Convergence analysis and approximation solution for the coupled fractional convection-diffusion equations

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Abstract

By using maximum principle approach, the existence, uniqueness and stability of a coupled fractional partial differential equations is studied. A new fractional characteristic finite difference scheme is given for solving the coupled system. This method is based on shifted Grünwald approximation and Diethelm's algorithm. We obtain the optimal convergence rate for this scheme and drive the stability estimates. The results are justified by implementing an example of the fractional order time and space dependent in concept of the complex Lévy motion. Also, the numerical results are examined for disinfection and sterilization of tetanus. ©2016 All rights reserved.

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1. Introduction

Recently, fractional calculus is a powerful tool to investigate the dynamics of complex systems in different sciences such as fluid mechanics, economic and biology (for example, see [4, 11, 13, 14, 15, 19, 21, 22] and therein references). Many of biology researchers have used to model real process by fractional calculus. For instance, it is developed a fractional-order mathematical model for a human root dentin [18]. Also, in biology, it is proved that the membranes of cells of biological organism have fractional-order electrical conductance [5]. In [7], it is shown that fractional derivatives embody essential features of cell rheological behavior. In addition in [1], it is presented that modeling the

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behavior of brainstem vestibule oculomotor neurons by fractional ordinary differential equation has more advantages rather than integer-order differential equation.

The coupled fractional dynamics system (CFDS) is a governing equation of Lévy motion that it describes several interesting biological phenomena. For example, some modern of epidemics or Avian influenza can spread around the world in a few weeks and seem to follow a non-Gaussian, scale-free dynamics [1]. In hydrology system, such equations have been shown to govern pulse propagation along orthogonal polarization axes in Lévy motion and in wavelength-division-multiplexed systems [2, 23]. The concept of the fractional equations can define as a model of beam propagation or water wave interactions. Fractional solitary waves in these equations are often called vector solutions in the literature as they generally contain two components. In all the above physical situations, collision of vector solutions is an important issue. These fractional equations have been studied intensively in recent years. Moreover, it has been shown that passing-through collision and vector solutions can also bounce off each other or trap each other. The stationary forms of these equations are investigated by a number of authors and a physical problem was introduce by Benson in [2]. As for the numerical methods for this type of problems, the finite difference method has been considered by several authors in various settings, see e.g.[23].

In this paper, as a result of a special characteristic scheme, artificial diffusion is added only in the fractional characteristic direction so that internal layers are not smeared out when the added diffusion removes oscillations near boundary layers. Stability of the scheme of characteristics has played a very important role in fractional partial differential equation. These methods were first introduced by Russell in [10, 20]; Douglas [8] and Su in [23]. The fractional fluid problems are further studied and mathematically developed, e.g. the fractional partial differential equations and the convection-diffusion problems in [23].

We consider the following form of the *coupled fractional system of Lévy motion equations* (CF-SLM) for $(x,t) \in \Omega_T = \Omega \times [0,T]$,

$$\begin{cases} \omega_{11}(x,t)\frac{\partial^{\beta_{11}}\psi_1}{\partial t^{\beta_{11}}} + \nu_{11}(x,t)\frac{\partial^{\beta_{11}}\psi_1}{\partial x^{\beta_{11}}} - \varepsilon_{11}\left(D_{+,11}(t)\frac{\partial^{\alpha_{11}}\psi_1}{\partial_+ x^{\alpha_{11}}} + D_{-,11}(t)\frac{\partial^{\alpha_{11}}\psi_1}{\partial_- x^{\alpha_{11}}}\right) \\ +\omega_{12}(x,t)\frac{\partial^{\beta_{12}}\psi_2}{\partial t^{\beta_{12}}} + \nu_{12}(x,t)\frac{\partial^{\beta_{12}}\psi_2}{\partial x^{\beta_{12}}} - \varepsilon_{12}\left(D_{+,12}(t)\frac{\partial^{\alpha_{12}}\psi_2}{\partial_+ x^{\alpha_{12}}} + D_{-,12}(t)\frac{\partial^{\alpha_{12}}\psi_2}{\partial_- x^{\alpha_{12}}}\right) = S_1(x,t), \\ \\ \omega_{21}(x,t)\frac{\partial^{\beta_{21}}\psi_1}{\partial t^{\beta_{21}}} + \nu_{21}(x,t)\frac{\partial^{\beta_{21}}\psi_1}{\partial x^{\beta_{21}}} - \varepsilon_{21}\left(D_{+,21}(t)\frac{\partial^{\alpha_{21}}\psi_1}{\partial_+ x^{\alpha_{21}}} + D_{-,21}(t)\frac{\partial^{\alpha_{21}}\psi_1}{\partial_- x^{\alpha_{21}}}\right) \\ +\omega_{22}(x,t)\frac{\partial^{\beta_{22}}\psi_2}{\partial t^{\beta_{22}}} + \nu_{22}(x,t)\frac{\partial^{\beta_{22}}\psi_2}{\partial x^{\beta_{22}}} - \varepsilon_{22}\left(D_{+,22}(t)\frac{\partial^{\alpha_{22}}\psi_2}{\partial_+ x^{\alpha_{22}}} + D_{-,22}(t)\frac{\partial^{\alpha_{22}}\psi_2}{\partial_- x^{\alpha_{22}}}\right) = S_2(x,t), \\ \\ \psi_1(x,0) = \widetilde{\psi}_1(x), \quad \psi_2(x,0) = \widetilde{\psi}_2(x), \qquad x \in \Omega, \\ \\ \psi_1(x,t) = \psi_2(x,t) = 0, \qquad t \in (0,T], \quad \& \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where ψ_1 and ψ_2 are the wave amplitudes in two polarizations, $\Omega = [x_L, x_R] \subset \mathbb{R}$ is a bounded domain with boundary $\partial\Omega$ and $0 < \varepsilon_{i,j} \ll 1$, $i,j = 1,2$ are small damping factors which control the diffusion. The advection dominance of system (1.1) are shown by $0 \leq \beta_{i,j} < 1$, $1 < \alpha_{i,j} \leq 2$, $i,j = 1,2$ whose are the orders of fractional diffusion and finally $\{\omega_{i,j}(x,t), \nu_{i,j}(x,t)\}_{i,j=1}^2$ are drifts of the process for representing the convection velocity. The equation (1.1) is associated with the following notations:

$$D_{+,i,j}(t) = \frac{(1+\gamma)D_{i,j}(t)}{2}, \quad D_{-,i,j}(t) = \frac{(1-\gamma)D_{i,j}(t)}{2}, \qquad i,j = 1,2. \tag{1.2}$$

Where $0 < D_{min} \leq D_{-,i,j}(t), D_{+,i,j}(t) \leq D_{max} < \infty$ are the time-dependent coefficients of dispersion such that $-1 \leq \gamma \leq 1$ shows the relative weight of forward versus backward transition probability; $\{S_i(x,t)\}_{i=1}^2$ are the source and sink terms; also the left-side $(+)$ and the the right-side $(-)$ Riemann-Liouville fractional derivatives of different orders of a function $\phi(x)$ for $x \in [x_L, x_R]$ are defined as follows:

$$\frac{\partial^\sigma \phi}{\partial_+ x^\sigma} = \frac{1}{\Gamma(\rho - \sigma)} \frac{d^\rho}{dx^\rho} \int_{x_L}^x \frac{\phi(\xi)}{(x-\xi)^{\sigma-\rho+1}} d\xi, \quad \rho - 1 < \sigma < \rho, \tag{1.3}$$

$$\frac{\partial^\sigma \phi}{\partial_- x^\sigma} = \frac{1}{\Gamma(\rho - \sigma)} \frac{(-d)^\rho}{dx^\rho} \int_x^{x_R} \frac{\phi(\xi)}{(-x+\xi)^{\sigma-\rho+1}} d\xi, \quad \rho - 1 < \sigma < \rho, \tag{1.4}$$

where $\rho \in \{1, 2\}$ and $\Gamma(.)$ is the Gamma function. Also, the modified Riemann-Liouville derivative is defined as

$$\frac{\partial^\sigma \phi}{\partial x^\sigma} = \frac{1}{\Gamma(1 - \sigma)} \frac{d}{dx} \int_{x_L}^x (x-\xi)^{-\sigma}(\phi(\xi) - \phi(x_L)) d\xi, \quad 0 \leq \sigma < 1, \tag{1.5}$$

Moreover, we state (1.3), (1.4) and (1.5) for time in (1.1). The authors of [3] provided an important justification for the above model equations for simulating the epidemiology of tetanus in Italy from 1955 to 1982.

For simplifying the system (1.1) is written in the following matrix form

$$\begin{cases} \mathcal{F}_{\alpha,\varepsilon}^\beta(\psi(x,t)) = \mathcal{S}(x,t), & \text{in} \quad (x,t) \in \Omega_T, \\ \psi(x,0) = \psi_0(x), & x \in \Omega, \\ \psi(x,t) = 0, & t \in [0,T], \quad x \in \partial\Omega, \end{cases} \tag{1.6}$$

where

$$\mathcal{F}_{\alpha,\varepsilon}^\beta(.) = \left(\mathcal{F}_{\alpha_{i,j},\varepsilon_{i,j}}^{\beta_{i,j}}(.) \right)_{i,j=1}^2, \tag{1.7}$$

$$\mathcal{F}_{\alpha_{i,j},\varepsilon_{i,j}}^{\beta_{i,j}}(.) = \omega_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}(.)}{\partial t^{\beta_{i,j}}} + \nu_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}(.)}{\partial x^{\beta_{i,j}}} - \varepsilon_{i,j}\left(D_{+,i,j}(t)\frac{\partial^{\alpha_{i,j}}(.)}{\partial_+ x^{\alpha_{i,j}}} + D_{-,i,j}(t)\frac{\partial^{\alpha_{i,j}}(.)}{\partial_- x^{\alpha_{i,j}}} \right),$$

$$\mathcal{S}(x,t) = \left(S_1(x,t), S_2(x,t) \right)^T, \psi(x,t) = \left(\psi_1(x,t), \psi_2(x,t) \right)^T \text{ and } \psi_0(x) = \left(\widetilde{\psi}_1(x), \widetilde{\psi}_2(x) \right)^T.$$

Formulas of finite differences have been the most dominating methods in the numerical study of different equations, see, e.g. [11] and the references therein. In the most recent studies, the focus has been moved towards some aspects of finite characteristics approach. But, connective terms of the type included in (1.1) are not considered elsewhere. Also, in this paper, we observe the new fractional characteristic finite difference based on stability estimates and convergence analysis for the approximation solution of (1.6) which have not been studied elsewhere.

The contents of the article are as follows. We use maximum principle theorem for such equations in the second Section. In Section 3, we use the fractional characteristic finite difference scheme(FCFD) based on the shifted Grünwald formula and Diethelm's algorithm. Section 4 is devoted to study of the stability estimates and the proof of convergence rates. In the Section 5, we present our results by performing the algorithm on some examples for different $\alpha_{i,j}, \beta_{i,j}$.

2. Existence, uniqueness and stability

The Hölder space $C^{0,\gamma}(\Xi)$ is defined to be the subspace of $C(\Xi)$ functions that are Hölder continuous with the exponent γ. For $0 < \eta < \gamma \leq 1$, we have the obvious relations

$$C^{0,1}(\Xi) \subset C^{0,\gamma}(\Xi) \subset C^{0,\eta}(\Xi) \subset C(\Xi).$$

With the norm

$$\|v\|_{C^{0,\gamma}(\Xi)} = \|v\|_{C(\Xi)} + \sup\{\frac{|v(x) - v(y)|}{\|x - y\|^{\gamma}}| \quad x, y \in \Xi, \quad x \neq y\},$$

the space $C^{0,\gamma}(\Xi)$ becomes a Banach space. We now consider the following generalization of the mixed initial-boundary value problem of (1.1):

$$\begin{cases} \mathcal{L}_{i\alpha,\varepsilon}^{\beta}(\psi_1, \psi_2) = S_i(x,t), & \text{in} \quad (x,t) \in \Omega_T, \quad i = 1, 2 \\ \psi(x, 0) = \psi_0(x), & x \in \Omega, \\ \psi(x, t) = 0, & t \in [0, T], \quad x \in \partial\Omega, \end{cases} \tag{2.1}$$

where

$$\mathcal{L}_{i\alpha,\varepsilon}^{\beta}(\psi_1, \psi_2) = \sum_{j=1}^{2} \omega_{i,j}(x, t) \frac{\partial^{\beta_{i,j}} \psi_j}{\partial t^{\beta_{i,j}}} - \sum_{j=1}^{2} \varepsilon_{i,j} \left(D_{+,i,j}(t) \frac{\partial^{\alpha_{i,j}} \psi_j}{\partial_+ x^{\alpha_{i,j}}} + D_{-,i,j}(t) \frac{\partial^{\alpha_{i,j}} \psi_j}{\partial_- x^{\alpha_{i,j}}} \right)$$
$$+ \sum_{j=1}^{2} \nu_{i,j}(x, t) \frac{\partial^{\beta_{i,j}} \psi_j}{\partial x^{\beta_{i,j}}}.$$

Theorem 2.1. *Let ψ_1, ψ_2 be smooth and assume that $\mathcal{L}_{i\alpha,\varepsilon}^{\beta}(\psi_1, \psi_2) \leq 0$ in Ω_T, $0 < \beta_{i,j} < 1$ and $\beta = \min \beta_{i,j}$. Then*

(i) *ψ_1 and ψ_2 attain theirs maximum on the parabolic boundary Γ_p i.e., the boundary of Ω_T minus the interior of the top part of this boundary, $\Omega \times \{t = T\}$,*

(ii) *the solution of (2.1) satisfies*

$$\|\psi\|_{C^{0,\beta}(\Omega_T)} \leq max\{\|\widetilde{\psi}_1\|_{C^{0,\beta}(\Omega)}, \|\widetilde{\psi}_2\|_{C^{0,\beta}(\Omega)}\} \tag{2.2}$$
$$+ T\|S\|_{C^{0,\beta}(\Omega_T)},$$

and finally

(iii) *the problem (2.1) has at most one solution which is bounded.*

Proof.

(i) If this were not true, then the maximum would be attained either at an interior point of $\Omega \times (0, T)$ or at a point of $\Omega \times \{0, T\}$, i.e. at a point $(\overline{x}, \overline{t}) \in \Omega \times [0, T]$, and we would have

$$\psi_i(\overline{x}, \overline{t}) = \max_{\Omega \times [0,T]} \psi_i = M > m = \max_{\partial\{\Omega \times \{0,T\}\}} \psi_i, \quad i = 1, 2.$$

By our assumptions we have

$$\mathcal{L}_{i\alpha,\varepsilon}^{\beta}(\psi_1, \psi_2) \leq 0 \qquad \Omega \times (0, T).$$

On the other hand, at the point $(\overline{x}, \overline{t})$, where $\overline{\psi_i}$ takes its maximum, we have

$$-\mathcal{L}_{i\alpha,\varepsilon}^{\beta}(\psi_1, \psi_2) \geq 0,$$

and $\frac{\partial \overline{\psi_i}(\overline{x}, \overline{t})}{\partial t} = 0$ if $\overline{t} < T$ or $\frac{\partial \overline{\psi_i}(\overline{x}, \overline{t})}{\partial t} \geq 0$ if $\overline{t} = T$, so that in both cases

$$\mathcal{L}_{i\alpha,\varepsilon}^{\beta}(\psi_1, \psi_2) \geq 0.$$

This is a contraction and thus shows our claim .

(ii) In fact as we shall see, the solution of this problem may be expressed as

$$w(t) = w_0(t)w_1(t)E^\alpha(t)[\phi(x)] + \int_0^t E^\alpha(t-s)[\widetilde{f}(s)]ds, \qquad (2.3)$$

where we write $w(t)$ for $w(.,t)$ and similarly for $\widetilde{f}(s)$ and $E^\alpha(t)[\ ] = (4\pi t)^{-1/2}\int_0^1 \frac{e^{-|x-y|^2/4t}}{|x-y|^\alpha}[\ ]dy$ is a linear operator. This formula represents the solution of the inhomogeneous equations and is referred as Duhael's principle [23]. Clearly, since $E^\alpha(t)$ is bounded in L_2-norm, the right hand side of (2.3) is well defined. It is clear that $\|E^\alpha(t)[g]\|_{C^\alpha} \leq \|g\|_{C^\alpha}$ for $t > 0$. Then we obtain (2.2). By (2.2) we obtain the stability estimate.

(iii) If there were two solutions of (2.3), then their difference would be a solution with initial data zero. It suffices therefore to show that the only bounded solution w of homogeneous (2.3) (that is $\widetilde{f} = 0$) is $w = 0$, or that and if (x_0, t_0) is an arbitrary point in $[0,1] \times [0,T]$, and $\epsilon > 0$ is arbitrary, then $|w(x_0, t_0)| \leq \epsilon$. We introduce the auxiliary function $\widetilde{w}(x,t) = \frac{|x|^2+2}{|x_0|^2+2}$. Let now $H_\pm(x,t) = -\epsilon\widetilde{w}(x,t) \pm w(x,t)$. We conclude $H_{\pm t} - \Im^\alpha(H_\pm) = 0$ in $[0,1] \times [0,T]$. Since w is bounded we have $w(x,t) \leq M$ on $[0,1] \times [0,T]$ for some M. Defining $R^2 = \max(|x_0(1 - x_0)|, M(|x_0(1 - x_0)| + 2t_0)/\epsilon)$, we have $H_\pm(x,t) \leq -\epsilon\frac{R^2}{|x_0(1-x_0)|+2t_0} + M \leq 0$, if $|x| = R$, and $H_\pm(x,0) = \frac{-\epsilon|x_0(1-x_0)|}{|x_0(1-x_0)|+2t_0} \leq 0$, for $x \in [0,1]$. Hence we may apply (i) and conclude $H_\pm(x,t) \leq 0$, for $(x,t) \in [0,1] \times [0,T]$. In particular, at (x_0, t_0) we have $\pm w(x_0, t_0) = H_\pm(x_0, t_0) + \epsilon \leq \epsilon$, which completes the proof.

□

3. Discretization by new fractional characteristic finite difference scheme

The implementation of characteristic finite difference scheme is given by many authors (see e.g. [24]) and Diethelm's algorithm is given by [6] . We combine two numerical schemes Shifted Grunwald and Diethelm's algorithm to solve the quite complex coupled FDE. Let

$$d\psi_i = \omega_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}(\psi_i)}{\partial t^{\beta_{i,j}}} + \nu_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}(\psi_i)}{\partial x^{\beta_{i,j}}} \quad i,j = 1,2 \qquad (3.1)$$

and we assume that the characteristic directions associated with fractional operators $\omega_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}(\psi_i)}{\partial t^{\beta_{i,j}}} + \nu_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}(\psi_i)}{\partial x^{\beta_{i,j}}}$ be denoted by $\tau_{i,j} = \tau_{i,j}(x)$ where

$$\frac{\partial^{\beta_{i,j}}[\ ]}{\partial \tau_{i,j}^{\beta_{i,j}}} = \frac{\omega_{i,j}(x,t)}{\chi_{i,j}(x,t)}\frac{\partial^{\beta_{i,j}}[\ ]}{\partial t^{\beta_{i,j}}} + \frac{\nu_{i,j}(x,t)}{\chi_{i,j}(x,t)}\frac{\partial^{\beta_{i,j}}[\ ]}{\partial x^{\beta_{i,j}}} \quad i,j = 1,2 \qquad (3.2)$$

and

$$\chi_{i,j}(x,t) = \left((\omega_{i,j}(x,t))^2 + (\nu_{i,j}(x,t))^2\right)^{1/2}. \qquad (3.3)$$

Then, the matrix form of equation (1.6) can be written in the following form

$$\mathcal{F}_{\alpha_{i,j}, \varepsilon_{i,j}}^{\beta_{i,j}}\psi_i = \overbrace{\chi_{i,j}(x,t)\frac{\partial^{\beta_{i,j}}\psi_i}{\partial \tau^{\beta_{i,j}}}}^{=\text{\textcircled{R}}} - \underbrace{\varepsilon_{i,j}\left(D_{+,i,j}(t)\frac{\partial^{\alpha_{i,j}}\psi_i}{\partial_+x^{\alpha_{i,j}}} + D_{-,i,j}(t)\frac{\partial^{\alpha_{i,j}}\psi_i}{\partial_-x^{\alpha_{i,j}}}\right)}_{=\text{\textcircled{S}}} \quad i,j = 1,2. \qquad (3.4)$$

Thus, we compute two terms of (3.4) i.e. Ⓡ by Diethelm's algorithm and Ⓢ by the approximate characteristics. In each of the procedure to be treated below we shall consider a time step $t^n = n\Delta t$ for a positive integer n, $\Delta t = \frac{T}{N}$, $N \in \mathbb{Z}^+$ and for any $x \in [x_L, x_R]$ we define a fractional backward characteristic tracking by:

$$r^{\beta_{i,j}}(t; x; t^{n+1}) = x + \left(-\frac{\nu_{i,j}(x, t^{n+1})}{\omega_{i,j}(x, t^{n+1})}\right)^{\frac{1}{\beta_{i,j}}}(t^{n+1} - t), \qquad (3.5)$$

$$\overline{x}_{\beta_{i,j}} = r^{\beta_{i,j}}(t^n; x; t^{n+1}) = x + \left(-\frac{\nu_{i,j}(x, t^{n+1})}{\omega_{i,j}(x, t^{n+1})}\right)^{\frac{1}{\beta_{i,j}}}\Delta t. \qquad (3.6)$$

Also, we shall consider a space step $x^m = m\Delta x$ for a positive integer m, $\Delta x = \frac{x_R - x_L}{M}$, and $M \in \mathbb{Z}^+$. Therefore, we write:

$$\begin{aligned}
Ⓡ &= \omega_{i,j}(x, t)\frac{\partial^{\beta_{i,j}}\psi_i}{\partial t^{\beta_{i,j}}} + \nu_{i,j}(x, t)\frac{\partial^{\beta_{i,j}}\psi_i}{\partial x^{\beta_{i,j}}} \\
&= \omega_{i,j}(x, t)\frac{t^{-n\beta_{i,j}}}{\Gamma(-\beta_{i,j})}\int_0^1 \frac{\overbrace{\psi_i(x, t^n - t^n\eta)}^{=g_1(x,\eta)}}{\eta^{\beta_{i,j}+1}}\,d\eta \\
&\quad + \nu_{i,j}(x, t)\frac{x^{-m\beta_{i,j}}}{\Gamma(-\beta_{i,j})}\int_0^1 \frac{\overbrace{\psi_i(x^m - x^m\eta, t)}^{=g_2(\eta,t)}}{\eta^{\beta_{i,j}+1}}\,d\eta.
\end{aligned}$$

Hence, according to Diethelm's algorithm [6], we have:

$$Ⓡ = \omega_{i,j}(x, t)\frac{(t^n)^{-\beta_{i,j}}}{\Gamma(-\beta_{i,j})}\left(Q_n(g_1) + R_n(g_1)\right) + \nu_{i,j}(x, t)\frac{(x^m)^{-\beta_{i,j}}}{\Gamma(-\beta_{i,j})}\left(Q_m(g_2) + R_m(g_2)\right), \qquad (3.7)$$

where

$$Q_n(g_1) = \sum_{k=0}^{n} \lambda_{k,n}^{i,j}g_1(x, \frac{k}{n}) \approx \int_0^1 g_1(x, w)w^{-\beta_{i,j}-1}dw, \quad R_n(g_1) = \mathcal{O}((\Delta t)^{2-\beta_{i,j}}),$$

$$Q_m(g_2) = \sum_{k=0}^{m} \lambda_{k,m}^{i,j}g_2(\frac{k}{m}, t) \approx \int_0^1 g_2(w, t)w^{-\beta_{i,j}-1}dw, \quad R_m(g_2) = \mathcal{O}((\Delta x)^{2-\beta_{i,j}}),$$

and in the following explicit expressions for the weights $\lambda_{k,q}^{i,j}$, $q = n$ or $q = m$ are given (cf. [8]):

$$\lambda_{k,q}^{i,j} = \frac{1}{\beta_{i,j}(1 - \beta_{i,j})q^{-\beta_{i,j}}}\begin{cases} -1 & \text{for } k = 0 \\ 2k^{1-\beta_{i,j}} - (k-1)^{1-\beta_{i,j}} - (k+1)^{1-\beta_{i,j}} & \text{for } k = 1, 2, ..., q-1 \\ (-1 + \beta_{i,j})k^{-\beta_{i,j}} - (k-1)^{1-\beta_{i,j}} + k^{1-\beta_{i,j}} & \text{for } k = q. \end{cases} \qquad (3.8)$$

For computing Ⓢ, we use the shifted Grünwald approximations in discrete forms at each nodes x introduced in [4, 9]:

$$\begin{cases} \frac{\partial^{\alpha_{i,j}}\psi_i(x,t)}{\partial_+ x^{\alpha_{i,j}}} = \frac{1}{(\Delta x)^{\alpha_{i,j}}} \sum_{k=0}^{\lfloor l_1 \rfloor +1} g_k^{(\alpha_{i,j})} \psi_i(x - (k-1)\Delta x, t) + a_1 \frac{\partial^{\alpha_{i,j}+1}\psi_i(x,t)}{\partial_+ x^{\alpha_{i,j}+1}} \Delta x + \mathcal{O}((\Delta x)^2), \\[3mm] \frac{\partial^{\alpha_{i,j}}\psi_i(x,t)}{\partial_- x^{\alpha_{i,j}}} = \frac{1}{(\Delta x)^{\alpha_{i,j}}} \sum_{k=0}^{\lfloor l_2 \rfloor +1} g_k^{(\alpha_{i,j})} \psi_i(x + (k-1)\Delta x, t) + b_1 \frac{\partial^{\alpha_{i,j}+1}\psi_i(x,t)}{\partial_+ x^{\alpha_{i,j}+1}} h + \mathcal{O}((\Delta x)^2), \end{cases} \quad (3.9)$$

where $l_1 = \frac{x - x_L}{\Delta x}$ and $l_2 = \frac{x_R - x}{\Delta x}$, with $\lfloor l_1 \rfloor$ and $\lfloor l_2 \rfloor$ being the floor of l_1 and l_2, respectively. Also, a_1 and b_1 are constant and independent of h, ψ_i, x or t. Moreover, the coefficients $g_k^{(\alpha_{i,j})}$ are evaluated recursively.

$$\begin{cases} g_0^{(\alpha_{i,j})} = 1, \\[3mm] g_k^{(\alpha_{i,j})} = \left(1 - \frac{\alpha+1}{k}\right) g_{k-1}^{(\alpha_{i,j})} \quad \text{for } k \geq 1. \end{cases} \quad (3.10)$$

Therefore, the following lemma plays an important role in this method.

Lemma 3.1. *By the above discretization and the fractional characteristic finite difference method for* (1.7) *we have the following system:*

$$A^{n+1}\Psi^{n+1} = \sum_{k=1}^{n} B^k \Psi^k + S^{n+1}, \quad (3.11)$$

where $A^{n+1} = \begin{bmatrix} A_{11}^{n+1} & A_{12}^{n+1} \\ A_{21}^{n+1} & A_{22}^{n+1} \end{bmatrix}$, $\Psi^{n+1} = \begin{bmatrix} \Psi_1^{n+1} \\ \Psi_2^{n+1} \end{bmatrix}$, $B^k = \begin{bmatrix} B_{11}^k & B_{12}^k \\ B_{21}^k & B_{22}^k \end{bmatrix}$ *and* $S^{n+1} = \begin{bmatrix} S_1^{n+1} \\ S_2^{n+1} \end{bmatrix}$.

Remark 3.2. We stress on the fact that the above notations are introduced in the proof.

Proof. By (3.4) - (3.9) and the fully discretization form of (1.6), we conclude that

$$\mathcal{F}_{\alpha_{i,j},\varepsilon_{i,j}}^{\beta_{i,j}}\psi_i(x,t) \mid_{(x^l,t^{n+1})}$$

$$= \textcircled{R} + \textcircled{S} \simeq \widetilde{\mathcal{F}_{\alpha_{i,j},\varepsilon_{i,j}}^{\beta_{i,j}}\psi_i}$$

$$= \omega_{i,j}(x^l, t^{n+1})\frac{(t^n)^{-\beta_{i,j}}}{\Gamma(-\beta_{i,j})} \sum_{k=0}^{n} \lambda_{k,n}^{i,j} \left(\frac{k}{n}\right)^{-\beta_{i,j}-1} \frac{\psi_i(x^l, t^n - t^n(\frac{k}{n})) + \psi_i(\overline{x}_{\beta_{i,j}}, t^n - t^n(\frac{k}{n}))}{2}$$

$$+ \nu_{i,j}(x^l, t^{n+1})\frac{(x^m)^{-\beta_{i,j}}}{\Gamma(-\beta_{i,j})} \sum_{k=0}^{m} \lambda_{k,m}^{i,j} \left(\frac{k}{m}\right)^{-\beta_{i,j}-1} \psi_i(x^l - x^l(\frac{k}{m}), t^{n+1}) \quad (3.12)$$

$$- \frac{\varepsilon_{i,j}D_{+,i,j}^{n+1}}{(\Delta x)_{i,j}^{\alpha}} \sum_{k=0}^{l+1} g_k^{(\alpha_{i,j})} \psi_i(x^{l-k+1}, t^{n+1})$$

$$- \frac{\varepsilon_{i,j}D_{-,i,j}^{n+1}}{(\Delta x)_{i,j}^{\alpha}} \sum_{k=0}^{M-l+1} g_k^{(\alpha_{i,j})} \psi_i(x^{l+k-1}, t^{n+1}),$$

where $D_{-,i,j}^{n+1} = D_{-,i,j}(t^{n+1})$ for $i, j \in \{1,2\}$, $l = 1, 2, ..., M-1$, $n = 0, 1, ..., N-1$, $\psi_1(x^l, 0) = \psi_{10}(x^l)$, $\psi_2(x^l, 0) = \psi_{20}(x^l)$, $\psi_{10}(x), \psi_{20}(x)$ and $S_j^{n+1,l} = S_j(x^l, t^{n+1})$ are given functions.

Let $f_{i,j}^k = \lambda_{k,n}^{i,j} \left(\frac{k}{n}\right)^{-\beta_{i,j}-1}$, $\delta_{i,j}^{l,n+1} = \omega_{i,j}(x^l, t^{n+1})\frac{(t^n)^{-\beta_{i,j}}}{2\Gamma(-\beta_{i,j})}$, $\gamma_{i,j}^{l,n+1} = \nu_{i,j}(x^l, t^{n+1})\frac{(x^m)^{-\beta_{i,j}}}{\Gamma(-\beta_{i,j})}$, $\xi_{i,j}^{n+1} = \frac{\varepsilon_{i,j}D_{-,i,j}^{n+1}}{(\Delta x)_{i,j}^{\alpha}}$ and $\eta_{i,j}^{n+1} = \frac{\varepsilon_{i,j}D_{+,i,j}^{n+1}}{(\Delta x)_{i,j}^{\alpha}}$. Thus, we have the following system:

$$
\left\{
\begin{aligned}
&\delta_{1,j}^{l,n+1} \sum_{k=0}^{n} f_{1,j}^{k} \psi_1(x^l, t^n - t^n(\tfrac{k}{n})) + \delta_{1,j}^{l,n+1} \sum_{k=0}^{n} f_{1,j}^{k} \psi_1(\overline{x}_{\beta_{1,j}}, t^n - t^n(\tfrac{k}{n})) \\
&+\gamma_{1,j}^{l,n+1} \sum_{k=0}^{m} f_{1,j}^{k} \psi_1(x^l - x^l(\tfrac{k}{m}), t^{n+1}) - \eta_{1,j}^{n+1} \sum_{k=0}^{l+1} g_k^{(\alpha_{1,j})} \psi_i(x^{l-k+1}, t^{n+1}) \\
&-\xi_{1,j}^{n+1} \sum_{k=0}^{M-l+1} g_k^{(\alpha_{1,j})} \psi_1(x^{l+k-1}, t^{n+1}) \\
&+\delta_{2,j}^{2,n+1} \sum_{k=0}^{n} f_{2,j}^{k} \psi_2(x^l, t^n - t^n(\tfrac{k}{n})) + \delta_{2,j}^{2,n+1} \sum_{k=0}^{n} f_{2,j}^{k} \psi_2(\overline{x}_{\beta_{2,j}}, t^n - t^n(\tfrac{k}{n})) \\
&+\gamma_{2,j}^{2,n+1} \sum_{k=0}^{m} f_{2,j}^{k} \psi_2(x^l - x^l(\tfrac{k}{m}), t^{n+1}) - \eta_{2,j}^{n+1} \sum_{k=0}^{l+1} g_k^{(\alpha_{2,j})} \psi_2(x^{l-k+1}, t^{n+1}) \\
&-\xi_{2,j}^{n+1} \sum_{k=0}^{M-l+1} g_k^{(\alpha_{2,j})} \psi_2(x^{l+k-1}, t^{n+1}) = S_j^{n+1,l}, \\
&\quad j = 1, 2.
\end{aligned}
\right. \tag{3.13}
$$

We let the node functions $\Psi(x^l, t^n) = \Psi_l^n = (\Psi_{1,l}^n, \Psi_{2,l}^n)$ be the numerical approximation to the true solution $\psi(x^l, t^n) = \psi_l^n = (\psi_{1,l}^n, \psi_{2,l}^n)$ and $\Psi(\overline{x}_{\beta_{i,j}}, t^n) = \overline{\Psi}^n = (\overline{\Psi}_1^n, \overline{\Psi}_2^n)$ is evaluated by the Courant number(see for more details [20]). As a result, we obtain the above iterative formula i.e. (3.11) by:

$$
a_{l,k,i,j}^{n+1} = \left\{
\begin{aligned}
&1 + (\delta_{i,j}^{l,n+1} \gamma_{i,j}^{l,n+1}) f_{i,j}^{k} - (\xi_{i,j}^{n+1} + \eta_{i,j}^{n+1})\varepsilon_{i,j} g_1^{(\alpha_{i,j})}, && l = k, \\
&(\delta_{i,j}^{l,n+1} \gamma_{i,j}^{l,n+1}) f_{i,j}^{k} - (\xi_{i,j}^{n+1} g_2^{(\alpha_{i,j})} + \eta_{i,j}^{n+1} g_0^{(\alpha_{i,j})})\varepsilon_{i,j}, && l = k-1, \\
&(\delta_{i,j}^{l,n+1} \gamma_{i,j}^{l,n+1}) f_{i,j}^{k} - (\xi_{i,j}^{n+1} g_0^{(\alpha_{i,j})} + \eta_{i,j}^{n+1} g_2^{(\alpha_{i,j})})\varepsilon_{i,j}, && l = k+1, \\
&(\delta_{i,j}^{l,n+1} \gamma_{i,j}^{l,n+1}) f_{i,j}^{k} - \xi_{i,j}^{n+1} \varepsilon_{i,j} g_{k-l+1}^{(\alpha_{i,j})}, && l < k-1, \\
&(\delta_{i,j}^{l,n+1} \gamma_{i,j}^{l,n+1}) f_{i,j}^{k} - \eta_{i,j}^{n+1} \varepsilon_{i,j} g_{l-k+1}^{(\alpha_{i,j})}, && l > k-1,
\end{aligned}
\right. \tag{3.14}
$$

where $A_{i,j}^{n+1} = (a_{l,k,i,j}^{n+1})$, $i, j = 1, 2$. Also, we obtain $B_{i,j}^k = (b_{l,k,i,j}^k)$ such that

$$
b_{l,k,i,j}^k = \left\{
\begin{aligned}
&1 - Cr_l^{n,*,i,j}, && l = k - [Cr_l^{n,*,i,j}], \\
&Cr_l^{n,*,i,j}, && l = k - [Cr_l^{n,*,i,j}] - 1, \\
&0 && \text{otherwise,}
\end{aligned}
\right. \tag{3.15}
$$

where $Cr_l^{n,i,j} = \left(-\dfrac{v_{i,j}(x^l, t^n)}{w_{i,j}(x^l, t^n)}\right)^{\frac{1}{\beta_{i,j}}} \dfrac{\Delta t}{\Delta x}$ is the Courant number [20, 23] and $Cr_l^{n,*,i,j}$ is the fractional part of the Courant number. □

4. Stability and the convergence analysis

In the section, we analyze the stability and the convergence behavior of the characteristic finite difference method.

Theorem 4.1. *The characteristic finite difference scheme (3.11) is unconditionally stable in the L^∞ norm for $1 < \alpha_{i,j} \le 2$ and $0 < \beta_{i,j} \le 1$. In particular matrices A^{n+1} and B^k define in (3.14) and (3.15) below satisfy*

$$
\|(A^{n+1})^{-1} B^n\|_\infty \le 1, \|(A_{i,j}^{n+1})^{-1}\|_\infty \le 1 \quad i, j = 1, 2. \tag{4.1}
$$

Proof. According to (3.15), $0 < \beta_{i,j} \le 1$ and the results of the Courant number in [20, 23], we conclude that $\|B_{i,j}^k\|_\infty \le 1$, $i, j = 1, 2$. Also, if we use (3.14) and $1 < \alpha_{i,j} \le 2$ than $A_{i,j}^{n+1}$ is

diagonally dominant by rows and $det(A_{i,j}^{n+1}) \neq 0$, therefore we have $\|(A_{i,j}^{n+1})^{-1}\|_\infty \leq 1$. Finally, we referred to literature in linear algebra and matrix theory, e.g. G. Golub, [12] and this conclude the proof.   □

Theorem 4.2. *Assume that the true solution of problem* (1.1) *and the numerical solution of the fractional characteristic finite difference scheme of problem* (3.11) *denote by* $\psi(x,t)$, $\Psi_h(x,t)$ *respectively. Then, the following order of error estimate holds for* $\Delta t = \Delta x = h$, *and for the residual* $R[\psi(x,t)] = \mathcal{F}_{\alpha,\varepsilon}^\beta(\psi(x,t)) - \mathcal{S}(x,t)$ *we have:*

$$e^h := \|R[\Psi(x,t)]\|_2 = \mathcal{O}(h^{2-\beta}), \quad \beta = \max_{i,j=1,2} \beta_{i,j}.$$

Proof. We consider (3.11) and by using (3.7) and (3.9), we obtain the global truncation error see Lemma 2.1 [8] and Diethelm's algorithm [6]. Hence, for finding the global truncation error, we apply Theorem 4.1 and norm equivalences in $\mathbb{R}^n$ (e.g. G. Golub, [12]) .   □

5. Experimental results and some real world applications

We recall that in 1827, Robert Brown looked through a microscope at particles trapped in cavities inside pollen grains in water, he noted that the particles moved through the water but was not able to determine the mechanisms that caused this motion. Atoms and molecules had long been theorized as the constituents of matter, and many decades later, Albert Einstein published a paper in 1905 that explained in precise detail how the motion that Brown had observed was a result of the pollen being moved by individual water molecules. This explanation of Brownian motion served as definitive confirmation that atoms and molecules actually exist, and was further verified experimentally by Jean Perrin in 1908. The mathematical model of Brownian motion has numerous real-world applications. For instance, disinfection areas and sterilization in health care facilities are often cited [16, 17].

In this section, we examine the results by an example of fractional model of disinfection and sterilization of tetanus. By Euclidean distance map, we portrait an three dimension of area into the $[x_L, x_R]$. Also, we assumed that ψ_1, ψ_2, S_1 and S_2 denote the the density (number of particles per unit volume) of infective and susceptible, recovered and vaccinated individuals in the population for an area with environmental conditions, respectively. Finally, we report the best results for this example. Furthermore, Carducci et al. [3] show that they consider an application on tetanus, and this example is not a purely mathematical test case without any practical use.

We observe that if we choose $\omega_{i,j}(x,t) = -\nu_{i,j}(x,t) = \sin^2(x+t)$, $S_1(x,t) = e^{-(x+t)} - e^{-(1+t)}$, $S_2(x,t) = e^t - e^{-(x^2-t-1)}$, $\epsilon_{i,j}D_{+,i,j} = \epsilon_{i,j}D_{-,i,j} = 10^{-3}$, $\alpha_{i,j} = 1.8$ and $\beta_{i,j} = 0.2$, $i,j = 1,2$. Then the following couple system arises in dynamics [16, 17]. Therefore, the system (1.1) for $n = m$, $x_L = 0$ and $x_R = T = 1$ can be written in a matrix form as

$$\begin{cases} \mathcal{F}_{\alpha,\varepsilon}^\beta(\psi(x,t)) = \mathcal{S}(x,t), & \text{in} \quad (x,t) \in \Omega_T, \\ \psi(x,0) = x, & x \in [0,1], \\ \psi(0,t) = 0, & t \in [0,1], \\ \psi(1,t) = 0, & t \in [0,1]. \end{cases} \tag{5.1}$$

Moreover, we carry out the above algorithm, by an AMD Opteron computer where 15 Gigabytes RAM memory with 2.2 GHz CPU has been used for these experiments. Let $\Psi(x,t)$ be an approximated solution for this algorithm then the error quantity is $e^h = \|R[\Psi(x,t)]\|_2$. The pointwise error quantity

and the discrete norm of error for this approximation method with reference solution is given in Figure 1 and the order of convergence rates are investigated in Figures of 3–5. Also, we show the global assembled matrix in Figure 2. We know that the similar band matrix occurs in many areas of linear algebra. Because of the simple description of the matrix operation and eigenvalues/eigenvectors given, we confirm the stability and convergence analysis for this method. From a computational point of view, working with band matrices is always preferential to working with similarly dimensioned square matrices. A band matrix can be likened in complexity to a rectangular matrix whose row dimension is equal to the bandwidth of the band matrix. Thus the work involved in performing operations such as multiplication falls significantly, often leading to huge savings in terms of calculation time and complexity. Finally, by using Figures, we conclude that $e^h = \|R[\Psi(x,t)]\|_2 = \mathcal{O}(h^{2-\beta})$.
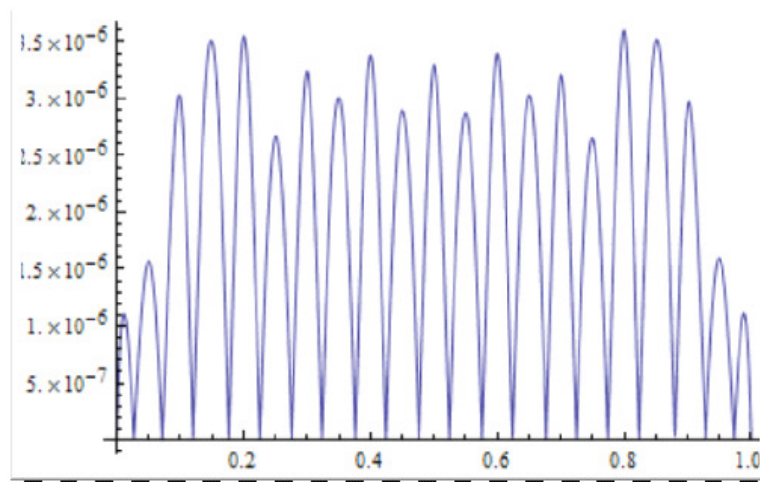

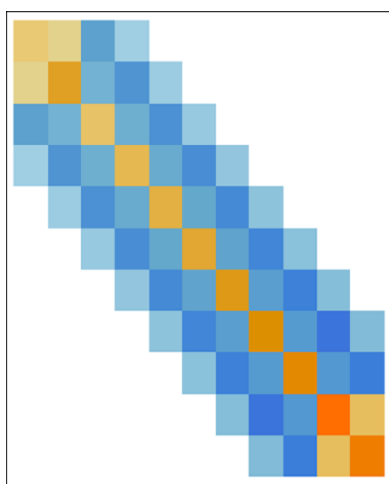Figure 1: Plot of error function $e^h = \|R[\Psi(x,t)]\|_2$ for $n = 25$.
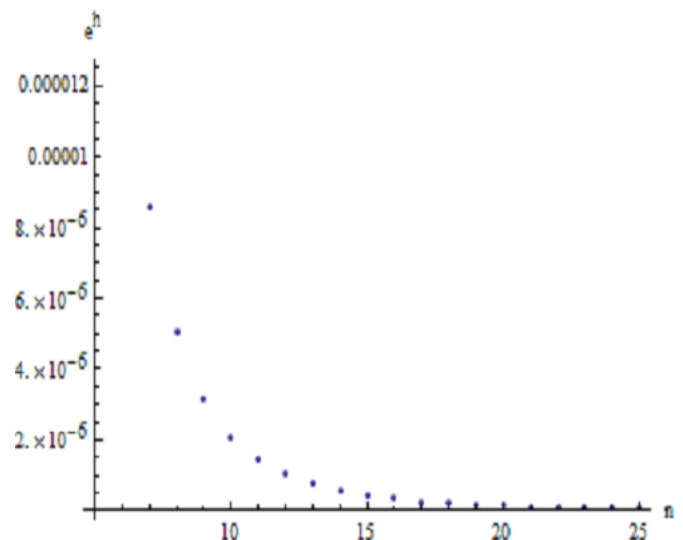

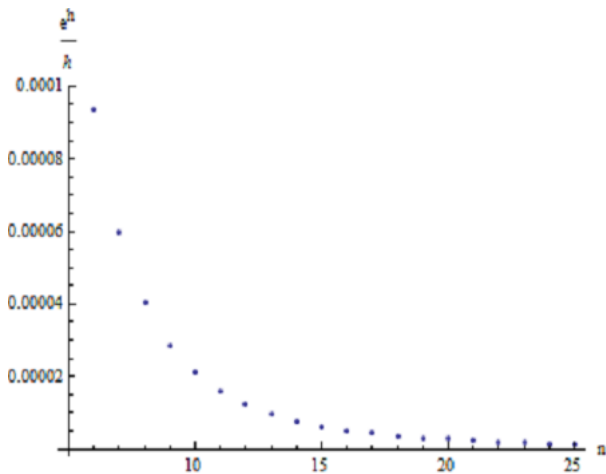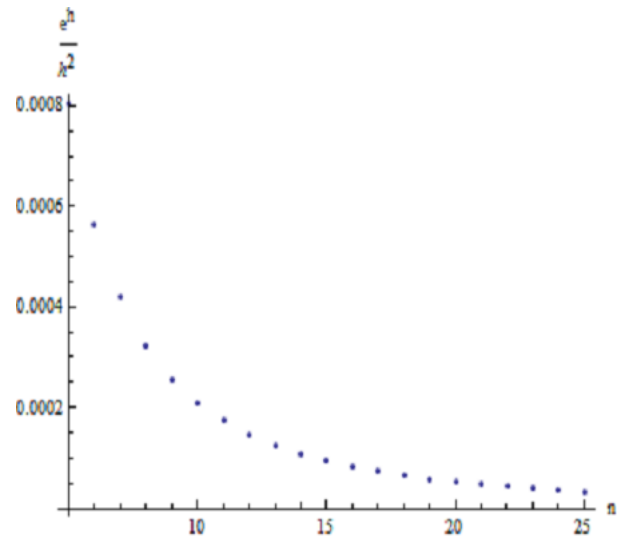Figure 2: Graph of matrix $A_{i,j}^n$.

Figure 3: Plot of convergence rate for e^h.

Figure 4: Plot of convergence rate for $\frac{e^h}{h}$.

Figure 5: Plot of convergence rate for $\frac{e^h}{h^2}$.

6. Conclusion

To this end, we have constructed a fractional characteristic finite difference based on the shifted Grünwald and Diethelm formulas in space and time for solving the fractional coupled system of equations. Stability and convergence analysis are very powerful mathematical tools in this problem by this method. We investigated them by some theorems and numerical experimental results.

Acknowledgments

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