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# Some fixed point results in partially ordered E metric space

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**Abstract**

The existence and uniqueness of the fixed point theorem for self mapping meeting certain contractive conditions in partially ordered $E$ metric spaces with non-normal positive cone $E^+$ of a real normed space $E$ with empty interior are investigated in this research.

**Keywords:** Fixed point, positive cone, normed space.

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## 1. Introduction

Over the last few decades, fixed point theory in normed spaces has been applied to a wide range of optimization problems, dynamical systems, economy, fractals, computer science, variational inequalities and many other fields. In 1911 Luitzen E. Brouwer published a famous paper discussing results on fixed point theory [11]. Many of its proofs were later adapted in a topological sense. Banach, 1922, presented a method for finding the fixed point of a self operator in complete metric spaces in a systematic manner [7]. Later, a great deal of work on variants and generalizations was published to improve the Banach contraction principle by modifying the topology of the space or acting on the contraction requirement [1, 6, 8, 12, 17, 21, 23].

Huang and Zhang presented the notion of cone metric space with a fresh point of view in which Cauchy and convergent sequences were analyzed in terms of the interior points with respect to the cone partial ordering [16]. Many mathematicians followed Huang's lead and focused on fixed point problems in such spaces (see [2–4, 14–19] and the references therein).

Rezapour and Hamlbarani [22] extended the notion of $K$-metric spaces and convergence in an ordered Banach space $X$ with a solid cone $E^+$ without normality assumptions. Beg et al. established the concept of topological vector space-valued (tvs-valued) cone metric space in 2009, [10], to convey the above mentioned principles in a more extended framework.

To introduce the concept of $E$-metric space, the authors in [5] changed the definition of cone metric space replacing ordered Banach space with ordered vector space. Most fixed point issues in cone metric

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spaces are embedded in solid cones, which are cones with non empty interior. Unfortunately there were just a few results that took non-solid cones into account [9, 20].

Fortunately, by embedding non-solid cones that contain semi-interior points in E-metric spaces, Basile et al. [9] established the concept of the semi-interior point and took fixed point results in E-metric spaces into consideration. Embedding such cones in the setting of E-metric spaces, Mehmood et al. [20] and Huang [14], obtained some fixed theorems in 2019.

As the examples in [9] illustrate, the class of cones with semi-interior point and empty interior is wider than the one with nonempty interior. Moreover, fixed points results for ordered normed spaces also hold for this wider class of cones with semi-interior points, which is quite fascinating.

In this article, we study and prove some fixed point results in normed spaces with reference to a new class of cones, namely cones with semi-interior points, generalizing some of the existence results in [5, 20].

In the situation of cone metric spaces, a number of articles have recently appeared proving some fixed-point theorems of generalized contractive mappings. The following definitions were introduced by Huang [16] are presented in particular.

**Definition 1.1** ([16])**.** Let $E^+$ be a closed and convex nonempty subset of the real normed space $E$ and $0_E$ be the zero of $E$. If the following conditions are satisfied,

(1) if $y \in E^+$ and $\alpha > 0$, then $\alpha y \in E^+$;
(2) if $y \in E^+$ and $-y \in E^+$, then $y = 0_E$,
then $E^+$ is called a positive cone.

**Definition 1.2** ([16])**.** If $E^+$ is the positive cone of a real normed space $E$, then a partial order " $\preccurlyeq$ " on $E$ is defined as: for $x, z \in E$, $x \preccurlyeq z$ if and only if $z - x \in E^+$. Clearly $0_E \preccurlyeq z \in E^+$ if and only if $z \in E^+$.

**Definition 1.3** ([5])**.** A real vector space $E$ with a partial order " $\preccurlyeq$ "is called an ordered vector space if the following conditions hold:

(1) $x, y, z \in E$ and $x \preccurlyeq z$ imply $x + y \preccurlyeq z + y$;
(2) $\lambda > 0, y \in E$ and $0_E \preccurlyeq y$ imply $0_E \preccurlyeq \lambda y$.

**Definition 1.4** ([16])**.** If $E^+$ is the positive cone of a normed ordered space $E$, then $E^+$is called:

(1) solid if $\mathrm{int}(E^+) \neq \phi$;
(2) normal if there exists a constant $K > 0$ such that, for $x, z \in E$, and $0_E \preccurlyeq x \preccurlyeq z$, then, $\|x\| \preccurlyeq K\|y\|$.

The definition of an E-metric space is given as follows.

**Definition 1.5** ([5])**.** Let $X \neq \phi$ be a set and $E$ be an ordered vector space over the set of real numbers. A function $d^E : X \times X \to E$ is called an ordered E-metric on $X$, if for all $x, y, z \in X$, we have

(1) $d^E(x, z) \succcurlyeq 0_E$ and $d^E(x, z) = 0_E$ if and only if $x = z$;
(2) $d^E(x, z) = d^E(z, x)$;
(3) $d^E(x, z) \preccurlyeq d^E(x, y) + d^E(y, z)$.

The pair $(X, d^E)$ is, then known as E-metric space.

Now, consider a normed ordered space $E$ and $(E, d^E)$ be the E-metric space. Let $x \in E$ and $c \in \mathrm{int}(E^+)$. Let us denote by $B(x_0, c) \subset E$, the open ball with radius $c$ and center $x_0$,

$$B(x_0, c) = \left\{y \in E;\ d^E(y, x_0) \ll c\right\}.$$

The definitions of convergent and Cauchy sequences in the E-metric space are given below, assuming $\mathrm{int}(E^+) \neq \phi$.

**Definition 1.6** ([5])**.** If $(X, d^E)$ is an E-metric space, then

(1) a sequence $(x_n)$ in $X$ is said to be convergent to a point $x \in X$ if for any $c \in \mathrm{int}(E^+)$, there exists a positive integer $K$ such that $d^E(x_n, x) \ll c$; for all $n \geqslant K$, simply we write $\lim_{n\to\infty} x_n \to x$, or $x_n \to x$;
(2) a sequence $(x_n)$ in $X$ is said to be Cauchy if for any $c \in \mathrm{int}(E^+)$, there exists a positive integer $N$ such that $d^E(x_n, x_m) \ll c$, for all $n, m \geqslant N$.

## 2. Generalized E-metric space

Let $E$ be a real ordered normed space ordered by it's positive cone $E^+$. If $\mathfrak{B}_+ = \mathfrak{B} \cap E^+$, where $\mathfrak{B}$ is the closed ball of $E$ of radius 1, a point $x_0 \in E$ is called a semi-interior point in $E^+$ if there exists a positive real number $r > 0$ such that $x_0 - r\mathfrak{B}_+ \subseteq E^+$. It is easy to see that any interior point is a semi-interior point. Let $(E^+)^\ominus = \{y \in E^+ : y \text{ is a semi-interior point}\}$. Then a partial order relation " $\lll$ " can be defined on $E^+$ as if $x, z \in E^+$, then $x \lll z$ if and only if $z - x \in (E^+)^\ominus$. Clearly $x \in (E^+)^\ominus$ iff $0_E \lll x$.

The following result (proposition 2.2 in [9]) is a useful characterization of semi -interior points.

**Proposition 2.1** ([9])**.** *Let $E^+$ be the cone of the ordered space $E$. Then the point $y_0$ is a semi-interior point of $E^+$ iff there exists a positive real number $b$ such that $by_0$ is an upper bound of $\mathfrak{B}_+$, the positive part of $\mathfrak{B}$.*

In this section omitting the assumption of non-solid cones of the main results in the literature, we present the notion of semi-interior points of a cone in E-metric spaces providing some nontrivial examples to insure the existence and applications of this notion.

The following example (example 2.3 in [9]) shows that there exists an ordered normed space $E$ ordered by a generating closed cone $E^+$ with $(E^+)^\ominus = \phi$ and $\text{int}(E^+) = \phi$.

**Example 2.2** ([9])**.** Let $X = \{(x_n), x_n \neq 0 \text{ for finitely many terms}\}$ endowed with the norm $\|\mathbf{x}\| = \max\{|x_n| : n \in \mathbb{N}\}$. If $E$ is ordered by the point-wise ordering, then the positive cone $E^+ = \{\mathbf{a} \in E : a(i) \geqslant 0 \text{ for any } i\}$ does not have any semi-interior point. If $\mathbf{y} = \{(i)\} \in (E^+)^\ominus$, then $\mathfrak{B}_+$ is bounded above by $ky$ for some positive number $k$. Since $e_j \in \mathfrak{B}_+, \mathbf{e}_j(i) = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases}$, we have that $y(j) \geqslant \frac{1}{k} e_j(j) = \frac{1}{k} > 0$ for any $j$. This is a contradiction. Hence $(E^+)^\ominus = \phi$ does not have semi-interior points.

Clearly, interior points of $E^+$ are semi-interior points of $E^+$ but the converse is not true shown by the example (example 2.5 in [9]).

**Example 2.3** ([9])**.** Let $X_n = \mathbb{R}^2$ ordered point-wise and endowed with norm $\|.\|_n$, where

$$\|(x,y)\|_n = \begin{cases} |x| + |y|, & \text{if } xy \geqslant 0, \\ \max\{|x|,|y|\} - \frac{n-1}{n}\min\{|x|,|y|\}, & \text{if } xy < 0. \end{cases}$$

It is easy to show that the unit ball of $X_n$, $\mathfrak{B}_n$ is the polygon $D_n$ with vertices $(-n,n), (-1,0), (0,-1), (n,-n), (1,0), (0,1)$.

Let $E = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{N}}, x_n = (x_n^1, x_n^2) \in X_n \text{ and } \|x_n\|_n \leqslant m_x, m_x > 0 \text{ depends on } x\right\}$. Suppose that $E$ is ordered by the use of $E^+ = \{\mathbf{y} = (y_n) \in E : y_n \in \mathbb{R}^2_+ \text{ for any } n\}$ and normed by $\|y\|_\infty = \sup\limits_{n\in\mathbb{N}} \|y_n\|_n$.

Let $X = E^+ - E^+$ be the subspace of $E$ generated by $E^+$ ordered by $X^+ = E^+$. Now if $\mathbf{1} = (y_n) \in X, y_n = (1,1)$ for every $n$, then $\mathbf{1}$ cannot be an interior point of $X^+$. In fact, if for any positive integer $k$, let $\mathbf{x} = (x_n)$ of $X$ with $x_m = (-2,2)$ and $x_n = (0,0)$ for any $n \neq m$. It is easy to show that $\|x\|_\infty = \frac{2}{m}$ and $\mathbf{1} + \mathbf{x} \notin X^+$. Hence $\mathbf{1} + \lambda\mathfrak{B}_+ \nsubseteq X^+$, for any $\lambda > 0$. Hence $\mathbf{1}$ cannot be an interior point of the space $X^+$. Similarly one can show that $\text{int}(X^+) = \phi$ and the point $\mathbf{1} = (y_n)$ is a semi interior point of $X^+$.

The following Proposition (Proposition 2.4 in [9]) shows that under certain conditions semi-interior point of a closed and generating cone is an interior point.

**Proposition 2.4** ([9])**.** *Let $E$ be a Banach space and $E^+$ be its closed generating cone. Then semi-interior points of $E^+$ are interior points of $E^+$.*

In particular, by Proposition 2.4, the space $X$ in Example 2.3 is not complete.

Now we give the definitions of $e$-convergence and the $e$-Cauchy convergence criteria in the frame work of the non-solid cone metric space $E$.

**Definition 2.5** ([20])**.** Suppose that $E$ is an ordered normed space such that $(E^+)^\ominus \neq \phi$ and $(X, d^E)$ is an $E$-metric space. Then

(1) a sequence $(a_k)$ in $X$ is said to be $e$-convergent if there exists a point $x \in X$ and for every $e \ggg 0_E$, there exists an integer $M > 0$ such that $d^E(a_k, x) \lll e$; for all $k \geqslant M$, simply we write $\lim\limits_{k\to\infty} a_k \stackrel{e}{=} x$, $a_k \stackrel{e}{\to} x$;

(2) a sequence $(a_k)$ in $X$ is said to be $e$-Cauchy, if for every $e \ggg 0_E$, there exists an integer $M > 0$ such that $d^E(x_n, x_k) \lll e$, for all $n, k \geqslant M$;

(3) if every $e$-Cauchy sequence in $X$ is $e$-convergent, the space $(X, d^E)$ is said to be an $e$-complete metric space.

## 3. Fixed point results

The authors of [5] updated the definition of cone metric space by introducing the notion of $E$-metric space by substituting ordered Banach space with ordered vector space. Most of the fixed point results in cone or $E$-metric spaces are embedded in solid cones.

Fortunately, Basile et al. [9] established the concept of the semi-interior point in 2017, and used it to incorporate fixed point findings in $E$-metric spaces by embedding non-solid cones with semi-interior points.

In this section we embed a class of cones with semi-interior point and empty interior to obtain certain fixed point results in ordered normed spaces in the setting of $E$-metric spaces using some new contractions that generalize some of the following results.

**Theorem 3.1** ([20])**.** *Let $E$ be an ordered normed space such that $(E^+)^\ominus \neq \phi$ and $(X, d^E)$ be an $E$-complete metric space. Suppose $S : X \to X$ is a self mapping that satisfies*

$$d^E(Sx, Sy) \preccurlyeq \alpha d^E(x, y)$$

*for some $\alpha \in [0, 1)$ and all $x, y$ in $X$. Then for each $x \in X$, the iterative sequence $(S^n x)_{n\geqslant 0}$ converges to the unique fixed point of $S$.*

**Theorem 3.2** ([20])**.** *Let $E$ be an ordered normed space such that $(E^+)^\ominus \neq \phi$ and $(X, d^E)$ be an $E$-complete metric space. Suppose $S : X \to X$ is a self mapping that satisfies*

$$d^E(Sx, Sy) \preccurlyeq \alpha \left(d^E(Sx, x) + d^E(Sy, y)\right)$$

*for all $x, y \in X$ and some $\alpha \in [0, \frac{1}{2})$. Then for each $x \in X$, the iterative sequence $(S^n x)_{n\geqslant 0}$ converges to the unique fixed point of $S$.*

**Theorem 3.3** ([20])**.** *Let $E$ be an ordered normed space such that $(E^+)^\ominus \neq \phi$ and $(X, d^E)$ be an $E$-complete metric space. Suppose $S : X \to X$ is a self mapping that satisfies*

$$d^E(Sx, Sy) \preccurlyeq \alpha \left(d^E(Sx, y) + d^E(Sy, x)\right)$$

*for all $x, y \in X$ and some $\alpha \in [0, \frac{1}{2})$. Then for each $x \in X$, the iterative sequence $(S^n x)_{n\geqslant 0}$ converges to the unique fixed point of $S$.*

**Theorem 3.4** ([14])**.** *Let $E$ be an ordered normed space such that $(E^+)^\ominus \neq \phi$ and $(X, d^E)$ be an $E$-complete metric space. Suppose $S : X \to X$ is a self mapping that satisfies*

$$d^E(Sx, Sy) \preccurlyeq \lambda_1 d^E(x, y) + \lambda_2 d^E(x, Sx) + \lambda_3 d^E(y, Sy) + \lambda_4 d^E(x, Sy) + \lambda_5 d^E(y, Sx)$$

*for all $x, y \in X$, where $\lambda_j > 0$ $(j = 1, \ldots, 5)$ and $0 \leqslant \sum\limits_{i=1}^{5} \lambda_i < 1$. Then*

(1) *for each $x \in X$, the iterative sequence $(S^n x)_{n\geqslant 0}$ converges to the unique fixed point of $S$;*

(2) *the Picard's iteration is* S*-stable;*
(3) $\{d^E(y_n, Sy_n)\}$ *is an* e*-sequence iff* $\{d^E(y_{n+1}, Sy_n)\}$ *is an* e*-sequence.*

We begin by our first main results.

**Theorem 3.5.** *Let* $E$ *be an ordered normed space such that* $(E^+)^\ominus \neq \phi$ *and* $(X, d^E)$ *be an* E*-complete metric space. Suppose* $T : X \to X$ *is a self mapping that satisfies*

$$d^E(Tx, Ty) \preccurlyeq h \max\left\{d^E(x, y), d^E(Tx, x), d^E(Ty, y), d^E(Tx, y), d^E(Ty, x)\right\}$$

*for all* $x, y \in X$ *and some* $h \in [0, \frac{1}{2})$*. Then for each* $x \in X$*, the iterative sequence* $(T^n x)_{n\geqslant 0}$ *converges to the unique fixed point of* $T$*.*

*Proof.* Let $x_0$ be an arbitrary point of X and $(x_n)$ be an iterative sequence defined by $x_{n+1} = Tx_n = T^{n+1}x_0$ with $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
d^E(x_n, x_{n+1}) &= d^E(Tx_{n-1}, Tx_n)\\
&\preccurlyeq h \max\left\{\begin{array}{c} d^E(x_{n-1}, x_n), d^E(Tx_n, x_n), d^E(Tx_{n-1}, x_{n-1}),\\ d^E(Tx_n, x_{n-1}), d^E(Tx_{n-1}, x_n)\end{array}\right\}\\
&= h \max\left\{\begin{array}{c} d^E(x_{n-1}, x_n), d^E(x_{n+1}, x_n), d^E(x_n, x_{n-1}), d^E(x_{n+1}, x_{n-1}),\\ d^E(x_n, x_n)\end{array}\right\}\\
&= h \max\left\{d^E(x_n, x_{n-1}), d^E(x_{n+1}, x_n), d^E(x_{n+1}, x_{n-1})\right\}\\
&\preccurlyeq h \max\left\{d^E(x_n, x_{n-1}), d^E(x_{n+1}, x_n), d^E(x_{n+1}, x_n) + d^E(x_n, x_{n-1})\right\}.
\end{aligned}$$

Since $h \in [0, \frac{1}{2})$ and $d^E(x_n, x_{n-1}) \succcurlyeq 0_E$, it follows that both $d^E(x_n, x_{n+1})$ and

$$d^E(x_n, x_{n-1}) \neq \max\left\{d^E(x_n, x_{n-1}), d^E(x_{n+1}, x_n), d^E(x_{n+1}, x_n) + d^E(x_n, x_{n-1})\right\}.$$

Hence,

$$d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1}) = \max\left\{d^E(x_n, x_{n-1}), d^E(x_{n+1}, x_n), d^E(x_{n+1}, x_n) + d^E(x_n, x_{n-1})\right\}.$$

Thus

$$d^E(x_n, x_{n+1}) \preccurlyeq h\left(d^E(x_{n+1}, x_n) + d^E(x_n, x_{n-1})\right),$$

and

$$d^E(x_n, x_{n+1}) \preccurlyeq \frac{h}{1-h} d^E(x_{n-1}, x_n).$$

Consequently if we put $k = \frac{h}{1-h}$, we get

$$d^E(x_n, x_{n+1}) \preccurlyeq k d^E(x_{n-1}, x_n) \preccurlyeq k^2 d^E(x_{n-2}, x_{n-1}) \preccurlyeq \cdots \preccurlyeq k^n d^E(x_0, x_1).$$

Now for $n > m$,

$$\begin{aligned}
d^E(x_m, x_n) &\preccurlyeq d^E(x_m, x_{m+1}) + d^E(x_{m+1}, x_{m+2}) + d^E(x_{m+2}, x_{m+3}) + \cdots + d^E(x_{n-1}, x_n)\\
&\preccurlyeq \left(k^m + k^{m+1} + k^{m+2} + \cdots + k^{n-1}\right) d^E(x_1, x_0)\\
&= k^m\left(1 + k + k^2 + \cdots + k^{n-m-1}\right) d^E(x_1, x_0)\\
&= k^m\left(\frac{1 - k^{n-m}}{1-k}\right) d^E(x_1, x_0).
\end{aligned}$$

Now we are going to prove that $(x_n)$ is an $e$-Cauchy sequence. Let $e \ggg 0_E$ be given, and take $\mu > 0$ such that $e - \mu\mathfrak{B}_+ \subseteq E^+$, and an integer $k_1$ such that $k^m\left(\frac{1-k^{n-m}}{1-k}\right)d^E(x_1,x_0) \in \frac{\mu}{2}\mathfrak{B}_+$ for all $n, m \geqslant k_1$. Therefore,

$$e - k^m\left(\frac{1-k^{n-m}}{1-k}\right)d^E(x_1,x_0) - \frac{\mu}{2}\mathfrak{B}_+ \subseteq e - \mu\mathfrak{B}_+ \subseteq E^+,$$

for all $n, m \geqslant k_1$. Hence,

$$d^E(x_n,x_m) \preccurlyeq k^m\left(\frac{1-k^{n-m}}{1-k}\right)d^E(x_1,x_0) \lll e.$$

This implies that the sequence $(x_n)$ is an $e$-Cauchy. But $X$ is an $e$-complete metric space. Then there exists $x \in X$ such that $x_n$ converges to $x$. So for $e \ggg 0_E$, we can choose $k_2 \in \mathbb{N}$, with $d^E(x_{n-1},x_n) \lll \frac{e}{3}$ and $d^E(x,x_{n-1}) \lll \frac{e}{3}$ for all $n \geqslant k_2$. Now

$$\begin{aligned}
d^E(Tx,x) &\preccurlyeq d^E(Tx,x_n) + d^E(x_n,x)\\
&= d^E(x_n,x) + d^E(Tx,Tx_{n-1})\\
&\preccurlyeq d^E(x_n,x) + h\max\left\{\begin{array}{c} d^E(x,x_{n-1}), d^E(Tx_{n-1},x_{n-1}),\\ d^E(Tx,x), d^E(Tx,x_{n-1}) + d^E(Tx_{n-1},x)\end{array}\right\}\\
&\preccurlyeq d^E(x,x_n) + h\max\left\{\begin{array}{c} d^E(x,x_{n-1}), d^E(x_n,x_{n-1}),\\ d^E(Tx,x), d^E(Tx,x_{n-1}), d^E(x_n,x)\end{array}\right\},
\end{aligned}$$

which must be studied as following cases.

Case 1: If the maximum occurs at $d^E(x_{n-1},x)$ or $d^E(x,x_n)$, then

$$d^E(Tx,x) \preccurlyeq \frac{e}{3} + h\left(\frac{e}{3}\right) \lll e.$$

Case 2: If the maximum occurs at $d^E(Tx,x)$, then

$$d^E(Tx,x) \preccurlyeq d^E(x_n,x) + hd^E(Tx,x),$$

therefore,

$$(1-h)\,d^E(Tx,x) \preccurlyeq d^E(x_n,x), \quad d^E(Tx,x) \preccurlyeq \frac{1}{1-h}d^E(x_n,x) \preccurlyeq \left(\frac{1}{1-h}\right)\frac{e}{3} \lll e,$$

since $\frac{1}{1-h} < 2$.

Case 3: If the maximum occurs at $d^E(x_n,x_{n-1})$, then

$$\begin{aligned}
d^E(Tx,x) &\preccurlyeq d^E(x_n,x) + hd^E(x_n,x_{n-1})\\
&\preccurlyeq d^E(x_n,x) + h\left(d^E(x,x_{n-1}) + d^E(x,x_n)\right)\\
&\preccurlyeq (1+h)d^E(x,x_n) + hd^E(x_{n-1},x)\\
&\preccurlyeq (1+h)\frac{e}{3} + h\frac{e}{3} = \frac{1+2h}{3}e \lll e,
\end{aligned}$$

since $h < \frac{1}{2}$.

Case 4: If the maximum is at $d^E(Tx,x_{n-1})$, then

$$d^E(Tx,x) \preccurlyeq d^E(x_n,x) + hd^E(Tx,x_{n-1}) \preccurlyeq d^E(x_n,x) + h\left(d^E(x,x_{n-1}) + d^E(Tx,x)\right),$$

which implies that

$$(1-h)d^E(Tx,x) \preccurlyeq d^E(x_n,x_n) + hd^E(x,x_{n-1}).$$

Hence,

$$d^E(x,Tx) \preccurlyeq \frac{1}{1-h} d^E(x,x_n) + \frac{h}{1-h} d^E(x_{n-1},x) \preccurlyeq \left(\frac{1}{1-h}\right)\frac{e}{3} + \left(\frac{h}{1-h}\right)\frac{e}{3} \lll e.$$

Since $d^E(x,Tx) \lll \frac{\epsilon}{m}$ for any $\frac{\epsilon}{m} \ggg 0_E$, and $m \in \mathbb{N}$, it follows that $\frac{\epsilon}{m} - d^E(x,Tx) \in E^+$ for all $m \in \mathbb{N}$. Therefore $-d^E(Tx,x) \in E^+$, and $d^E(x,Tx) \in E^+$. Hence $d^E(x,Tx) = 0_E$ and $x = Tx$.

To prove uniqueness, let $z \in X$ be such that $z = Tz$. Then consider

$$\begin{aligned} d^E(x,z) &= d^E(Tx,Tz) \\ &\preccurlyeq h \max\left\{d^E(x,z), d^E(Tx,x), d^E(Tz,z), d^E(Tz,x), d^E(z,Tx)\right\} \\ &= h \max\left\{d^E(x,z), d^E(x,x), d^E(z,z), d^E(x,z), d^E(z,x)\right\} \\ &= h \max\left\{d^E(x,z), 0_E, 0_E, d^E(x,z), d^E(z,x)\right\} \\ &= h d^E(x,z), \end{aligned}$$

which implies $d^E(x,z) = 0_E$ . □

**Corollary 3.6.** *Let* $E$ *be an ordered normed space such that* $(E^+)^{\ominus} \neq \phi$ *and* $(X, d^E)$ *is an* $E$*-complete metric space. For* $e \ggg 0_E$ *and* $x_0 \in X$*, set* $B(x_0,e) = \left\{x \in X : d^E(x_0,x) \lll e\right\}$*. Suppose* $T : X \to X$ *is a self mapping that satisfies*

$$d^E(Tx,Ty) \preccurlyeq h \max\left\{d^E(x,y), d^E(Tx,x), d^E(Ty,y), d^E(Tx,y), d^E(Ty,x)\right\}$$

*for any* $x,y \in B(x_0,e), h \in [0,\frac{1}{2})$ *and* $d^E(x_0,Tx_0) \preccurlyeq (1-h)e$*. Then* $T$ *has a unique fixed point in* $B(x_0,e)$*.*

*Proof.* Using Theorem 3.5 if we prove that $B(x_0,e)$ is $e$-complete, and $Tx \in B(x_0,e)$ for any $x \in B(x_0,e)$, we are done. If $\{x_n\}$ is an $e$-Cauchy sequence in $B(x_0,e)$, then $\{x_n\}$ is an $e$-Cauchy sequence in $X$. But $X$ is $e$-complete. Hence for some $x \in X$, $\lim_{n\to\infty} x_n \overset{e}{=} x$. Thus $d^E(x_n,x) \to 0_E$. Since the inequality

$$d^E(x,x_0) \preccurlyeq d^E(x,x_n) + d^E(x_n,x_0)$$

implies that $d^E(x_0,x) \lll e$ and $x \in B(x_0,e)$, it follows that for every $x \in \mathcal{B}(x_0,e)$,

$$d^E(x_0,Tx) \preccurlyeq d^E(Tx_0,x_0) + d^E(Tx_0,Tx) \preccurlyeq d^E(Tx_0,x_0) + h \max\left\{\begin{array}{c} d^E(x_0,x), d^E(x_0,Tx), d^E(x,Tx), \\ d^E(x_0,Tx_0), d^E(x,Tx_0). \end{array}\right\}.$$

We have the following cases.

Case1: If

$$\max\left\{d^E(x_0,x), d^E(x_0,Tx), d^E(x,Tx), d^E(x_0,Tx_0), d^E(x,Tx_0)\right\} = d^E(x_0,x),$$

then

$$d^E(x_0,Tx) \lll (1-h)e + he = e.$$

Case 2: If

$$\max\left\{d^E(x_0,x), d^E(x_0,Tx), d^E(x,Tx), d^E(x_0,Tx_0), d^E(x,Tx_0)\right\} = d^E(x_0,Tx),$$

then

$$d^E(x_0,Tx) \preccurlyeq d^E(Tx_0,x_0) + h d^E(x_0,Tx)$$

or

$$d^E(x_0,Tx) \preccurlyeq \frac{1}{1-h} d^E(Tx_0,x_0) \lll \frac{1}{1-h}(1-h)\, e = e.$$

The other cases are similar. Hence $Tx \in B(x_0,e)$. □

**Corollary 3.7.** *Let* $E$ *be an ordered normed space such that* $(E^+)^{\ominus} \neq \phi$ *and* $(X, d^E)$ *is an* $E$*-complete metric space. Suppose that for some integer* $n > 0$*, the mapping* $T : X \to X$ *satisfies*

$$d^E(T^n x, T^n y) \preccurlyeq h \max\left\{d^E(x,y), d^E(x,Tx), d^E(y,Ty), d^E(x,Ty), d^E(y,Tx)\right\}$$

*for all* $x, y \in X$*, and* $h \in [0, \frac{1}{2})$*. Then* $T$ *has a unique fixed point in* $X$*.*

*Proof.* Using Theorem 3.5, the point $x_0$ is the unique fixed point of $T^n$. Since, $T^n(Tx_0) = T(T^n x_0) = Tx_0$, it follows that $Tx_0$ is a fixed point of $T^n$. Hence $x_0 = Tx_0$. □

**Theorem 3.8.** *Let* $E$ *be an ordered normed space such that* $(E^+)^{\ominus} \neq \phi$ *and* $(X, d^E)$ *is an* $E$*-complete metric space. Suppose* $T : X \to X$ *is a self mapping that satisfies*

$$d^E(Tx, Ty) \preccurlyeq h \max\left\{d^E(x,y), d^E(x,Tx), d^E(y,Ty), \alpha d^E(x,Ty) + \beta d^E(y,Tx)\right\}$$

*for all* $x, y \in X$*, and some* $h \in [0, \frac{1}{2})$ *with* $\alpha, \beta > 0$*,* $\alpha + \beta = 1$*. Then for each* $x \in X$*, the iterative sequence* $(T^n x)_{n \geqslant 0}$ *converges to the unique fixed point of* $T$*.*

*Proof.* Let $x_0$ be an arbitrary point of $X$ and $(x_n)$ be an iterative sequence defined by as $x_{n+1} = Tx_n = T^{n+1} x_0$ with $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
d^E(x_{n+1}, x_n) &= d^E(Tx_n, Tx_{n-1}) \\
&\preccurlyeq h \max\left\{\begin{array}{c} d^E(x_n, x_{n-1}), d^E(x_n, Tx_n), d^E(x_{n-1}, Tx_{n-1}), \\ \alpha d^E(x_n, Tx_{n-1}) + \beta d^E(x_{n-1}, Tx_n) \end{array}\right\} \\
&= h \max\left\{\begin{array}{c} d^E(x_n, x_{n-1}), d^E(x_n, x_{n+1}), d^E(x_{n-1}, x_n), \\ \alpha d^E(x_n, x_n) + \beta d^E(x_{n-1}, x_{n+1}) \end{array}\right\} \\
&= h \max\left\{d^E(x_n, x_{n-1}), d^E(x_n, x_{n+1}), \beta d^E(x_{n-1}, x_{n+1})\right\} \\
&\preccurlyeq h \max\left\{\begin{array}{c} d^E(x_n, x_{n-1}), d^E(x_n, x_{n+1}), \\ \beta\left(d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1})\right) \end{array}\right\}.
\end{aligned}$$

Since $h \in [0, \frac{1}{2})$, then $d^E(x_n, x_{n+1}) \neq \max\left\{\begin{array}{c} d^E(x_n, x_{n-1}), d^E(x_n, x_{n+1}), \\ \beta(d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1})) \end{array}\right\}$. Hence either

$$d^E(x_n, x_{n-1}) = \max\left\{d^E(x_n, x_{n-1}), d^E(x_n, x_{n+1}), \beta(d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1}))\right\},$$

which implies

$$d^E(x_{n+1}, x_n) \preccurlyeq h d^E(x_{n-1}, x_n),$$

and

$$d^E(x_{n+1}, x_n) \preccurlyeq h d^E(x_n, x_{n-1}) \preccurlyeq h^2 d^E(x_{n-1}, x_{n-2}) \preccurlyeq \cdots \preccurlyeq h^n d^E(x_1, x_0),$$

or

$$\beta\left(d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1})\right) = \max\left\{d^E(x_n, x_{n-1}), d^E(x_n, x_{n+1}), \beta(d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1}))\right\}.$$

Thus

$$d^E(x_{n+1}, x_n) \preccurlyeq h\beta\left(d^E(x_{n-1}, x_n) + d^E(x_n, x_{n+1})\right),$$

and

$$d^E(x_{n+1}, x_n) \preccurlyeq \frac{h\beta}{1 - h\beta} d^E(x_n, x_{n-1}).$$

Consequently if $k = \frac{h\beta}{1-h\beta}$, we have

$$d^E(x_{n+1}, x_n) \preccurlyeq k d^E(x_n, x_{n-1}) \preccurlyeq k^2 d^E(x_{n-1}, x_{n-2}) \preccurlyeq \cdots \preccurlyeq k^n d^E(x_1, x_0).$$

Now for $n > m$ and $\delta = \min(h, k)$,

$$\begin{aligned} d^E(x_m, x_n) &\preccurlyeq d^E(x_m, x_{m+1}) + d^E(x_{m+1}, x_{m+2}) + \cdots + d^E(x_{n-1}, x_n) \\ &\preccurlyeq \left(\delta^m + \delta^{m+1} + \delta^{m+2} + \cdots + \delta^{n-1}\right) d^E(x_1, x_0) \\ &= \delta^m \left(1 + \delta + \cdots + \delta^{n-m-1}\right) d^E(x_1, x_0) \\ &= \delta^m \left(\frac{1-\delta^{n-m}}{1-\delta}\right) d^E(x_1, x_0). \end{aligned}$$

Now let us prove that $(x_n)$ is an $e$-Cauchy sequence. Let $e \ggg 0_E$ be given, and take $\mu > 0$ such that $e - \mu\mathfrak{B}_+ \subseteq E^+$, and an integer $k_1$ such that $\delta^m \left(\frac{1-\delta^{n-m}}{1-\delta}\right) d^E(x_1, x_0) \in \frac{\mu}{2}\mathfrak{B}_+$ for all $n, m \geqslant k_1$. Therefore,

$$e - \delta^m \left(\frac{1-\delta^{n-m}}{1-\delta}\right) d^E(x_1, x_0) - \frac{\mu}{2}\mathfrak{B}_+ \subseteq e - \mu\mathfrak{B}_+ \subseteq E^+,$$

for all $n, m \geqslant k_1$. Hence

$$d^E(x_n, x_m) \preccurlyeq \delta^m \left(\frac{1-\delta^{n-m}}{1-\delta}\right) d^E(x_1, x_0) \lll e.$$

This implies that the sequence $(x_n)$ is an e-Cauchy. But X is an $e$-complete metric space. Then the sequence $x_n$ converges to some point $x \in X$. So for $e \ggg 0_E$, we can choose $k_2 \in \mathbb{N}$, with $d^E(x_{n-1}, x_n) \lll \frac{e}{4}$ and $d^E(x, x_{n-1}) \lll \frac{e}{4}$ for all $n \geqslant k_2$. Now

$$\begin{aligned} d^E(x, Tx) &\preccurlyeq d^E(x, x_n) + d^E(x_n, Tx) \\ &= d^E(x, x_n) + d^E(Tx_{n-1}, Tx) \\ &\preccurlyeq d^E(x, x_n) + h \max \left\{ \begin{array}{c} d^E(x_{n-1}, x), d^E(x_{n-1}, Tx_{n-1}), d^E(x, Tx), \\ \alpha d^E(x_{n-1}, Tx) + \beta d^E(x, Tx_{n-1}) \end{array} \right\}, \end{aligned}$$

which must be studied as following cases.

Case 1: If the maximum occurs at $d^E(x_{n-1}, x)$, then

$$d^E(x, Tx) \preccurlyeq \frac{e}{4} + h\left(\frac{e}{4}\right) \lll e.$$

Case 2: If the maximum occurs at $d^E(x, Tx)$, then

$$d^E(x, Tx) \preccurlyeq d^E(x, x_n) + hd^E(x, Tx).$$

Therefore,

$$(1-h)\, d^E(x, Tx) \preccurlyeq d^E(x, x_n), \quad d^E(x, Tx) \preccurlyeq \frac{1}{1-h} d^E(x, x_n) \preccurlyeq \left(\frac{1}{1-h}\right) \frac{e}{4} \lll e,$$

since $\frac{1}{1-h} < 2$.

Case 3: If the maximum occurs at $d^E(x_{n-1}, x_n)$, then

$$\begin{aligned} d^E(x, Tx) &\preccurlyeq d^E(x, x_n) + hd^E(x_{n-1}, x_n) \\ &\preccurlyeq d^E(x, x_n) + h\left(d^E(x_{n-1}, x) + d^E(x, x_n)\right) \\ &\preccurlyeq (1+h) d^E(x, x_n) + hd^E(x_{n-1}, x) \\ &\preccurlyeq (1+h)\frac{e}{4} + h\frac{e}{4} = \frac{1+2h}{4} e \lll e, \end{aligned}$$

since $h < \frac{1}{2}$.

Case 4: If the maximum is at $\alpha d^E(x_{n-1}, Tx) + \beta d^E(x, x_n)$, then

$$\begin{aligned} d^E(x, Tx) &\preccurlyeq d^E(x, x_n) + h\left(\alpha d^E(x_{n-1}, Tx) + \beta d^E(x, x_n)\right) \\ &\preccurlyeq d^E(x, x_n) + h\left(\alpha\left(d^E(x_{n-1}, x) + d^E(x, Tx)\right) + \beta d^E(x, x_n)\right), \end{aligned}$$

which implies that

$$(1 - \alpha h) d^E(x, Tx) \preccurlyeq (1 + \beta h)\, d^E(x, x_n) + \alpha h d^E(x_{n-1}, x).$$

Hence,

$$d^E(x, Tx) \preccurlyeq \frac{1 + \beta h}{1 - \alpha h} d^E(x, x_n) + \frac{\alpha h}{1 - \alpha h} d^E(x_{n-1}, x) \preccurlyeq \left(\frac{1 + \beta h}{1 - \alpha h}\right)\frac{e}{4} + \left(\frac{\alpha h}{1 - \alpha h}\right)\frac{e}{4} \lll e.$$

Since $d^E(x, Tx) \lll \frac{\epsilon}{m}$ for any $\frac{\epsilon}{m} \ggg 0_E$ and $m \in \mathbb{N}$, then $\frac{\epsilon}{m} - d^E(x, Tx) \in E^+$ for all $m \in \mathbb{N}$. This implies that $-d^E(x, Tx) \in E^+$, and $d^E(x, Tx) \in E^+$. Therefore $d^E(x, Tx) = 0_E$. Hence $x = Tx$.

To prove uniqueness, let $y \in E$ be such that $y = Ty$. Then consider

$$\begin{aligned} d^E(x, y) &= d^E(Tx, Ty) \\ &\preccurlyeq h \max\left\{d^E(x, y), d^E(x, Tx), d^E(y, Ty), \alpha d^E(x, Ty) + \beta d^E(y, Tx)\right\} \\ &= h \max\left\{d^E(x, y), d^E(x, x), d^E(y, y), \alpha d^E(x, y) + \beta d^E(y, x)\right\} \\ &= h \max\left\{d^E(x, y), 0_E, 0_E, (\alpha + \beta) d^E(x, y)\right\} \\ &= h d^E(x, y), \end{aligned}$$

which implies $d^E(x, y) = 0_E$ . □

Similarly one can prove the following theorem.

**Theorem 3.9.** *Let $E$ be an ordered normed space such that $(E^+)^{\ominus} \neq \phi$ and $(X, d^E)$ is an $E$-complete metric space. Suppose $T : X \to X$ is a self mapping that satisfies*

$$d^E(Tx, Ty) \preccurlyeq h \max\left\{d^E(x, y), d^E(x, Ty), d^E(y, Tx), \alpha d^E(x, Tx) + \beta d^E(y, Ty)\right\}$$

*for all $x, y \in X$, and some $h \in [0, \frac{1}{2})$ with $\alpha, \beta > 0$, $\alpha + \beta = 1$. Then for each $x \in X$, the iterative sequence $(T^n x)_{n \geqslant 0}$ converges to the unique fixed point of $T$.*

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