J. Math. Computer Sci., 27 (2022), 105–117

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Coefficient bounds for Al-Oboudi type bi-univalent functions connected with a modified sigmoid activation function and k-Fibonacci numbers

Ala Amourah[a], Basem Aref Frasin[b,*], Sondekola Rudra Swamy[c], Yerragunta Sailaja[d]

[a]Department of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid, Jordan.
[b]Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan.
[c]Department of Computer Science and Engineering, RV College of Engineering, Bengaluru, 560 059, Karnataka, India.
[d]Department of Mathematics, RV College of Engineering, Bengaluru, 560 059, Karnataka, India.

Abstract

Using the Al-Oboudi type operator, we present and investigate two special families of bi-univalent functions connected with the activation function $\phi(s) = 2/(1+e^{-s})$, $s \in \mathbb{R}$ and k-Fibonacci numbers. We derive the bounds on initial coefficients and the Fekete-Szegö functional for functions of the type $g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j$ in these introduced families. Furthermore, we present interesting observations of the results investigated.

Keywords: Fekete-Szegö inequality, regular function, Sigmoid function, Fibonacci numbers, bi-univalent function.

2020 MSC: 30C45, 33C45, 11B65.

1. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\}$ be the sets of real numbers and positive integers, respectively. Let $\mathbb{C}$ be the sets of complex numbers and the set of normalized regular functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ is symbolized by $\mathcal{A}$. Such a function $g \in \mathcal{A}$ has the expansion about the origin in the form

$$g(z) = z + d_2 z^2 + d_3 z^3 + \cdots = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1.1}$$

and the set of all elements of $\mathcal{A}$ that are univalent in $\mathfrak{D}$ is symbolized by $\mathcal{S}$. The famous Koebe theorem (see[12]) ensures that every function $g \in \mathcal{S}$ has an inverse g^{-1} satisfying $g^{-1}(g(z)) = z$, $z \in \mathfrak{D}$, $g(g^{-1}(\omega)) = \omega$, $|\omega| < r_0(g)$, $\omega \in \mathfrak{D}$ and $1/4 \leqslant r_0(g)$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2 d_3 + d_4)\omega^4 + \cdots . \tag{1.2}$$

*Corresponding author
Email addresses: alaammour@yahoo.com (Ala Amourah), bafrasin@yahoo.com (Basem Aref Frasin), mailtoswamy@rediffmail.com (Sondekola Rudra Swamy), sailajay@rvce.edu.in (Yerragunta Sailaja)

doi: 10.22436/jmcs.027.02.02

Received: 2021-12-08 *Revised:* 2021-12-21 *Accepted:* 2022-01-15

A function g of $\mathcal{A}$ is called bi-univalent in $\mathfrak{D}$ if g^{-1} and g are both in $\mathcal{S}$. Let Σ denote the set of bi-univalent functions having the form (1.1). The functions $\frac{z}{1-z}$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and $-\log(1-z)$ are in the class Σ, but the Koebe function as well as $\frac{z}{1-z^2}$ and $z-\frac{z^2}{2}$ (members of $\mathcal{S}$) are not members the class Σ. Historically investigations of the family Σ begun fifty years ago by Lewin [30] and Brannan and Clunie [9]. Few years later, Tan [45] found the upper bounds for few coefficients of bi-univalent functions. Brannan and Taha [10] presented certain subfamilies of Σ similar to known subfamilies of convex and starlike functions of order σ $(0 \leqslant \sigma < 1)$ in $\mathfrak{D}$. Many results concerning initial bounds for some special families of Σ have been found in [6, 11, 23, 39]. However the general coefficient bounds for many of the special families of functions $g \in \Sigma$ are not completely addressed.

In the paper [19], Falcón and Plaza have investigated the sequence of k-Fibonacci number $\{F_{k,j}\}_{j=0}^{\infty}$, $k \in \mathbb{R}^+$, defined by

$$F_{k,j+1} = kF_{k,j} + F_{k,j-1},\ j \in \mathbb{N} \tag{1.3}$$

with $F_{k,0} = 0, F_{k,1} = 1$, and

$$F_{k,j} = \frac{(k-t_k)^j - t_k^j}{\sqrt{k^2+4}} \quad \text{where} \quad t_k = \frac{k-\sqrt{k^2+4}}{2}. \tag{1.4}$$

If $k = 1$, then we get the familiar Fibonacci numbers F_j.

Özgür and Sokól [34] in 2015 proved that if

$$\widetilde{p}_k(z) = \frac{1+t_k^2 z^2}{1-kt_k z - t_k^2 z^2},\ z \in \mathfrak{D}, \tag{1.5}$$

then

$$\begin{aligned}\widetilde{p}_k(z) &= 1 + (F_{k,0}+F_{k,2})t_k z + (F_{k,1}+F_{k,3})t_k^2 z^2 + \cdots \\ &= 1 + k\,t_k z + (k^2+2)t_k^2 z^2 + \cdots,\end{aligned}$$

where $t_k = \frac{k-\sqrt{k^2+4}}{2}$. Clearly if $\widetilde{p}_k(z) = 1 + \Sigma \widetilde{p}_{k,j} z^j$, then we have

$$\widetilde{p}_{k,j} = (F_{k,j-1} + F_{k,j+1})\, t_k^j,\ j \in \mathbb{N}.$$

The bounds for first two coefficients and the celebrated Fekete- Szegö inequality were found for bi-univalent functions connected with certain polynomials like (p,q)-Lucas polynomials, Chebyshev polynomials, Fibonacci polynomials, Gegenbauer polynomials and Horadam polynomials. We also note that these polynomials as well as their extensions, are potentially very important in a variety of physical, statistical, engineering, and mathematical disciplines. Additional information related to these polynomials can be found in [2, 4, 7, 8, 21, 22, 46]. More about the estimates on initial coefficient bounds and the solution of Fekete- Szegö problem for functions in Σ linked with k-Fibonacci numbers can be seen in [5, 13, 14, 24, 26, 27].

The recent research trend is the study of functions of Σ associated with any of the above mentioned polynomials using well-known operators, which can be seen in the research papers [1, 15, 25, 32, 36, 40–44]. Generally interest was shown to estimate the first two coefficients and the Fekete-Szegö inequality for the introduced families of Σ using known operators.

In Mathematics, an activation function plays an important role for scientist and engineers. The function that maps the net input to the output signal value is known as the activation function. Certain functions such as the identity function, the step function, the sigmoid function, the hyperbolic tangent function etc. are widely used as activation function in an artificial neural network. Some artificial neural network training algorithm requires that the activation function be continuous and differentiable. The step function is not suitable for such cases. The sigmoid function $\xi(s) = 1/(1+e^{-s})$, $s \in \mathbb{R}$ has the nice property that they can approximate the step function to the desired extent without loosing its differentiability. The sigmoid function has the following advantages:

i) it increases monotonically;
ii) it gives real numbers between 0 and 1;
iii) it never loses information because it is one-to-one function; and
iv) it maps a very large domain into a small range of outputs. For more details about activation functions, see [28].

In the real-valued case, a lot of research has been devoted to this topic, but a very limited literature exists for complex valued neural networks, where most activation functions are generally developed in a split fashion (i.e., by considering the real and imaginary parts of the activation separately) or with simple phase-amplitude techniques (for more knowledge, see [38]). In this direction, Ezeafulukwe et al. [16] proposed and studied certain properties of complex-valued sigmoid function $G(z) = 1/(1+e^{-z})$, $z \in \mathfrak{D}$. In this article, they have found the starlikeness and convexity of a sigmoid function $G(z)$. Fadipe-Joseph et al. [18] have studied some properties of the modified sigmoid function $F(z) = 2/(1+e^{-z})$, $z \in \mathfrak{D}$. For more details, see [29, 33].

Let $\mathcal{A}_\phi$ be the set of regular functions of the form

$$g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j,$$

where $\phi(s)$ is the real-valued modified sigmoid function defined by

$$\phi(s) = 2/(1+e^{-s}),\ s \in \mathbb{R}. \tag{1.6}$$

Clearly $\phi(0) = 1$ and hence $\mathcal{A}_1 := \mathcal{A}$ (see [17]).

Definition 1.1. For $g_\phi \in \mathcal{A}_\phi$, $n \in \mathbb{N}_0$, $\beta \geqslant 0$, an Al-Oboudi type operator $D^n_\beta : \mathcal{A}_\phi \to \mathcal{A}_\phi$, is defined by $D^0_\beta g_\phi(z) = g_\phi(z)$, $D^1_\beta g_\phi(z) = (1-\beta)g_\phi(z) + \beta z g'_\phi(z), \ldots, D^n_\beta g_\phi(z) = D_\beta(D^{n-1}_\beta g_\phi(z))$, $z \in \mathfrak{D}$.

Remark 1.2. If $g_\phi(z) = z + \Sigma\phi(s) d_j z^j \in \mathcal{A}_\phi$, then

$$D^n_\beta g_\phi(z) = z + \sum_{j=2}^{\infty} (1+(j-1)\beta)^n \phi(s) d_j z^j, \quad z \in \mathfrak{D}.$$

If $\phi(s) = 1$, then we obtain the operator due to Al-Oboudi [3], which reduces to the operator presented by Sălăgean in [37], when $\beta = 1$.

For regular functions g and f in $\mathfrak{D}$, g is said to subordinate to f, if there is a Schwarz function ψ in $\mathfrak{D}$, $|\psi(z)| < 1$, $\psi(0) = 0$ such that $g(z) = f(\psi(z))$, $z \in \mathfrak{D}$. This is indicated as $g \prec f$ or $g(z) \prec f(z)$. Specifically, when $f \in \mathcal{S}$ in $\mathfrak{D}$, then $g(z) \prec f(z) \iff g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Motivated by the recent papers on bi-univalent functions using the real-valued modified sigmoid function $\phi(s)$ given by (1.6), we present two new special families of Σ making use of the Al-Oboudi type operator which was precisely defined in the paper [25], and k-Fibonacci numbers defined by the formula (1.3) with $F_{k,j}$ as in (1.4). We determine the initial coefficient estimates and also obtain the relevant connection to the celebrated Fekete-Szegö functional for functions in these new families.

Throughout this paper we assume that $T_k = k - (k^2+2)t_k$ with t_k as in (1.4), $\widetilde{p}_k$ as given by (1.5) and the modified sigmoid activation function $\phi(s)$ as in (1.6).

Definition 1.3. A function $g \in \Sigma$ having the power series as in (1.1) is said to be in the set $\mathcal{SR}\mathfrak{S}_\Sigma(\mu,\gamma,\ \beta,\ n,\ \phi(s),\widetilde{p}_k)$, if

$$\frac{z(D^n_\beta g_\phi(z))' + \mu z^2(D^n_\beta g_\phi(z))''}{(1-\gamma)z + \gamma D^n_\beta g_\phi(z)} \prec \widetilde{p}_k(z), \quad z \in \mathfrak{D},$$

and

$$\frac{\omega(D^n_\beta f_\phi(\omega))' + \mu\omega^2(D^n_\beta f_\phi(\omega))''}{(1-\gamma)\omega + \gamma D^n_\beta f_\phi(\omega)} \prec \widetilde{p}_k(\omega), \quad \omega \in \mathfrak{D},$$

where $\mu \geqslant 0$, $\beta \geqslant 0$, $0 \leqslant \gamma \leqslant 1$, $n \in \mathbb{N}_0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).

It is interesting to note the following subsets of $\mathcal{SR}\mathfrak{S}_\Sigma(\mu,\gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$ which are obtained for certain choices of μ and γ:

1. $\mathcal{SRK}_\Sigma(\gamma,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv \mathcal{SR}\mathfrak{S}_\Sigma(0,\gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$ is the set of functions $g \in \Sigma$ satisfying

$$\frac{z(D^n_\beta g_\phi(z))'}{(1-\gamma)z + \gamma D^n_\beta g_\phi(z)} \prec \widetilde{p}_k(z) \quad \text{and} \quad \frac{\omega(D^n_\beta f_\phi(\omega))'}{(1-\gamma)\omega + \gamma D^n_\beta f_\phi(\omega)} \prec \widetilde{p}_k(\omega), \quad z,\omega \in \mathfrak{D},$$

   where $\beta \geqslant 0$, $0 \leqslant \gamma \leqslant 1$, $n \in \mathbb{N}_0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).
2. When $\gamma = 0$, we have $\mathcal{SRL}_\Sigma(\mu,\ \beta,\ n,\ \phi(s),\widetilde{p}_k) \equiv \mathcal{SR}\mathfrak{S}_\Sigma(\mu,0,\ \beta,\ n,\phi(s),\widetilde{p}_k)$, the group of functions $g \in \Sigma$ satisfying

$$(D^n_\beta g_\phi(z))' + \mu z(D^n_\beta g_\phi(z))'' \prec \widetilde{p}_k(z) \text{ and } (D^n_\beta f_\phi(\omega))' + \mu\,\omega(D^n_\beta f_\phi(\omega))'' \prec \widetilde{p}_k(\omega),$$

   where $z,\omega \in \mathfrak{D}$, $n \in \mathbb{N}_0$, $\beta \geqslant 0$, $\mu \geqslant 0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).
3. $\mathcal{SRM}_\Sigma(\mu,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv \mathcal{SR}\mathfrak{S}_\Sigma(\mu,1,\ \beta,\ n,\phi(s),\widetilde{p}_k)$ is the set of functions $g \in \Sigma$ satisfying

$$\left(\frac{z(D^n_\beta g_\phi(z))'}{D^n_\beta g_\phi(z)}\right) + \mu\left(\frac{z(D^n_\beta g_\phi(z))''}{D^n_\beta g_\phi(z)}\right) \prec \widetilde{p}_k(z), \quad z \in \mathfrak{D}$$

   and

$$\left(\frac{\omega(D^n_\beta f_\phi(\omega))'}{D^n_\beta f_\phi(\omega)}\right) + \mu\left(\frac{\omega(D^n_\beta f_\phi(\omega))''}{D^n_\beta f_\phi(\omega)}\right) \prec \widetilde{p}_k(\omega), \quad \omega \in \mathfrak{D},$$

   where $n \in \mathbb{N}_0$, $\beta \geqslant 0$, $\mu \geqslant 0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).

Letting $n = 0$ and $\phi(s) = 1$ in the Definition 1.3, we get the family $\mathcal{SRN}_\Sigma(\gamma,\mu,\widetilde{p}_k) \equiv \mathcal{SR}\mathfrak{S}_\Sigma(\mu,\gamma,\beta,0,1,\widetilde{p}_k)$ of functions $g \in \Sigma$ satisfying

$$\frac{zg'(z) + \mu z^2 g''(z)}{(1-\gamma)z + \gamma g(z)} \prec \widetilde{p}_k(z) \quad \text{and} \quad \frac{\omega f'(\omega) + \mu\omega^2 f''(\omega)}{(1-\gamma)\omega + \gamma f(\omega)} \prec \widetilde{p}_k(\omega),$$

where $z,\omega \in \mathfrak{D}$, $0 \leqslant \gamma \leqslant 1$, $\mu \geqslant 0$ and $f(\omega) = g^{-1}(\omega)$ is as in (1.2).

Definition 1.4. A function $g \in \Sigma$ having the power series of the form (1.1) is said to belong to the set $\mathcal{SR}\mathfrak{B}_\Sigma(\gamma,\tau,\ \beta,\ n,\ \phi(s),\widetilde{p}_k)$, if

$$\frac{z[(D^n_\beta g_\phi(z))']^\tau}{(1-\gamma)z + \gamma\, D^n_\beta g_\phi(z)} \prec \widetilde{p}_k(z) \quad \text{and} \quad \frac{\omega[(D^n_\beta f_\phi(\omega))']^\tau}{(1-\gamma)\omega + \gamma\, D^n_\beta f_\phi(\omega)} \prec \widetilde{p}_k(\omega), \ z,\omega \in \mathfrak{D},$$

where $n \in \mathbb{N}_0$, $0 \leqslant \gamma \leqslant 1$, $\tau \geqslant 1$, $\beta \geqslant 0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).

Certain values of γ lead the family $SR\mathfrak{B}_\Sigma(\gamma,\tau,\ \beta,\ n,\phi(s),\widetilde{p}_k)$ to interesting sets as below:

1. $SRP_\Sigma(\tau,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SR\mathfrak{B}_\Sigma(0,\tau,\ \beta,\ n,\phi(s),\widetilde{p}_k)$ is the class of functions $g \in \Sigma$ satisfying

$$[(D_\beta^n g_\phi(z))']^\tau \prec \widetilde{p}_k(z),\ z \in \mathfrak{D} \quad \text{and} \quad [(D_\beta^n f_\phi(\omega))']^\tau \prec \widetilde{p}_k(\omega),\ \omega \in \mathfrak{D},$$

where $n \in \mathbb{N}_0$, $\tau \geqslant 1$, $\beta \geqslant 0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).

2. $SR\mathfrak{N}_\Sigma(\tau,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SR\mathfrak{B}_\Sigma(1,\tau,\ \beta,\ n,\phi(s),\widetilde{p}_k)$ is the family of functions $g \in \Sigma$ satisfying

$$\frac{z[(D_\beta^n g_\phi(z))']^\tau}{D_\beta^n g_\phi(z)} \prec \widetilde{p}_k(z),\ z \in \mathfrak{D} \quad \text{and} \quad \frac{\omega\,[(D_\beta^n f_\phi(\omega))']^\tau}{D^n f_\phi(\omega)} \prec \widetilde{p}_k(\omega),\ \omega \in \mathfrak{D},$$

where $n \in \mathbb{N}_0$, $\tau \geqslant 1$, $\beta \geqslant 0$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ as in (1.2).
$SR\mathfrak{N}_\Sigma(\tau,\ \beta,\ n\,\phi(s))$ is the family of Al-Oboudi type τ-bi-pseudo-starlike functions connected with a modified sigmoid activation function and k-Fibonacci numbers.

On taking $n = 0$ and $\phi(s) = 1$ in the Definition 1.4, we obtain the family $SRQ_\Sigma(\gamma,\tau,\widetilde{p}_k) \equiv SR\mathfrak{B}_\Sigma(\gamma,\tau,\ \beta,\ 0,\ 1,\widetilde{p}_k)$ of functions $g \in \Sigma$ satisfying

$$\frac{z[(g'(z))]^\tau}{(1-\gamma)z+\gamma\, g(z)} \prec \widetilde{p}_k(z),\ z \in \mathfrak{D} \quad \text{and} \quad \frac{\omega[(f'(\omega))]^\tau}{(1-\gamma)\omega+\gamma\, f(\omega)} \prec \widetilde{p}_k(\omega),\ \omega \in \mathfrak{D},$$

where $0 \leqslant \gamma \leqslant 1, \tau \geqslant 1$ and $g^{-1}(\omega) = f(\omega)$ is as in (1.2).

Remark 1.5. We note that

i) $SRM_\Sigma(0,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SRK_\Sigma(1,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SR\mathfrak{N}_\Sigma(1,\ \beta,\ n,\phi(s),\widetilde{p}_k)$;
ii) $SRP_\Sigma(1,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SRK_\Sigma(0,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SRL_\Sigma(0,\ \beta,\ n,\phi(s),\widetilde{p}_k)$; and
iii) $SRK_\Sigma(\gamma,\ \beta,\ n,\phi(s),\widetilde{p}_k) \equiv SR\mathfrak{B}_\Sigma(\gamma,\ 1,\ \beta,\ n,\ \phi(s),\widetilde{p}_k)$.

Remark 1.6. The family $SRN_\Sigma(0,1,\widetilde{p}_k) \equiv S_\Sigma^*(\widetilde{p}_k)$ was studied by Güney et al. [26], when $\mu = 0$ and $\gamma = 1$.

Remark 1.7. The family $SRQ_\Sigma(1,\tau,\widetilde{p}_1) \equiv S_\Sigma^*(\tau,\widetilde{p}_1)$ was investigated by Magesh et al. [31], when $\gamma = k = 1$.

We find the estimates for $|d_2|$, $|d_3|$ and also, fix the famous Fekete- Szegö problem [20] for functions belonging to classes $SR\mathfrak{S}_\Sigma(\mu,\gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$ and $SR\mathfrak{B}_\Sigma(\gamma,\tau,\ \beta,n,\ \phi(s),\widetilde{p}_k)$. We present few interesting cases and relavent connections of main results.

To prove our theorems, we need the below mentioned lemma.

Lemma 1.8 ([35]). *If the function* $p \in P$, *then* $|p_i| \leqslant 2$ *for each* i, *where* P *is the set of regular functions* p *in* $\mathfrak{D}$, *normalized by* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, *such that* $\Re(p(z)) > 0$, $z \in \mathfrak{D}$.

2. Estimates for the family $SR\mathfrak{S}_\Sigma(\mu,\gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$

We begin by obtaining the first two coefficients and the Fekete-Szegö bounds for functions in $SR\mathfrak{S}_\Sigma(\mu,\ \gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$.

Theorem 2.1. *If the function* $g \in SR\mathfrak{S}_\Sigma(\mu,\gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$, *then*

$$|d_2| \leqslant \frac{k\sqrt{k}\,|t_k|}{(1+\beta)^n\phi(s)\sqrt{|(\gamma^2-\gamma(2\mu+3)+3(2\mu+1))k^2t_k+(2(\mu+1)-\gamma)^2T_k|}}, \tag{2.1}$$

$$|d_3| \leqslant \frac{1}{(1+2\beta)^n\phi(s)}\left[\frac{k|t_k|}{3(2\mu+1)-\gamma}+\frac{k^3t_k^2}{|(\gamma^2-\gamma(2\mu+3)+3(2\mu+1))k^2t_k+(2(\mu+1)-\gamma)^2T_k|}\right], \tag{2.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k|t_k|}{(3(2\mu+1)-\gamma)(1+2\beta)^n\phi(s)}, & \left|1-\frac{(1+2\beta)^n\delta}{(1+\beta)^{2n}\phi(s)}\right| \leqslant J, \\ \frac{k^3t_k^2\left|1-\frac{(1+2\beta)^n\delta}{(1+\beta)^{2n}\phi(s)}\right|}{(1+2\beta)^n\phi(s)|(\gamma^2-\gamma(2\mu+3)+3(2\mu+1))k^2t_k+(2(\mu+1)-\gamma)^2T_k|}, & \left|1-\frac{(1+2\beta)^n\delta}{(1+\beta)^{2n}\phi(s)}\right| \geqslant J, \end{cases} \tag{2.3}$$

where $n \in \mathbb{N}_0, 0 \leqslant \gamma \leqslant 1,\ \mu \geqslant 0,\ \beta \geqslant 0$ *and*

$$J = \frac{1}{(3(2\mu+1)-\gamma)}\left|\gamma^2-\gamma(2\mu+3)+3(2\mu+1)+(2(\mu+1)-\gamma)^2\frac{T_k}{k^2t_k}\right|. \tag{2.4}$$

Proof. Let $p(z) = 1+p_1z+p_2z^2+\cdots$, and $p \prec \widetilde{p}_k$. Then there exists a regular function u with $|u(z)| < 1$ in $\mathfrak{D}$ and $p(z) = \widetilde{p}_k(u(z))$. Therefore, the function

$$m(z) = \frac{1-u(z)}{1+u(z)} = 1+u_1z+u_2z^2+\cdots$$

is in the set P. It now follows that

$$u(z) = \frac{m(z)-1}{m(z)+1} = \frac{u_1}{2}z+\left(u_2-\frac{u_1^2}{2}\right)\frac{z^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{z^3}{2}+\cdots$$

and

$$\begin{aligned} \widetilde{p}_k(u(z)) &= 1+\widetilde{p}_{k,1}\left(\frac{u_1z}{2}+\left(u_2-\frac{u_1^2}{2}\right)\frac{z^2}{2}+\cdots\right)+\widetilde{p}_{k,2}\left(\frac{u_1z}{2}+\left(u_2-\frac{u_1^2}{2}\right)\frac{z^2}{2}+\cdots\right)^2+\cdots \\ &= 1+\frac{\widetilde{p}_{k,1}u_1z}{2}+\left(\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)\widetilde{p}_{k,1}+\frac{u_1^2}{4}\widetilde{p}_{k,2}\right)z^2+\cdots. \end{aligned} \tag{2.5}$$

Also, there exists a regular function ν satisfying $|\nu(\omega)| < 1$ in $\mathfrak{D}$ such that $p(\omega) = \widetilde{p}_k(\nu(\omega))$. Therefore, $l(\omega) = \frac{1+\nu(\omega)}{1-\nu(\omega)} = 1+\nu_1\omega+\nu_2\omega^2+\cdots$ is in the class P. So it follows that

$$\widetilde{p}_k(\nu(\omega)) = 1+\frac{\widetilde{p}_{k,1}\nu_1\omega}{2}+\left(\frac{1}{2}\left(\nu_2-\frac{\nu_1^2}{2}\right)\widetilde{p}_{k,1}+\frac{\nu_1^2}{4}\widetilde{p}_{k,2}\right)\omega^2+\cdots. \tag{2.6}$$

Suppose $g \in \mathcal{SR}\mathfrak{S}_\Sigma(\mu,\gamma,\ \beta,n,\phi(s),\widetilde{p}_k)$. Then from Definition 1.3, we obtain

$$\frac{z(D_\beta^n g_\phi(z))'+\mu z^2(D_\beta^n g_\phi(z))''}{(1-\gamma)z+\gamma D_\beta^n g_\phi(z)} \prec \widetilde{p}_k(u(z)) \tag{2.7}$$

and

$$\frac{\omega(D_\beta^n f_\phi(\omega))'+\mu\omega^2(D_\beta^n f_\phi(\omega))''}{(1-\gamma)\omega+\gamma D_\beta^n f_\phi(\omega)} \prec \widetilde{p}_k(\nu(\omega)), \tag{2.8}$$

where $z,\ \omega \in \mathfrak{D}$ and $g_\phi^{-1}(\omega) = f_\phi(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (1.2). By virtue of (2.5), (2.6), (2.7), and (2.8), we obtain

$$(1+\beta)^n\phi(s)(2(\mu+1)-\gamma)\,d_2 = \frac{u_1kt_k}{2}, \tag{2.9}$$

$$\begin{aligned} (1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)d_3-(1+\beta)^{2n}\phi^2(s)(2(\mu+1)-\gamma)\gamma d_2^2 \\ = \frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)kt_k+\frac{u_1^2}{4}(k^2+2)t_k^2, \end{aligned} \tag{2.10}$$

$$-(1+\beta)^n\phi(s)(2(\mu+1)-\gamma)\,d_2=\frac{\nu_1 k\,t_k}{2}, \tag{2.11}$$

$$\begin{aligned}-(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)d_3+(1+\beta)^{2n}\phi^2(s)(\gamma^2-2(\mu+2)\gamma+6(2\mu+1))d_2^2\\=\frac{1}{2}\left(\nu_2-\frac{\nu_1^2}{2}\right)k\,t_k+\frac{\nu_1^2}{4}(k^2+2)t_k^2,\end{aligned} \tag{2.12}$$

From (2.9) and (2.11), we get

$$u_1=-\nu_1 \tag{2.13}$$

and also

$$2(1+\beta)^{2n}\phi^2(s)(2(\mu+1)-\gamma)^2d_2^2=\frac{(u_1^2+\nu_1^2)k^2\,t_k^2}{4}. \tag{2.14}$$

By adding (2.10) and (2.12), we obtain

$$2(1+\beta)^{2n}\phi^2(s)(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))d_2^2=\frac{1}{2}(u_2+\nu_2)k\,t_k-\frac{1}{4}(k\,t_k-(k^2+2)t_k^2)(u_1^2+\nu_1^2). \tag{2.15}$$

Substituting the value of $(u_1^2+\nu_1^2)$ from (2.14) in (2.15), we get

$$d_2^2=\frac{k^3\,t_k^2(u_2+\nu_2)}{4(1+\beta)^{2n}\phi^2(s)\,[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))k^2\,t_k+(2(\mu+1)-\gamma)^2T_k]}, \tag{2.16}$$

which gets (2.1), on using Lemma 1.8.

Subtracting (2.12) from (2.10) and using (2.13), we obtain

$$d_3=\frac{(1+\beta)^{2n}\phi(s)}{(1+2\beta)^n}d_2^2+\frac{k\,t_k(u_2-\nu_2)}{4(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)}. \tag{2.17}$$

Then in view of Lemma 1.8 and (2.16), (2.17) yields (2.2).

Using (2.16) in (2.17), for $\delta\in\mathbb{R}$, we get

$$|d_3-\delta d_2^2|=k\,|t_k|\left|\left(T(\delta)+\frac{1}{4(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)}\right)u_2+\left(T(\delta)-\frac{1}{4(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)}\right)\nu_2\right|,$$

where

$$T(\delta)=\frac{\left(\frac{(1+\beta)^{2n}\phi(s)}{(1+2\beta)^n}-\delta\right)k^2t_k}{4(1+\beta)^{2n}\phi^2(s)\,[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))k^2\,t_k+(2(\mu+1)-\gamma)^2T_k]}.$$

In view of (1.4), we conclude that

$$|d_3-\delta d_2^2|\leqslant\begin{cases}\frac{k\,|t_k|}{2(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)}, & 0\leqslant|T(\delta)|\leqslant\frac{1}{4(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)},\\ 4k\,|t_k||T(\delta)|, & |T(\delta)|\geqslant\frac{1}{4(1+2\beta)^n\phi(s)(3(2\mu+1)-\gamma)},\end{cases}$$

which gets (2.3) with J as in (2.4). This ends the proof. □

We now present few interesting observations of our result.

Corollary 2.2. *If the function* $g\in SRK_\Sigma(\gamma,\beta,n,\phi(s),\tilde{p}_k)$, *then*

$$|d_2|\leqslant\frac{k\sqrt{k}\,|t_k|}{(1+\beta)^n\phi(s)\sqrt{|(\gamma^2-3\gamma+3)k^2t_k+(2-\gamma)^2T_k|}},$$

$$|d_3| \leqslant \frac{1}{(1+2\beta)^n\phi(s)}\left[\frac{k|t_k|}{3-\gamma}+\frac{k^3t_k^2}{|(\gamma^2-3\gamma+3)k^2t_k+(2-\gamma)^2T_k|}\right],$$

and for some $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k|t_k|}{(3-\gamma)(1+2\beta)^n\phi(s)}, & \left|1-\frac{(1+2\beta)^n\delta}{(1+\beta)^{2n}\phi(s)}\right| \leqslant J_1, \\ \frac{k^3\,t_k^2\left|1-\frac{(1+2\beta)^n\delta}{(1+\beta)^{2n}\phi(s)}\right|}{(1+2\beta)^n|(\gamma^2-3\gamma+3)k^2\,t_k+(2-\gamma)^2\,T_k|\phi(s)}, & \left|1-\frac{(1+2\beta)^n\delta}{(1+\beta)^{2n}\phi(s)}\right| \geqslant J_1, \end{cases}$$

where

$$J_1=\frac{1}{(3-\gamma)}\left((\gamma^2-3\gamma+3)+(2-\gamma)^2\left|\frac{T_k}{k^2t_k}\right|\right).$$

Remark 2.3. Corollary 10 and Corollary 23 of [27] are particular cases of Corollary 2.2, when $\phi(s)=1$, $n=0$ and $\gamma=1$. Further, we get the results of Güney et al. [26, Corollaries 1 and 4], if we allow $k=1$.

Corollary 2.4. *If the function* $g \in SRL_\Sigma(\mu,\ \beta, n,\ \phi(s),\ \widetilde{p}_k)$, *then*

$$|d_2| \leqslant \frac{k\sqrt{k}\,|t_k|}{(1+\beta)^n\phi(s)\sqrt{|3(2\mu+1)k^2t_k+4(\mu+1)^2T_k|}},$$

$$|d_3| \leqslant \frac{1}{(1+2\beta)^n\phi(s)}\left[\frac{k|t_k|}{3(2\mu+1)}+\frac{k^3t_k^2}{|3(2\mu+1)k^2t_k+4(\mu+1)^2T_k|}\right],$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k|t_k|}{3(2\mu+1)(1+2\beta)^n\phi(s)}, & \left|1-\frac{(1+2\beta)^n\delta}{\phi(s)(1+\beta)^{2n}}\right| \leqslant J_2, \\ \frac{k^3\,t_k^2\left|1-\frac{(1+2\beta)^n\delta}{\phi(s)(1+\beta)^{2n}}\right|}{(1+2\beta)^n|3(2\mu+1)k^2t_k+4(\mu+1)^2T_k|\phi(s)}, & \left|1-\frac{(1+2\beta)^n\delta}{\phi(s)(1+\beta)^{2n}}\right| \geqslant J_2, \end{cases}$$

where

$$J_2=\frac{1}{3(2\mu+1)}\left(3(2\mu+1)+4(\mu+1)^2\left|\frac{T_k}{k^2t_k}\right|\right).$$

Corollary 2.5. *If the function* $g \in SRM_\Sigma(\mu,\ \beta, n, \phi(s),\ \widetilde{p}_k)$, *then*

$$|d_2| \leqslant \frac{k\sqrt{k}\,|t_k|}{\phi(s)(1+\beta)^n\sqrt{|(4\mu+1)k^2t_k+(2\mu+1)^2T_k|}},$$

$$|d_3| \leqslant \frac{1}{\phi(s)(1+2\beta)^n}\left[\frac{k|t_k|}{2(3\mu+1)}+\frac{k^3t_k^2}{|(4\mu+1)k^2t_k+(2\mu+1)^2T_k|}\right],$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k|t_k|}{2(3\mu+1)(1+2\beta)^n\phi(s)}, & \left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right| \leqslant J_3, \\ \frac{k^3t_k^2\left|1-\frac{\delta(1+2\beta)^n}{(1+\beta)^{2n}\phi(s)}\right|}{(1+2\beta)^n|(4\mu+1)k^2t_k+(2\mu+1)^2T_k|\phi(s)}, & \left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right| \geqslant J_3, \end{cases}$$

where

$$J_3=\frac{1}{2(3\mu+1)}\left((4\mu+1)+(2\mu+1)^2\left|\frac{T_k}{k^2\,t_k}\right|\right).$$

Remark 2.6. Corollaries 10 and 23 of Güney et al. [27] are particular cases of Corollary 2.5 , when $\phi(s)=1$, $n=0$ and $\mu=0$. Further, we get the results of Güney et al. [26, Corollaries 1 and 4], if we allow $k=1$.

Corollary 2.7. *If the function* $g \in SRN_{\Sigma}(\gamma, \mu, \widetilde{p}_k)$, *then*

$$|d_2| \leqslant \frac{k\sqrt{k}|t_k|}{\sqrt{|(\gamma^2-\gamma(2\mu+3)+3(2\mu+1))k^2t_k+(2(\mu+1)-\gamma)^2T_k|}},$$

$$|d_3| \leqslant \left[\frac{k|t_k|}{3(2\mu+1)-\gamma}+\frac{k^3t_k^2}{|(\gamma^2-\gamma(2\mu+3)+3(2\mu+1))k^2t_k+(2(\mu+1)-\gamma)^2T_k|}\right],$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k|t_k|}{(1+2\beta)^n)(3(2\mu+1)-\gamma)}, & |1-\delta| \leqslant J, \\ \frac{k^3t_k^2|1-\delta|}{|(\gamma^2-\gamma(2\mu+3)+3(2\mu+1))k^2t_k+(2(\mu+1)-\gamma)^2T_k|}, & |1-\delta| \geqslant J, \end{cases}$$

where J *is as in* (2.4).

3. Estimates for the family $SR\mathfrak{B}_{\Sigma}(\gamma, \tau, \beta, n, \phi(s), \widetilde{p}_k)$

We derive the initial Taylor-Maclaurin coefficients and the Fekete-Szegö inequality for functions in $SR\mathfrak{B}_{\Sigma}(\gamma, \tau, \beta, n, \phi(s), \widetilde{p}_k)$ in the following theorem.

Theorem 3.1. *Let* $0 \leqslant \gamma \leqslant 1$, $\tau \geqslant 1$, $\beta \geqslant 0$, *and* $n \in \mathbb{N}_0$. *If* $g \in SR\mathfrak{B}_{\Sigma}(\gamma, \tau, \beta, n, \phi(s), \widetilde{p}_k)$, *then*

$$|d_2| \leqslant \frac{k\sqrt{k}|t_k|}{\phi(s)(1+\beta)^n\sqrt{|(\gamma^2+(2\tau+1)(\tau-\gamma))k^2t_k+(2\tau-\gamma)^2T_k|}}, \tag{3.1}$$

$$|d_3| \leqslant \frac{1}{\phi(s)(1+2\beta)^n}\left[\frac{k|t_k|}{(3\tau-\gamma)}+\frac{k^3t_k^2}{|(\gamma^2+(2\tau+1)(\tau-\gamma))k^2t_k+(2\tau-\gamma)^2T_k|}\right], \tag{3.2}$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k|t_k|}{(3\tau-\gamma)(1+2\beta)^n\phi(s)}, & |1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}| \leqslant \Omega, \\ \frac{k^3t_k^2\left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|}{(1+2\beta)^n|(\gamma^2+(2\tau+1)(\tau-\gamma))k^2t_k+(2\tau-\gamma)^2T_k|\phi(s)}, & |1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}| \geqslant \Omega, \end{cases} \tag{3.3}$$

where

$$\Omega = \frac{1}{(3\tau-\gamma)}\left|\gamma^2+(2\tau+1)(\tau-\gamma)+(2\tau-\gamma)^2\left(\frac{T_k}{k^2t_k}\right)\right|. \tag{3.4}$$

Proof. Suppose $g \in SR\mathfrak{B}_{\Sigma}(\gamma, \tau, \beta, n, \phi(s), \widetilde{p}_k)$. Then from Definition 1.4, we have

$$\frac{z[(D_\beta^n g_\phi(z))']^\tau}{(1-\gamma)z+\gamma D_\beta^n g_\phi(z)} = \widetilde{p}_k(u(z)),\ z \in \mathfrak{D} \tag{3.5}$$

and

$$\frac{\omega[(D_\beta^n f_\phi(\omega))']^\tau}{(1-\gamma)\omega+\gamma D_\beta^n f_\phi(\omega)} = \widetilde{p}_k(v(\omega)),\ \omega \in \mathfrak{D}. \tag{3.6}$$

On account of (2.5), (2.6), (3.5), and (3.6), we have

$$(1+\beta)^n\phi(s)(2\tau-\gamma)d_2 = \frac{u_1kt_k}{2}, \tag{3.7}$$

$$\begin{aligned}(1+2\beta)^n\phi(s)(3\tau-\gamma)d_3+(1+\beta)^{2n}\phi^2(s)(\gamma^2-2\tau\gamma+2\tau(\tau-1))d_2^2 \\ = \frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)kt_k+\frac{u_1^2}{4}(k^2+2)t_k^2,\end{aligned} \tag{3.8}$$

$$-(1+\beta)^n\phi(s)(2\tau-\gamma)d_2=\frac{\nu_1 k\, t_k}{2}, \tag{3.9}$$

$$\begin{aligned} &-(1+2\beta)^n\phi(s)(3\tau-\gamma)d_3+(1+\beta)^{2n}\phi^2(s)(\gamma^2-2(\tau+1)\gamma+2\tau(\tau+2))d_2^2 \\ &\qquad=\frac{1}{2}\left(\nu_2-\frac{\nu_1^2}{2}\right)k\,t_k+\frac{\nu_1^2}{4}(k^2+2)t_k^2. \end{aligned} \tag{3.10}$$

Now by following the proof of Theorem 2.1 with respect to (2.9)-(2.12), the results (3.1)-(3.3) of this theorem are obtained from (3.7)-(3.10). □

We now present few interesting observations of our result.

Corollary 3.2. *If the function* $g\in SRP_\Sigma(\tau,\ \beta, n,\ \phi(s),\ \widetilde{p}_k)$ *, then*

$$|d_2|\leqslant\frac{k\sqrt{k}|t_k|}{\phi(s)(1+\beta)^n\sqrt{|(2\tau+1)\tau k^2\,t_k+4\tau^2T_k|}},$$
$$|d_3|\leqslant\frac{1}{\phi(s)(1+2\beta)^n}\left[\frac{k|t_k|}{3\tau}+\frac{k^3t_k^2}{|(2\tau+1)\tau k^2\,t_k+4\tau^2T_k|}\right],$$

and for $\delta\in\mathbb{R}$,

$$|d_3-\delta d_2^2|\leqslant\begin{cases}\frac{k|t_k|}{3\tau(1+2\beta)^n\phi(s)}, & \left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|\leqslant\Omega_1,\\ \frac{k^3t_k^2\left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|}{(1+2\beta)^n|(2\tau+1)\tau k^2\,t_k+4\tau^2T_k|\phi(s),} & \left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|\geqslant\Omega_1,\end{cases}$$

where

$$\Omega_1=\frac{1}{3}\left((2\tau+1)+4\tau\left|\frac{T_k}{k^2t_k}\right|\right).$$

Corollary 3.3. *If the function* $g\in SR\mathfrak{N}_\Sigma(\tau,\ \beta,\ n,\ \ \phi(s),\ \widetilde{p}_k)$*, then*

$$|d_2|\leqslant\frac{k\sqrt{k}|t_k|}{\phi(s)(1+\beta)^n\sqrt{|\tau(2\tau-1)k^2t_k+(2\tau-1)^2T_k|}},$$
$$|d_3|\leqslant\frac{1}{\phi(s)(1+2\beta)^n}\left[\frac{k|t_k|}{(3\tau-1)}+\frac{k^3t_k^2}{|\tau(2\tau-1)k^2t_k+(2\tau-1)^2T_k|}\right],$$

and for $\delta\in\mathbb{R}$,

$$|d_3-\delta d_2^2|\leqslant\begin{cases}\frac{k|t_k|}{(3\tau-1)(1+2\beta)^n\phi(s)}, & \left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|\leqslant\Omega_2,\\ \frac{k^3t_k^2\left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|}{(1+2\beta)^n|\tau(2\tau-1)k^2t_k+(2\tau-1)^2T_k|\phi(s)}, & \left|1-\frac{\delta(1+2\beta)^n}{\phi(s)(1+\beta)^{2n}}\right|\geqslant\Omega_2,\end{cases}$$

where

$$\Omega_2=\frac{1}{(3\tau-1)}\left(\tau(2\tau-1)+(2\tau-1)^2\left|\frac{T_k}{k^2t_k}\right|\right).$$

Remark 3.4. The results of Corollary 3.3 coincide with Theorem 2.3 of Magesh et al. [31] for $k=\phi(s)=1$ and $n=0$. Further, if we allow $\tau=1$, then we get the results of [26, Corollaries 1 and 4].

Corollary 3.5. *If the function* $g(z) \in SRQ_{\Sigma}(\gamma, \tau, \widetilde{p}_k)$, *then*

$$|d_2| \leqslant \frac{k\sqrt{k}\,|t_k|}{\sqrt{|(\gamma^2+(2\tau+1)(\tau-\gamma))k^2\,t_k+(2\tau-\gamma)^2T_k|}},$$

$$|d_3| \leqslant \left[\frac{k\,|t_k|}{(3\tau-\gamma)}+\frac{k^3\,t_k^2}{|(\gamma^2+(2\tau+1)(\tau-\gamma))k^2\,t_k+(2\tau-\gamma)^2T_k|}\right],$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \frac{k\,|t_k|}{3\tau-\gamma}, & |1-\delta| \leqslant \Omega, \\ \frac{k^3 t_k^2 |1-\delta|}{|(\gamma^2+(2\tau+1)(\tau-\gamma))k^2\,t_k+(2\tau-\gamma)^2T_k|}, & |1-\delta| \geqslant \Omega, \end{cases}$$

where Ω *is as in* (3.4).

4. Conclusion

Two special families of regular bi-univalent functions are introduced by using Al-Oboudi type operator connected with a modified sigmoid activation function and k-Fibonacci numbers. Bounds of the first two coefficients $|d_2|$, $|d_3|$ and the celebrated Fekete-Szegö functional have been fixed for each of the two families. Through corollaries of our main results, we have highlighted many interesting new consequences.

The special families examined in this research paper using Al-Oboudi type operator could inspire further research related to other aspects such as classes using q-derivative operator, meromorphic bi-univalent function classes linked with Al-Oboudi differential operator and classes using integro-differential operators.

Acknowledgement

The authors express their sincere thanks to the esteemed referee(s) for their careful readings, valuable suggestions and comments, which helped them to improve the presentation of the paper.

References

[1] I. Aldawish, T. Al-Hawary, B. A. Frasin, *Subclasses of bi-univalent functions defined by Frasin differential operator*, Mathematics, **8** (2020), 12 pages. 1
[2] T. Al-Hawary, A. Amourah and B.A. Frasin, *Fekete–Szegö inequality for bi-univalent functions by means of Horadam polynomials*, Bol. Soc. Mat. Mex. (3), **27** (2021), 12 pages. 1
[3] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Sci., **2004** (2004), 1429–1436. 1
[4] Ş. Altınkaya, S. G. Hamidi, J. M. Jahangiri, S. Yalçın, *Inclusion properties for bi-univalent functions of complex order defined by combining of Faber polynomial expansions and Fibonacci numbers*, An. Univ. Oradea Fasc. Mat., **27** (2020), 21–30. 1
[5] Ş. Altınkaya, S. Yalçın, S. Çakmak, *A subclass of bi-Univalent functions based on the Faber polynomial expansions and the Fibonacci numbers*, Mathematics, **160** (2019), 9 pages. 1
[6] A. Amourah, T. Al-Hawary, B. A. Frasin, *Application of Chebyshev polynomials to certain class of bi-Bazilevič functions of order $\alpha+i\beta$*, Afr. Mat., **32** (2021), 1059–1066. 1
[7] A. Amourah, B. A. Frasin, T. Abdeljawad, *Fekete-Szegö inequality for analytic and biunivalent functions subordinate to Gegenbauer polynomials*, J. Funct. Spaces, **2021** (2021), 7 pages. 1
[8] I. T. Awolere, A. T. Oladipo, *Coefficients of bi-univalent functions involving pseudo-starlikeness associated with Chebyshev polynomials*, Khayyam J. Math., **5** (2019), 140–149. 1
[9] D. A. Brannan, J. G. Clunie, *Aspects of contemporary complex analysis*, Academic Press, New York–London, (1980). 1
[10] D. A. Brannan, T. S. Taha, *On some classes of bi-univalent functions*, Studia Univ. Babeş-Bolyai Math., **31** (1986), 70–77. 1
[11] S. Bulut, *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C. R. Math. Acad. Sci. Paris, **352** (2014), 479–484. 1

[12] P. L. Duren, *Univalent functions*, Springer-Verlag, New York, (1983). 1
[13] J. Dziok, R. K. Raina, J. Sokól, *Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers*, Comput. Math. Appl., **61** (2011), 2605–2613. 1
[14] J. Dziok, R. K. Raina, J. Sokól, *On α-Convex functions related to shell-like functions connected with Fibonacci numbers*, Appl. Math. Compt., **218** (2011), 996–1002. 1
[15] S. M. El-Deeb, T. Bulboača, B. M. El-Matary, *Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative*, Mathematics, **8** (2020), 13 pages. 1
[16] U. A. Ezeafulukwe, M. Darus, O. A. Fadipe-Joseph, *On analytic properties of a sigmoid function*, Int. J. Math. Comput. Sci., **13** (2018), 171–178. 1
[17] O. A. Fadipe-Joseph, B. B. Kadir, S. E. Akinwumi, E. O. Adeniran, *Polynomial bounds for a class of univalent function involving sigmoid function*, Khayyam J. Math., **4** (2018), 88–101. 1
[18] O. A. Fadipe-Joseph, A. T. Oladdaipo, A. U. Ezeafulukwe, *Modified sigmoid function in univalent function theory*, Int. J. Math. Sci. Eng. Appl., **7** (2013), 313–317. 1
[19] S. Falcón, A. Plaza, *On the Fibonacci k-numbers*, Chaos Solitons Fractals, **32** (2007), 1615–1624. 1
[20] M. Fekete, G. Szegö, *Eine bemerkung über ungerade schlichte funktionen*, J. London Math. Soc., **8** (1933), 85–89. 1
[21] P. Filipponi, A. F. Horadam, *Derivative sequences of Fibonacci and Lucas polynomials*, In: Applications of Fibonacci Numbers, **4** (1991), 99–108. 1
[22] P. Filipponi, A. F. Horadam, *Second derivative sequence of Fibonacci and Lucas polynomials*, Fibonacci Quart., **31** (1993), 194–204. 1
[23] B. A. Frasin, M. K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24** (2011), 1569–1573. 1
[24] B. A. Frasin, S. R. Swamy, I. Aldawish, *A comprehensive family of biunivalent functions defined by k-Fibonacci numbers*, J. Funct. Spaces, **2021** (2021), 7 pages. 1
[25] B. A. Frasin, S. R. Swamy, J. Nirmala, *Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to k-Fibonacci numbers involving modified sigmoid activation function*, Afr. Mat., **32** (2021), 631–643. 1, 1
[26] H. Ö. Güney, G. Murugusundaramoorthy, J. Sokól, *Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers*, Acta Univ. Sapientiae Math., **10** (2018), 70–84. 1, 1.6, 2.3, 2.6, 3.4
[27] H. Ö. Güney, G. Murugusundaramoorthy, J. Sokól, *Certain subclasses of bi-univalent functions related to k-Fibonacci numbers*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **68** (2019), 1909–1921. 1, 2.3, 2.6
[28] S. Haykin, *Neural networks: a comprehensive foundation*, Second ed., an inprint of Pearson Education, India, (1999). iv
[29] M. G. Khan, B. Ahmad, G. Mirigusundaramoorthy, R. Chinram, W. K. Mashwani, *Applications of modified sigmoid functions to a class of starlike functions*, J. Funct. Spaces, **2020** (2020), 8 pages. 1
[30] M. Lewin, *On a coefficients problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18** (1967), 63–68. 1
[31] N. Magesh, C. Abirami, V. K. Balaji, *Certain classes of bi-univalent functions related to Shell-like curves connected with Fibonacci numbers*, Afr. Mat., **32** (2021), 185–198. 1.7, 3.4
[32] N. Magesh, P. K. Mamatha, S. R. Swamy, J. Yamini, *Horadam polynomial coefficient estimates for a class of λ-bi-pseudo-starlike functions*, Presented in National Conference on Recent Trends in Mathematics and its Applications (NCRTMA-2019), **2019** (2019), 7 pages. 1
[33] M. O. Oluwayemi, O. A. Fadipe-Joseph, U. A. Ezeafulukwe, *Certain subclass of univalent functions involving modified sigmoid function*, J. Phys.: Conf. Series, **1212** (2019), 6 pages. 1
[34] N. Y. Özgür, J. Sokól, *On starlike functions connected with k-Fibonacci numbers*, Bull. Malays. Math. Sci. Soc., **38** (2015), 249–258. 1
[35] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, (1975). 1.8
[36] F. M. Sakar, M. A. Aydoğan, *Initial bounds for certain subclasses of generalized Sălăgean type bi-univalent functions associated with the Horadam polynomials*, J. Qual. Measur. Anal., **15** (2019), 89–100. 1
[37] G. S. Sălăgean, *Subclasses of univalent functions*, Complex analysis—fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), **1983** (1983), 362–372. 1
[38] S. Scardapane, S. V. Vaerenbergh, A. Hussain, A. Uncini, *Complex-valued neural networks with non-parametric activation functions*, arXiv, **2018** (2018), 10 pages. 1
[39] H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23** (2010), 1188–1192. 1
[40] S. R. Swamy, S. Bulut, Y. Sailaja, *Some special families of holomorphic and Sălăgean type bi-univalent functions associated with Horadam polynomials involving modified sigmoid activation function*, Hacet. J. Math. Stat., **51** (2021), 710–720. 1
[41] S. R. Swamy, P. K. Mamatha, N. Magesh, J. Yamini, *Certain subclasses of bi-univalent functions defined by Sălăgean operator associated with the (p, q)-Lucas polynomials*, Adv. Math. Sci. J., **9** (2020), 6017–6025.
[42] S. R. Swamy, J. Nirmala, Y. Sailaja, *Some special families of holomorphic and Al-Oboudi type bi-univalent functions associated with (m, n)-Lucas polynomials involving modified sigmoid activation function*, South East Asian J. Math. Math. Sci., **17** (2021), 1–16.
[43] S. R. Swamy, J. Nirmala, Y. Sailaja, *Some special families of holomorphic and Al-Oboudi type bi-univalent functions associated with Horadam polynomials involving modified sigmoid activation function*, Electron. J. Math. Anal. Appl., **10** (2022), 29–41.

[44] S. R. Swamy, A. K. Wanas, Y. Sailaja, *Some special families of holomorphic and Sălăgean type bi-univalent functions associated with (m, n)-Lucas polynomials*, Comm. Math. Appl., **11** (2020), 563–574. 1

[45] D. L. Tan, *Coefficient estimates for bi-univalent functions*, Chinese Ann. Math. Ser. A, **5** (1984), 559–568. 1

[46] T. T. Wang, W. P. Zhang, *Some identities involving Fibonacci, Lucas polynomials and their applications*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **55** (2012), 95–103. 1