \in F_i(x)$, a contradiction. □

We next discuss majorized type maps motivated from the literature (see [10, 12]). Let Z and W be sets in a Hausdorff topological vector space with Z paracompact and W convex and compact.

Remark 2.8. In the setting we presented above recall (i). compact sets are paracompact; and (ii). if Ω is a compact subset of a topological vector space then $co(\Omega)$ is paracompact (see [4]).

Suppose $H : Z \to W$ and for each $x \in Z$ assume there exists a map $A_x : Z \to W$ and an open set U_x containing x with $H(z) \subseteq A_x(z)$ for every $z \in U_x$, $A_x : U_x \to W$ is upper semicontinuous with convex compact values. We claim there exists a (compact) map $\Psi : Z \to W$ with $H(z) \subseteq \Psi(z)$ for $z \in Z$ and $\Psi : Z \to W$ is upper semicontinuous with convex compact values. To see this note $\{U_x\}_{x \in Z}$ is an open covering of Z and since Z is paracompact there exists [5, 6] a locally finite open covering $\{V_x\}_{x\in Z}$ of Z with $x \in V_x$ and $V_x \subseteq U_x$ for $x \in Z$, and for each $x \in Z$ let

$$Q_x(z) = \begin{cases} A_x(z), & z \in V_x, \\ W, & z \in Z\backslash V_x. \end{cases}$$

Now Theorem 1.3 guarantees that $Q_x : Z \to W$ is upper semicontinuous with convex compact values. Next note $H(z) \subseteq Q_x(z)$ for every $z \in Z$ since if $z \in V_x$, then since $V_x \subseteq U_x$ and $H(w) \subseteq A_x(w)$ for $w \in U_x$ we have $H(z) \subseteq Q_x(z)$ whereas if $z \in Z\backslash V_x$, then it is immediate since $Q_x(z) = W$. Now define $\Psi : Z \to W$ by

$$\Psi(z) = \bigcap_{x\in Z} Q_x(z) \quad \text{for} \quad z \in Z.$$

Note $\Psi : Z \to W$ has convex compact values with $H(w) \subseteq \Psi(w)$ for $w \in Z$ since $H(z) \subseteq Q_x(z)$ for every $z \in Z$ (for each $x \in X$). It remains to show $\Psi : Z \to W$ is upper semicontinuous. Let $u \in Z$. There exists an open neighbourhood N_u of u such that $\{x \in Z : N_u \cap V_x \neq \emptyset\} = \{x_1, \dots, x_{n_u}\}$ (a finite set). Note if $x \notin \{x_1, \dots, x_{n_u}\}$, then $\emptyset = V_x \cap N_u$ so $Q_x(z) = W$ for $z \in N_u$ and so we have

$$\Psi(z) = \bigcap_{x\in Z} Q_x(z) = \bigcap_{j=1}^{n_u} Q_{x_j}(z) \quad \text{for} \quad z \in N_u.$$

Now for $j \in \{1, \dots, n_u\}$ note $Q_{x_j} : Z \to W$ is upper semicontinuous (so $Q^{\star}_{x_j} : N_u \to W$, the restriction of Q_{x_j} to N_u, is upper semicontinuous) so Theorem 1.4 guarantees that $\Psi : N_u \to W$ is upper semicontinuous (at u). Since N_u is open we have that $\Psi : Z \to W$ is upper semicontinuous (at u).

We will combine the above discussion with Theorems 2.6 and 2.7 to illustrate the method involved.

Theorem 2.9. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1, \dots, N\}$ suppose $H_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and for each $x \in X$ assume there exists a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x} : U_{i,x} \to X_i$ is upper semicontinuous with convex compact values, and also assume for each $w \in X$ there exists a $j_0 \in \{1, \dots, N\}$ (which does not depend on x) with $w_{j_0} \notin A_{j_0,x}(w)$. Suppose X is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Then there exists an $x \in X$ and a $i_0 \in \{1, \dots, N\}$ with $H_{i_0}(x) = \emptyset$.*

Proof. Let $i \in \{1, \dots, N\}$. From the discussion after Theorem 2.7 (with $Z = X$, $W = X_i$, $H = H_i$ and $A_x = A_{i,x}$) there exists a map $\Psi_i : X \to X_i$ with $H_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to X_i$ is upper semicontinuous with convex compact values: here $\{U_{i,x}\}_{x\in X}$ is an open covering of X so there exists a locally finite open covering $\{V_{i,x}\}_{x\in X}$ of X with $x \in V_{i,x}$ and $V_{i,x} \subseteq U_{i,x}$ for $x \in X$, and for each $x \in X$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x}, \\ X_i, & z \in X\backslash V_{i,x}, \end{cases}$$

and $\Psi_i : X \to X_i$ is

$$\Psi_i(z) = \bigcap_{x\in X} Q_{i,x}(z) \quad \text{for} \quad z \in X.$$

We now claim for each $w \in X$ there exists a $k \in \{1, \dots, N\}$ with $w_k \notin \Psi_k(w)$. To see this fix $w \in X$. From our assumption for each $x \in X$ there exists a $j_0 \in \{1, \dots, N\}$ (which does not depend on x) with $w_{j_0} \notin A_{j_0,x}(w)$. Now since $\{V_{j_0,x}\}_{x\in X}$ is a covering of X there exists an $x^{j_0} \in X$ with $w \in V_{j_0,x^{j_0}}$ so

$$\Psi_{j_0}(w) = \bigcap_{x\in X} Q_{j_0,x}(w) \subseteq Q_{j_0,x^{j_0}}(w) = A_{j_0,x^{j_0}}(w),$$

and as a result $w_{j_0} \notin \Psi_{j_0}(w)$. Thus our claim is true. Now apply Theorem 2.6 (with $F_i = \Psi_i$) so there exists an $x \in X$ and a $i_0 \in \{1, \dots, N\}$ with $\Psi_{i_0}(x) = \emptyset$. Now since $H_i(z) \subseteq \Psi_i(z)$ for $z \in X$, then $H_{i_0}(x) = \emptyset$. □

Theorem 2.10. *Let $\{X_i\}_{i=1}^N$ be a family of convex compact sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,\dots,N\}$ suppose $H_i : X \equiv \prod_{i=1}^N X_i \to X_i$ and for each $x \in X$ assume there exists a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x} : U_{i,x} \to X_i$ is upper semicontinuous with convex compact values, and also assume $w_i \notin A_{i,x}(w)$ for each $w \in X$. Suppose X is a Schauder admissible subset of the Hausdorff topological vector space $E \equiv \prod_{i=1}^N E_i$. Finally assume for each $i \in \{1,\dots,N\}$ that $U_i = \{x \in X : H_i(x) \neq \emptyset\}$ is open in X. Then there exists an $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1,\dots,N\}$.*

Proof. Let $i \in \{1,\dots,N\}$ and let $V_{i,x}$, $Q_{i,x}$ and Ψ_i be as in Theorem 2.9. We now claim that $w_i \notin \Psi_i(w)$ for each $w \in X$. To see this fix $w \in X$. Then there exists an $x^\star \in X$ with $w \in V_{i,x^\star}$ (recall $\{V_{i,x}\}_{x\in X}$ is a covering of X) so

$$\Psi_i(w) = \bigcap_{x\in X} Q_{i,x}(w) \subseteq Q_{i,x^\star}(w) = A_{i,x^\star}(w),$$

and thus since $w_i \notin A_{i,x^\star}(w)$ we have $w_i \notin \Psi_i(w)$, and our claim is true.

Define a map $G_i : X \to X_i$ by

$$G_i(x) = \begin{cases} \Psi_i(x), & x \in U_i = \{x \in X : H_i(x) \neq \emptyset\}, \\ X_i, & x \in X \backslash U_i. \end{cases}$$

Note G_i has nonempty convex compact values (to see the nonemptyness let $x \in X$ and note if $x \in X\backslash U_i$ then it is immediate whereas if $x \in U_i$, then $H_i(x) \neq \emptyset$ implies $\Psi_i(x) \neq \emptyset$ since $H_i(z) \subseteq \Psi_i(z)$ for $z \in X$) and $G_i : X \to X_i$ is upper semicontinuous from Theorem 1.3 (i.e., $G_i \in Kak(X, X_i)$). Let $G : X \to X$ be given by

$$G(x) = \prod_{j=1}^N G_j(x) \quad \text{for } x \in X.$$

Note $G \in Kak(X, X)$. Now Theorem 1.1 guarantees a $y \in X$ with $y \in G(y)) = \prod_{j=1}^N G_j(y)$, i.e., $y_i \in G_i(y)$ for $i \in \{1,\dots,N\}$. If there exists an $i_0 \in \{1,\dots,N\}$ with $H_{i_0}(y) \neq \emptyset$ then $y \in U_{i_0}$ so $y_{i_0} \in G_{i_0}(y) = \Psi_{i_0}(y)$, a contradiction. Thus $H_i(y) = \emptyset$ for all $i \in \{1,\dots,N\}$. □

Now we present collectively coincidence results in a variety of settings.

Theorem 2.11. *Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space E_i and $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \to Y_i$ and $F_i \in Kak(X, Y_i)$. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y, X_j)$. Then there exists an $x \in X$ and a $y \in Y$ with $y_j \in F_j(x)$ for $j \in \{1,\dots,N_0\}$ and $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$.*

Proof. Now Y is compact, $G_i \in DKT(Y, X_j)$ so for each $i \in \{1,\dots,N\}$ from [3, 4] there exists a continuous (single valued) selection $q_i : Y \to X_i$ of G_i with $q_i(y) \in G_i(y)$ for $y \in Y$ and there exists a finite set R_i of X_i with $q_i(Y) \subseteq co(R_i) \equiv Q_i$. Let $Q = \prod_{i=1}^N Q_i$ ($\subseteq X$) and note Q is compact. Let $F_i^\star$ denote the restriction of F_i to Q and let $F^\star(x) = \prod_{i=1}^{N_0} F_i^\star(x)$ for $x \in Q$. Note $F^\star \in Kak(Q, Y)$ (so in particular $F^\star \in Ad(Q, Y)$). Let $q(y) = \prod_{i=1}^N q_i(y)$ for $y \in Y$ and note $q : Y \to Q$ is continuous (note $q_i : Y \to Q_i$). Then $q\,F^\star \in Ad(Q, Q)$ (recall Ad maps are closed under compositions) and note Q is a compact convex in a finite dimensional subspace of $E = \prod_{i=1}^N E_i$, so Theorem 1.1 guarantees a $x \in Q$ with $x \in q(F^\star(x))$. Now let $y \in F^\star(x)$ with $x = q(y)$. Note $y \in F(x)$ so $y_j \in F_j(x)$ for all $j \in \{1,\dots,N_0\}$. Also $x_i = q_i(x) \in G_i(y)$ for $i \in \{1,\dots,N\}$. □

Remark 2.12.

(i). In Theorem 2.11 we could replace $G_j \in DKT(Y, X_j)$ with $G_j \in HLPY(Y, X_j)$.

(ii). In Theorem 2.11 we could replace $\{X_i\}_{i=1}^N, \{Y_i\}_{i=1}^{N_0}$ with $\{X_i\}_{i\in I}, \{Y_i\}_{i\in J}$ (where I and J are index sets); here to apply Theorem 1.1 we need to assume X is q-Schauder admissible.

(iii). The assumption $\{Y_i\}_{i=1}^{N_0}$ are convex sets is not needed in the statement of Theorem 2.11.

(iv). In the proof of Theorem 2.11 note $F^\star q \in Ad(Y,Y)$ so one could apply Theorem 1.1 if Y is a Schauder admissible subset of $\prod_{i=1}^{N_0} E_i$. This remark could also be applied to other results in this section.

Theorem 2.13. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and $F_i \in Kak(X,Y_i)$ and in addition assume there is a compact set K_i with $F_i(X) \subseteq K_i \subseteq Y_i$. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y,X_j)$. Then there exists an $x \in X$ and a $y \in Y$ with $y_j \in F_j(x)$ for $j \in \{1,\dots,N_0\}$ and $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$.*

Proof. Let q_i, R_i, Q_i, Q, $F_i^\star$ and $F^\star$ be as in Theorem 2.11. Note $F^\star \in Kak(Q,Y)$ (so in particular $F^\star \in Ad(Q,Y)$). Now $F_i^\star(Q) \subseteq F_i(X) \subseteq K_i$ for each $i \in \{1,\dots,N_0\}$ so $F^\star(Q) \subseteq K \equiv \prod_{i=1}^{N_0} K_i$. Let $q(y) = \prod_{i=1}^{N} q_i(y)$ for $y \in Y$ and note $q\,F^\star \in Ad(Q,Q)$ is a compact map. Also Theorem 1.1 guarantees a $x \in Q$ with $x \in q\,(F^\star(x))$ and now the result follows from the argument in Theorem 2.11. □

Theorem 2.14. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ is upper semicontinuous with convex compact values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y,X_j)$. Also assume for each $x \in X$ there exists an $i \in \{1,\dots,N_0\}$ with $F_i(x) \neq \emptyset$. Finally suppose for each $i \in \{1,\dots,N_0\}$ that $U_i = \{x \in X : F_i(x) \neq \emptyset\}$ is open in X. Then there exists an $x \in X$, a $y \in Y$ and a $j_0 \in \{1,\dots,N_0\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$.*

Proof. Let q_i, R_i, Q_i and Q be as in Theorem 2.11. Let $i \in \{1,\dots,N_0\}$ and define a mapping $\Phi_i : X \to Y_i$ by

$$\Phi_i(x)) = \begin{cases} F_i(x), & x \in U_i, \\ Y_i, & x \in X \backslash U_i. \end{cases}$$

Note Φ_i has nonempty convex compact values and $\Phi_i : X \to Y_i$ is upper semicontinuous from Theorem 1.3 (i.e., $\Phi_i \in Kak(X,Y_i)$). Let $\Phi : X \to Y$ be given by

$$\Phi(x) = \prod_{j=1}^{N_0} \Phi_j(x) \quad \text{for } x \in X.$$

Note $\Phi \in Kak(X,Y)$. Let $\Phi^\star$ denote the restriction of Φ to Q and note $\Phi^\star \in Kak(Q,Y)$. Let $q(y) = \prod_{i=1}^{N} q_i(y)$ for $y \in Y$ and note $q\,\Phi^\star \in Ad(Q,Q)$. Theorem 1.1 guarantees a $x \in Q$ with $x \in q\,(\Phi^\star(x))$. Now let $y \in \Phi^\star(x)$ with $x = q(y)$. Note $y \in \Phi(x)$ and $x_i = q_i(y) \in G_i(y)$ for $i \in \{1,\dots,N\}$. Now from our assumption there exists an $i_0 \in \{1,\dots,N_0\}$ with $F_{i_0}(x) \neq \emptyset$, i.e., $x \in U_{i_0}$ and so $i_0 \in \Phi_{i_0}(x) = F_{i_0}(x)$. □

Now we will rewrite Theorems 2.13 and 2.14 as maximal element type results.

Theorem 2.15. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ is upper semicontinuous with convex compact values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y,X_j)$. Now suppose either for each $(x,y) \in X \times Y$ there exists a $j \in \{1,\dots,N_0\}$ with $y_j \notin F_j(x)$ or for each $(x,y) \in X \times Y$ there exists an $i \in \{1,\dots,N\}$ with $x_i \notin G_i(y)$. Then there exists an $x \in X$ and a $i_0 \in \{1,\dots,N_0\}$ with $F_{i_0}(x) = \emptyset$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ we have $F_i(x) \neq \emptyset$ for all $i \in \{1,\dots,N_0\}$. Thus $F_i \in Kak(X,Y_i)$ for all $i \in \{1,\dots,N_0\}$. Now Theorem 2.11 guarantees a $x \in X$, a $y \in Y$ with $y_j \in F_j(x)$ for $j \in \{1,\dots,N_0\}$ and $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$, a contradiction. □

Theorem 2.16. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space and $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ is upper semicontinuous with convex compact values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y, X_j)$. Also assume for each $i \in \{1,\dots,N_0\}$ that $U_i = \{x \in X : F_i(x) \neq \emptyset\}$ is open in X. Now suppose either for each $(x,y) \in X \times Y$ we have $y_j \notin F_j(x)$ for all $j \in \{1,\dots,N_0\}$ or for each $(x,y) \in X \times Y$ there exists an $i \in \{1,\dots,N\}$ with $x_i \notin G_i(y)$. Then there exists an $x \in X$ with $F_i(x) = \emptyset$ for all $i \in \{1,\dots,N_0\}$.*

Proof. Suppose the conclusion is false. Then for each $x \in X$ there exists an $i \in \{1,\dots,N_0\}$ with $F_i(x) \neq \emptyset$. Now Theorem 2.14 guarantees a $x \in X$, a $y \in Y$ and a $j_0 \in \{1,\dots,N_0\}$ with $y_{j_0} \in F_{j_0}(x)$ and $x_i \in G_i(y)$ for $i \in \{1,\dots,N\}$, a contradiction. □

We will now use the discussion after Theorem 2.7 to obtain our final results.

Theorem 2.17. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets, and $X \equiv \prod_{i=1}^{N} X_i$ is paracompact. For each $i \in \{1,\dots,N_0\}$ suppose $H_i : X \to Y_i$ and for each $x \in X$ assume there exists a map $A_{i,x} : X \to Y_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x} : U_{i,x} \to Y_i$ is upper semicontinuous with convex compact values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y, X_j)$. Also assume either for each $x \in X$ and for each $(w,y) \in X \times Y$ there exists a $j_0 \in \{1,\dots,N_0\}$ (which does not depend on X) with $y_{j_0} \notin A_{j_0,x}(w)$ or for each $(x,y) \in X \times Y$ there exists an $i \in \{1,\dots,N\}$ with $x_i \notin G_i(y)$. Then there exists an $x \in X$ and a $i_0 \in \{1,\dots,N_0\}$ with $H_{i_0}(x) = \emptyset$.*

Proof. Let $i \in \{1,\dots,N_0\}$. From the discussion after Theorem 2.7 (with $Z = X$, $W = Y_i$, $H = H_i$ and $A_x = A_{i,x}$) there exists a map $\Psi_i : X \to Y_i$ with $H_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to Y_i$ is upper semicontinuous with convex compact values: here $\{U_{i,x}\}_{x\in X}$ is an open covering of X so there exists a locally finite open covering $\{V_{i,x}\}_{x\in X}$ of X (recall X is paracompact) with $x \in V_{i,x}$ and $V_{i,x} \subseteq U_{i,x}$ for $x \in X$, and for each $x \in X$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x}, \\ Y_i, & z \in X \backslash V_{i,x}, \end{cases}$$

and $\Psi_i : X \to Y_i$ is

$$\Psi_i(z) = \bigcap_{x\in X} Q_{i,x}(z) \quad \text{for} \quad z \in X.$$

We now claim for each $x \in X$ and for each $(w,y) \in X \times Y$ there exists a $j_0 \in \{1,\dots,N_0\}$ with $y_{j_0} \notin \Psi_{j_0}(w)$ if in the statement of Theorem 2.17 we have for each $x \in X$ and for each $(w,y) \in X \times Y$ there exists a $j_0 \in \{1,\dots,N_0\}$ (which does not depend on X) with $y_{j_0} \notin A_{j_0,x}(w)$. To see this fix $(w,y) \in X \times Y$. Now for each $x \in X$ there exists a $j_0 \in \{1,\dots,N\}$ (which does not depend on x) with $y_{j_0} \notin A_{j_0,x}(w)$. Now since $\{V_{j_0,x}\}_{x\in X}$ is a covering of X there exists an $x^{j_0} \in X$ with $w \in V_{j_0,x^{j_0}}$ so

$$\Psi_{j_0}(w) = \bigcap_{x\in X} Q_{j_0,x}(w) \subseteq Q_{j_0,x^{j_0}}(w) = A_{j_0,x^{j_0}}(w),$$

and as a result $y_{j_0} \notin \Psi_{j_0}(w)$. Thus our claim is true. Now apply Theorem 2.15 (with $F_i = \Psi_i$) so there exists an $x \in X$ and a $i_0 \in \{1,\dots,N_0\}$ with $\Psi_{i_0}(x) = \emptyset$. Now since $H_i(z) \subseteq \Psi_i(z)$ for $z \in X$, then $H_{i_0}(x) = \emptyset$. □

Theorem 2.18. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of convex sets each in a Hausdorff topological vector space, $\{Y_i\}_{i=1}^{N_0}$ is also a family of compact sets, and $X \equiv \prod_{i=1}^{N} X_i$ is paracompact. For each $i \in \{1,\dots,N_0\}$ suppose $H_i : X \to Y_i$ and for each $x \in X$ assume there exists a map $A_{i,x} : X \to Y_i$ and an open set $U_{i,x}$ containing x with $H_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$, $A_{i,x} : U_{i,x} \to Y_i$ is upper semicontinuous with convex compact values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y, X_j)$. Also assume for each $i \in \{1,\dots,N_0\}$ that $U_i = \{x \in X : F_i(x) \neq \emptyset\}$ is open in X. Finally suppose either for each $x \in X$ and for each $(w,y) \in X \times Y$ we have $y_j \notin A_{j,x}(w)$ for all $j \in \{1,\dots,N_0\}$ or for each $(x,y) \in X \times Y$ there exists an $i \in \{1,\dots,N\}$ with $x_i \notin G_i(y)$. Then there exists an $x \in X$ with $H_i(x) = \emptyset$ for all $i \in \{1,\dots,N_0\}$.*

Proof. Let $i \in \{1, \dots, N_0\}$ and let $V_{i,x}$, $Q_{i,x}$ and Ψ_i be as in Theorem 2.17. We now claim for each $(w, y) \in X \times Y$ we have $y_j \notin \Psi_j(w)$ for $j \in \{1, \dots, N_0\}$ if in the statement of Theorem 2.18 we have for each $x \in X$ and for each $(w, y) \in X \times Y$ we have $y_j \notin A_{j,x}(w)$ for all $j \in \{1, \dots, N_0\}$. To see this fix $(w, y) \in X \times Y$ and fix $i \in \{1, \dots, N_0\}$. Now for each $x \in X$ we have $y_j \notin A_{j,x}(w)$. Since $\{V_{i,x}\}_{x \in X}$ is a covering of X there exists an $x^\star \in X$ with $w \in V_{i,x^\star}$ so

$$\Psi_i(w) = \bigcap_{x \in X} Q_{i,x}(w) \subseteq Q_{i,x^\star}(w) = A_{i,x^\star}(w),$$

and thus since $y_i \notin A_{i,x^\star}(w)$ we have $y_i \notin \Psi_i(w)$, and our claim is true.

Let $i \in \{1, \dots, N_0\}$ and define a map $\Phi_i : X \to Y_i$ by

$$\Phi_i(x) = \begin{cases} \Psi_i(x), & x \in U_i = \{x \in X : H_i(x) \neq \emptyset\}, \\ Y_i, & x \in X \backslash U_i. \end{cases}$$

Note Φ_i has nonempty convex compact values and $\Phi_i : X \to Y_i$ is upper semicontinuous from Theorem 1.3 (i.e., $\Phi_i \in Kak(X, Y_i)$). Let $\Phi : X \to Y$ be given by

$$\Phi(x) = \prod_{j=1}^{N} \Phi_j(x) \quad \text{for } x \in X.$$

Note $\Phi \in Kak(X, Y)$. Let q_i, R_i, Q_i and Q be as in Theorem 2.11 and let $\Phi^\star$ denote the restriction of Φ to Q and note $\Phi^\star \in Kak(Q, Y)$. Now let $q(y) = \prod_{i=1}^{N} q_i(y)$ for $y \in Y$ and note $q : Y \to Q$ is continuous so $q\,\Phi^\star \in Ad(Q, Q)$. Theorem 1.1 guarantees a $x \in Q$ with $x \in q\,(\Phi^\star(x))$. Let $y \in \Phi^\star(x)$ with $x = q(y)$, i.e., $y \in \Phi(x)$ with $x_i = q_i(y)$ for $i \in \{1, \dots, N_0\}$. For $i \in \{1, \dots, N_0\}$ we have $x_i = q_i(y) \in G_i(y)$ which is a contradiction if we assume in the statement of Theorem 2.18 that for each $(x, y) \in X \times Y$ there exists an $i \in \{1, \dots, N\}$ with $x_i \notin G_i(y)$. Suppose there exists an $i_0 \in \{1, \dots, N_0\}$ with $H_{i_0}(x) \neq \emptyset$. Then $x \in U_{i_0}$ so we have $y_{i_0} \in \Phi_{i_0}(x) = \Psi_{i_0}(x)$, which is a contradiction if we assume in the statement of Theorem 2.18 that for each $z \in X$ and for each $(w, y) \in X \times Y$ we have $y_j \notin A_{j,z}(w)$ for all $j \in \{1, \dots, N_0\}$. As a result $H_i(x) = \emptyset$ for all $i \in \{1, \dots, N_0\}$. □

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