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Approximate solution of the special type differential equation of higher order using Taylor's series

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Abstract

We study the approximate solution of the special type n^{th} order linear differential equation by applying initial and boundary conditions using Taylor's series formula. That is, we prove the sufficient condition for the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the special type linear differential equation of higher order with initial and boundary conditions using Taylor's series formula.

Keywords: Mittag-Leffler-Hyers-Ulam stability, Mittage-Leffler-Hyers-Ulam-Rassias stability, linear differential equations, initial and boundary conditions, Taylor's series formula.

2020 MSC: 34K20, 26D10, 44A10, 39B82, 34A40, 39A30.

1. Introduction

The theory of stability is an important branch of the qualitative theory of differential equations. In 1940, Ulam [37] posed a problem concerning the stability of functional equation: "Give conditions in order for a linear function near an approximately linear function to exist". One year later, Hyers [15] provided an answer to problem of Ulam for Cauchy additive functional equation based on Banach spaces. After that, many mathematicians have contributed to the development of the Ulam's problem to other functional equations on various spaces in different directions [2, 5–9, 16, 27, 31, 34, 35].

Ulam's recent problem has been generalization by substituting functional equations with differential equations: The differential equation $\phi\left(h, \zeta, \zeta', \zeta'', \cdots, \zeta^{(n)}\right) = 0$ has the Hyers-Ulam stability (Shortly denote: HU stability) if for a given $\epsilon > 0$ and a function ζ such that

$$\left|\phi\left(h, \zeta, \zeta', \zeta'', \cdots, \zeta^{(n)}\right)\right| \leqslant \epsilon,$$

then there exists a solution ζ_a of the differential equation such that $|\zeta(s) - \zeta_a(s)| \leqslant K(\epsilon)$ and $\lim_{\epsilon \to 0} K(\epsilon) = 0$.

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Oblaza seems to be the first author who has investigated the HUS of linear differential equations (see [28, 29]). Thereafter, Alsina and Ger published their paper [4], which handles the HU stability of the linear differential equation $\psi'(s) = \psi(s)$.

In recent years, many authors are studying the HUS of differential equations, and a number of mathematicians are paying attention to the new results of the HUS of differential equations, which were extended to the first order, second order and higher orders in [1, 10, 13, 14, 17, 18, 20–23, 26, 30, 32, 33, 36].

Recently, Murali et al. [25] have investigated the HU stability of the linear differential equation of higher order using Fourier transform method.

In 2014, Alqifiary and Jung [3] proved the Hyers-Ulam stability of the following linear differential equation $y''(x) + \beta(x)y(x) = 0$ with boundary conditions $y(a) = 0 = y(b)$ or with initial conditions $y(a) = 0 = y'(a)$.

In the next year, Huang et al. [12] are investigated the generalized superstability of differential equations of nth-order with initial conditions and investigate the generalized superstability of differential equations of second order in the form of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ and the superstability of linear differential equations with constant coefficients with initial conditions.

These days, the HU stability of differential equation is investigated and the investigation is ongoing. Very recently, Murali et al. [19] studied the Hyers-Ulam stability for the third order linear ordinary differential equation of the form

$$x'''(t) + (p(t) - \alpha(t))x(t) = 0.$$

Motivated and linked by the above result, our main is to generalize the result reported in [3, 11, 12, 19] (see also [24]). That is, we are going study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the special type n^{th} order linear differential equation of the form

$$\zeta^{(n)}(s) + (\ell(s) - \mu(s))\zeta(s) = 0, \tag{1.1}$$

with boundary conditions

$$\zeta(\imath) = \zeta(\jmath) = 0, \tag{1.2}$$

and initial conditions

$$\zeta(\imath) = \zeta'(\imath) = \zeta''(\imath) = \cdots = \zeta^{(n-1)}(\imath) = 0, \tag{1.3}$$

where $\zeta \in C^n(I)$, $\ell(s) \in C(I)$, and $\mu(s)$ is a bounded for all sufficiently large s in $\mathbb{R}$, whereas $I = [\imath, \jmath]$, $-\infty < \imath < \jmath < \infty$ using Taylor's series.

2. Preliminaries

Definition 2.1. A Mittag-Leffler function of one parameter is defined as

$$E_\nu(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\nu k + 1)},$$

where $s, \nu \in \mathbb{C}$ and $Re(\nu) > 0$.

Definition 2.2. We call the differential equation (1.1) has HU stability with boundary conditions (1.2), if there exists a $K > 0$, such that the conditions are holds: For each $\epsilon > 0$, $\zeta \in C^n([\imath, \jmath])$ satisfies the differential inequality

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon,$$

with $\zeta(\imath) = \zeta(\jmath) = 0$, then there exists some $\psi \in C^n([\imath, \jmath])$ satisfies the differential equation

$$\psi^{(n)}(s) + (\ell(s) - \mu(s))\,\psi(s) = 0,$$

with $\psi(\imath) = \psi(\jmath) = 0$, such that $|\zeta(s) - \psi(s)| \leqslant K\epsilon$.

Definition 2.3. We call the differential equation (1.1) is said to have HUS with initial conditions (1.3), if there exists a $K > 0$, such that the following properties holds: For each $\epsilon > 0$, and $\zeta \in C^n([\imath, \jmath])$ satisfies the differential inequality

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon,$$

with initial conditions $\zeta(\imath) = \zeta'(\imath) = \zeta''(\imath) = \cdots = \zeta^{(n-1)}(\imath) = 0$, then there exists some $\psi \in C^n([\imath, \jmath])$ satisfies

$$\psi^{(n)}(s) + (\ell(s) - \mu(s))\,\psi(s) = 0,$$

with $\psi(\imath) = \psi'(\imath) = \psi''(\imath) = \cdots = \psi^{(n-1)}(\imath) = 0$, such that $|\zeta(s) - \psi(s)| \leqslant K\epsilon$.

If the preceding definitions is also correct when we putting ϵ by $\phi(s)\epsilon$, where $\phi : I \to [0, \infty)$ are functions not depending on $\zeta(s)$ and $\psi(s)$ explicitly, then we say that the corresponding differential equation has the generalized HUS (or the HUR stability).

Definition 2.4. The linear differential equation (1.1) has Mittag-Leffler-Hyers-Ulam stability with boundary conditions (1.2), if there exists a real number $K > 0$, such that the conditions are holds: For each $\epsilon > 0$, $\zeta \in C^n([\imath, \jmath])$ satisfies the differential inequality

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon E_\nu(s),$$

with boundary condition (1.2), then there exists some $\psi \in C^n([\imath, \jmath])$ satisfies

$$\psi^{(n)}(s) + (\ell(s) - \mu(s))\,\psi(s) = 0,$$

with (1.2), such that $|\zeta(s) - \psi(s)| \leqslant K\epsilon E_\nu(s)$.

Definition 2.5. The differential equation (1.1) has Mittag-Leffler-Hyers-Ulam stability with initial conditions (1.3), if there is a positive real number K, such that the properties are exists: For each $\epsilon > 0$ and $\zeta \in C^n([\imath, \jmath])$ satisfying the differential inequality

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon E_\nu(s),$$

with initial conditions (1.3), then there exists some $\psi \in C^n([\imath, \jmath])$ satisfies

$$\psi^{(n)}(s) + (\ell(s) - \mu(s))\,\psi(s) = 0,$$

with (1.3) such that $|\zeta(s) - \psi(s)| \leqslant K\epsilon E_\nu(s)$.

If the preceding definitions is also correct when we setting $\epsilon E_\nu(s)$ with $\phi(s)\ \epsilon E_\nu(s)$, where $\phi : I \to [0, \infty)$ are functions not depending on $\zeta(s)$ and $\psi(s)$ explicitly, then we say that the corresponding differential equation has the Mittag-Leffler-Hyers-Ulam-Rassias stability.

3. Mittag-Leffler-Hyers-Ulam stability

In the following theorems, we prove the Mittag-Leffler-Hyers-Ulam stability of the linear differential equation (1.1) with (1.2) and (1.3).

Theorem 3.1. *If* $\max|\ell(s) - \mu(s)| < \dfrac{n!\,2^n}{(\jmath - \imath)^n}$ *for* $s \in [\imath, \jmath]$. *Then, the differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam with boundary conditions* (1.2).

Proof. For each $\epsilon > 0$, there exists $\zeta \in C^n([\imath, \jmath])$, such that

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon E_\nu(s), \tag{3.1}$$

where $E_\nu(s)$ is a Mittag-Leffler function and $\zeta(\imath) = \zeta(\jmath) = 0$. Let us define $M = \max\{|\zeta(s)| : s \in [\imath, \jmath]\}$. Since $\zeta(\imath) = \zeta(\jmath) = 0$, there exists $s_0 \in (\imath, \jmath)$ such that $|\zeta(s_0)| = M$. By Taylor's formula, we get

$$\zeta(\imath) = \zeta(s_0) + \zeta'(s_0)(s_0 - \imath) + \frac{\zeta''(s_0)}{2!}(s_0 - \imath)^2 + \cdots + \frac{\zeta^{(n)}(\gamma)}{n!}(s_0 - \imath)^n, \tag{3.2}$$

$$\zeta(\jmath) = \zeta(s_0) + \zeta'(s_0)(\jmath - s_0) + \frac{\zeta''(s_0)}{2!}(\jmath - s_0)^2 + \cdots + \frac{\zeta^{(n)}(\delta)}{n!}(\jmath - s_0)^n. \tag{3.3}$$

We have $\zeta(\imath) = 0$, and so equation (3.2) becomes

$$\zeta(s_0) + \zeta'(s_0)(s_0 - \imath) + \frac{\zeta''(s_0)}{2!}(s_0 - \imath)^2 + \cdots + \frac{\zeta^{(n)}(\gamma)}{n!}(s_0 - \imath)^n = 0.$$

Thus, we have $\left|\zeta^{(n)}(\gamma)\right| = \frac{n!\,M}{(s_0 - \imath)^n}$. Similarly, from $\zeta(\jmath) = 0$ the relation (3.3) can be converted to

$$\zeta(s_0) + \zeta'(s_0)(\jmath - s_0) + \frac{\zeta''(s_0)}{2!}(\jmath - s_0)^2 + \cdots + \frac{\zeta^{(n)}(\delta)}{n!}(\jmath - s_0)^n = 0.$$

So, we have $\left|\zeta^{(n)}(\delta)\right| = \frac{n!\,M}{(\jmath - s_0)^n}$. On the other hand, for $s_0 \in (\imath, \frac{\imath+\jmath}{2}]$, we obtain

$$\frac{n!\,M}{(s_0 - \imath)^n} \geqslant \frac{n!\,M}{\frac{(\jmath-\imath)^n}{2^n}} = \frac{n!\,2^n\,M}{(\jmath - \imath)^n}. \tag{3.4}$$

Now, if $s_0 \in [\frac{\imath+\jmath}{2}, \jmath)$, then

$$\frac{M\,n!}{(s_0 - \jmath)^n} \geqslant \frac{n!\,M}{\frac{(\jmath-\imath)^n}{2^n}} = \frac{n!\,2^n\,M}{(\jmath - \imath)^n}. \tag{3.5}$$

Using (3.4) and (3.5), one can obtain $\max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{2^n n!} \max\left|\zeta^{(n)}(s)\right|$. Hence,

$$\begin{aligned}
\max|\zeta(s)| &\leqslant \frac{(\jmath - \imath)^n}{2^n n!}\left\{\max\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s) - (\ell(s) - \mu(s))\,\zeta(s)\right|\right\} \\
&\leqslant \frac{(\jmath - \imath)^n}{2^n n!}\left\{\max\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| + \max|(\ell(s) - \mu(s))|\ \max|\zeta(s)|\right\}.
\end{aligned}$$

Now, let us choose $\lambda = \frac{(\jmath - \imath)^n}{2^n n!} \max|(\ell(s) - \mu(s))|$. Then, we obtain that

$$\max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{2^n n!}\epsilon E_\nu(s) + \lambda\ \max|\zeta(s)| \ \Rightarrow\ \max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{2^n\ n!\ (1 - \lambda)}\ \epsilon E_\nu(s).$$

Choosing $K = \frac{(\jmath - \imath)^n}{2^n\ n!\ (1 - \lambda)}$. So, we have $\max|\zeta(s)| \leqslant K\epsilon E_\nu(s)$. Obviously, $\psi_0(s) \equiv 0$ is a solution of

$$\zeta^{(n)}(s) - (\ell(s) - \mu(s))\,\zeta(s) = 0,$$

with boundary conditions $\zeta(\imath) = \zeta(\jmath) = 0$. Therefore, $|\zeta(s) - \psi_0(s)| \leqslant K\epsilon E_\nu(s)$. Thus the differential equation (1.1) has Mittag-Leffler-Hyers-Ulam with boundary conditions (1.2). □

Now, we prove the Mittag-Leffler-HU stability of a differential equation (1.1) with initial conditions (1.3).

Theorem 3.2. *If* $\max|(\ell(s)-\mu(s))| < \dfrac{n!}{(\jmath-\imath)^n}$ *for* $s \in [\imath,\jmath]$*. Then, the differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam stability with initial conditions* (1.3)*.*

Proof. For each $\epsilon > 0$, there exists $\zeta \in C^n([\imath,\jmath])$, such that

$$\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s)\right| \leqslant \epsilon E_\nu(s),$$

where $E_\nu(s)$ is a Mittag-Leffler function. By Taylor's formula, we arrive at

$$\zeta(s) = \zeta(\imath) + \zeta'(\imath)(s-\imath) + \frac{\zeta''(\imath)}{2!}(s-\imath)^2 + \cdots + \frac{\zeta^{(n)}(\xi)}{n!}(s-\imath)^n. \tag{3.6}$$

Using the condition (1.3), then (3.6) becomes $\zeta(s) = \dfrac{\zeta^{(n)}(\xi)}{n!}(s-\imath)^n$ and thus

$$\max|\zeta(s)| \leqslant \max\left|\zeta^{(n)}(s)\right| \frac{(\jmath-\imath)^n}{n!},$$

so, we obtain

$$\begin{aligned}\max|\zeta(s)| &\leqslant \frac{(\jmath-\imath)^n}{n!}\left\{\max\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s) - (\ell(s)-\mu(s))\,\zeta(s)\right|\right\}\\ &\leqslant \frac{(\jmath-\imath)^n}{n!}\left\{\max\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s)\right| + \max|(\ell(s)-\mu(s))|\ \max|\zeta(s)|\right\}.\end{aligned}$$

Now, let us choose $\eta = \dfrac{(\jmath-\imath)^n}{n!}\ \max|(\ell(s)-\mu(s))|$. Then

$$\max|\zeta(s)| \leqslant \frac{(\jmath-\imath)^n}{n!}\epsilon E_\nu(s) + \eta\ \max|\zeta(s)|.$$

Hence, we have $\max|\zeta(s)| \leqslant K\ \epsilon E_\nu(s)$, where $K = \dfrac{(\jmath-\imath)^n}{n!\ (1-\eta)}$. Hence, $\max|\zeta(s)| \leqslant K\epsilon E_\nu(s)$. Obviously, $\psi_0(s) \equiv 0$ is a solution to $\zeta^{(n)}(s) - (\ell(s)-\mu(s))\,\zeta(s) = 0$ with the initial conditions

$$\zeta(\imath) = \zeta'(\imath) = \zeta''(\imath) = \cdots = \zeta^{(n-1)}(\imath) = 0.$$

Thus, $|\zeta(s)-\psi_0(s)| \leqslant K\epsilon E_\nu(s)$. Hence the differential equation (1.1) has Mittag-Leffler-Hyers-Ulam in the presence of initial conditions (1.3). □

In the following corollaries, we prove the Hyers-Ulam stability of the linear differential equation (1.1) with (1.2) and (1.3). If we replace $\epsilon E_\nu(s)$ by ϵ in the inequality (3.1), we will have the HUS.

Corollary 3.3. *If* $\max|\ell(s)-\mu(s)| < \dfrac{n!\ 2^n}{(\jmath-\imath)^n}$ *for* $s \in [\imath,\jmath]$*. Then, the differential equation* (1.1) *has Hyers-Ulam stability with boundary conditions* (1.2)*.*

Proof. For every $\epsilon > 0$, there exists $\zeta \in C^n[\imath,\jmath]$, if

$$\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s)\right| \leqslant \epsilon,$$

where $E_\nu(s)$ is a Mittag-Leffler function and $\zeta(\imath) = \zeta(\jmath) = 0$. Let us define $M = \max\{|\zeta(s)| : s \in [\imath, \jmath]\}$. Since $\zeta(\imath) = \zeta(\jmath) = 0$, there exists $s_0 \in (\imath, \jmath)$ such that $|\zeta(s_0)| = M$. By Taylor's formula, we have

$$\zeta(\imath) = \zeta(s_0) + \zeta'(s_0)(s_0 - \imath) + \frac{\zeta''(s_0)}{2!}(s_0 - \imath)^2 + \cdots + \frac{\zeta^{(n)}(\gamma)}{n!}(s_0 - \imath)^n, \tag{3.7}$$

$$\zeta(\jmath) = \zeta(s_0) + \zeta'(s_0)(\jmath - s_0) + \frac{\zeta''(s_0)}{2!}(\jmath - s_0)^2 + \cdots + \frac{\zeta^{(n)}(\delta)}{n!}(\jmath - s_0)^n, \tag{3.8}$$

where $\gamma \in (a, s_0)$ and $\delta \in (s_0, b)$. Thus, we have $\left|\zeta^{(n)}(\gamma)\right| = \frac{n!\, M}{(s_0 - \imath)^n}$. Now, let us take $s_0 \in (\imath, \frac{\imath+\jmath}{2}]$, we get

$$\frac{n!\, M}{(s_0 - \imath)^n} \geqslant \frac{n!\, M}{\frac{(\jmath-\imath)^n}{2^n}} = \frac{2^n\, n!\, M}{(\jmath - \imath)^n}. \tag{3.9}$$

So, we have $\left|\zeta^{(n)}(\delta)\right| = \frac{n!\, M}{(s_0 - \jmath)^n}$. Now, let us take $s_0 \in [\frac{\imath+\jmath}{2}, \jmath)$, then

$$\frac{M\, n!}{(s_0 - \jmath)^n} \geqslant \frac{n!\, M}{\frac{(\jmath-\imath)^n}{2^n}} = \frac{2^n\, n!\, M}{(\jmath - \imath)^n}. \tag{3.10}$$

Using (3.9) and (3.10), one can reach $\max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{2^n n!} \max\left|\zeta^{(n)}(s)\right|$. Hence,

$$\max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{2^n n!}\epsilon + \lambda \max|\zeta(s)|,$$

where $\lambda = \frac{(\jmath - \imath)^n}{2^n n!} \max|(\ell(s) - \mu(s))|$. We get

$$\max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{2^n\, n!\, (1 - \lambda)}\, \epsilon,$$

where $K = \frac{(\jmath - \imath)^n}{2^n\, n!\, (1 - \lambda)}$, therefore, we have $\max|\zeta(s)| \leqslant K\epsilon$. Clearly, $\psi_0(s) \equiv 0$ is a solution of the linear differential equation $\zeta^{(n)}(s) - (\ell(s) - \mu(s))\,\zeta(s) = 0$ with boundary conditions (1.2). Hence, $|\zeta(s) - \psi_0(s)| \leqslant K\epsilon$. Thus the linear differential equation (1.1) has HU stability with boundary conditions (1.2). □

Corollary 3.4. *Let* $\max|(\ell(s) - \mu(s))| < \frac{n!}{(\jmath - \imath)^n}$ *for* $s \in [\imath, \jmath]$. *Then, the differential equation* (1.1) *has Hyers-Ulam stability with of initial conditions* (1.3).

Proof. For each $\epsilon > 0$, there exists $\zeta \in C^n([\imath, \jmath])$, such that

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon.$$

By Taylor's formula, we arrive at

$$\zeta(s) = \zeta(\imath) + \zeta'(\imath)(s - \imath) + \frac{\zeta''(\imath)}{2!}(s - \imath)^2 + \cdots + \frac{\zeta^{(n)}(\xi)}{n!}(s - \imath)^n. \tag{3.11}$$

Using the condition (1.3), then (3.11) becomes $\zeta(s) = \frac{\zeta^{(n)}(\xi)}{n!}(s - \imath)^n$ and thus

$$\max|\zeta(s)| \leqslant \max\left|\zeta^{(n)}(s)\right| \frac{(\jmath - \imath)^n}{n!},$$

so, we obtain

$$\begin{aligned}\max|\zeta(s)| &\leqslant \frac{(\jmath-\imath)^n}{n!}\left\{\max\left|\zeta^{(n)}(s)+(\ell(s)-\mu(s))\,\zeta(s)-(\ell(s)-\mu(s))\,\zeta(s)\right|\right\}\\ &\leqslant \frac{(\jmath-\imath)^n}{n!}\left\{\max\left|\zeta^{(n)}(s)+(\ell(s)-\mu(s))\,\zeta(s)\right|+\max|(\ell(s)-\mu(s))|\ \max|\zeta(s)|\right\}.\end{aligned}$$

Now, let us choose $\eta=\frac{(\jmath-\imath)^n}{n!}\ \max|(\ell(s)-\mu(s))|$. Then

$$\max|\zeta(s)|\leqslant\frac{(\jmath-\imath)^n}{n!}\epsilon+\eta\ \max|\zeta(s)|.$$

Hence, we have $\max|\zeta(s)|\leqslant K\epsilon$, where $K=\frac{(\jmath-\imath)^n}{n!\,(1-\eta)}$. Obviously, $\psi_0(s)\equiv 0$ is a solution of the linear differential equation $\zeta^{(n)}(s)-(\ell(s)-\mu(s))\,\zeta(s)=0$ with the initial conditions (1.3). Thus,

$$|\zeta(s)-\psi_0(s)|\leqslant K\epsilon.$$

Hence, the differential equation (1.1) has Hyers-Ulam stability with (1.3). □

4. Mittag-Leffler-Hyers-Ulam-Rassias stability

In the following theorems, we will establish the Mittag-Leffler-Hyers-Ulam-Rassias of the linear differential equation (1.1) with (1.2) and (1.3).

Theorem 4.1. *Suppose if* $\max|\ell(s)-\mu(s)|<\frac{n!\,2^n}{(\jmath-\imath)^n}$ *for* $s\in[\imath,\jmath]$*, then the linear differential equation of higher order* (1.1) *has Mittag-Leffler-Hyers-Ulam-Rassias stability with boundary conditions* (1.2)*.*

Proof. For all $\epsilon>0$, there is a $\zeta\in C^n([\imath,\jmath])$ and $\phi:\mathbb{R}\to[0,\infty)$ such that

$$\left|\zeta^{(n)}(s)+(\ell(s)-\mu(s))\,\zeta(s)\right|\leqslant\epsilon\phi(s)E_\nu(s), \tag{4.1}$$

where $E_\nu(s)$ is a Mittag-Leffler function with $\zeta(\imath)=\zeta(\jmath)=0$. Let us define $M=\max\{|\zeta(s)|\ :\ s\in[\imath,\jmath]\}$. Since $\zeta(\imath)=\zeta(\jmath)=0$, there exists $s_0\in(\imath,\jmath)$ such that $|\zeta(s_0)|=M$. By Taylor's formula, we get

$$\zeta(\imath)=\zeta(s_0)+\zeta'(s_0)(s_0-\imath)+\frac{\zeta''(s_0)}{2!}(s_0-\imath)^2+\cdots+\frac{\zeta^{(n)}(\gamma)}{n!}(s_0-\imath)^n, \tag{4.2}$$

$$\zeta(\jmath)=\zeta(s_0)+\zeta'(s_0)(\jmath-s_0)+\frac{\zeta''(s_0)}{2!}(\jmath-s_0)^2+\cdots+\frac{\zeta^{(n)}(\delta)}{n!}(\jmath-s_0)^n. \tag{4.3}$$

We have $\zeta(\imath)=0$, and so equation (4.2) becomes

$$\zeta(s_0)+\zeta'(s_0)(s_0-\imath)+\frac{\zeta''(s_0)}{2!}(s_0-\imath)^2+\cdots+\frac{\zeta^{(n)}(\gamma)}{n!}(s_0-\imath)^n=0.$$

Thus, we have $\left|\zeta^{(n)}(\gamma)\right|=\frac{n!\,M}{(s_0-\imath)^n}$. Similarly, from $\zeta(\jmath)=0$ the relation (3.3) can be converted to

$$\zeta(s_0)+\zeta'(s_0)(\jmath-s_0)+\frac{\zeta''(s_0)}{2!}(\jmath-s_0)^2+\cdots+\frac{\zeta^{(n)}(\delta)}{n!}(\jmath-s_0)^n=0.$$

So, we have $\left|\zeta^{(n)}(\delta)\right| = \dfrac{n!\,M}{(\jmath - s_0)^n}$. On the other hand, for $s_0 \in (\imath, \frac{\imath+\jmath}{2}]$, we obtain

$$\frac{n!\,M}{(s_0-\imath)^n} \geqslant \frac{n!\,M}{\frac{(\jmath-\imath)^n}{2^n}} = \frac{2^n\,n!\,M}{(\jmath-\imath)^n}. \tag{4.4}$$

Now, if $s_0 \in [\frac{\imath+\jmath}{2}, \jmath)$, then

$$\frac{M\,n!}{(s_0-\jmath)^n} \geqslant \frac{n!\,M}{\frac{(\jmath-\imath)^n}{2^n}} = \frac{2^n\,n!\,M}{(\jmath-\imath)^n}. \tag{4.5}$$

Using (4.4) and (4.5), one can reach $\max|\zeta(s)| \leqslant \dfrac{(\jmath-\imath)^n}{2^n n!} \max\left|\zeta^{(n)}(s)\right|$. Hence,

$$\begin{aligned}\max|\zeta(s)| &\leqslant \frac{(\jmath-\imath)^n}{2^n n!}\left\{ \max\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s) - (\ell(s)-\mu(s))\,\zeta(s)\right|\right\}\\ &\leqslant \frac{(\jmath-\imath)^n}{2^n n!}\left\{\max\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s)\right| + \max|(\ell(s)-\mu(s))|\ \max|\zeta(s)|\right\}.\end{aligned}$$

Now, let us choose $\lambda = \dfrac{(\jmath-\imath)^n}{2^n n!} \max|(\ell(s)-\mu(s))|$. Then, we obtain that

$$\max|\zeta(s)| \leqslant \frac{(\jmath-\imath)^n}{2^n n!}\epsilon\phi(s)E_\nu(s) + \lambda\ \max|\zeta(s)| \ \Rightarrow\ \max|\zeta(s)| \leqslant \frac{(\jmath-\imath)^n}{2^n\ n!\ (1-\lambda)}\ \epsilon\phi(s)E_\nu(s).$$

Consider $K = \dfrac{(\jmath-\imath)^n}{2^n\ n!\ (1-\lambda)}$. So, we have $\max|\zeta(s)| \leqslant K\epsilon\phi(s)E_\nu(s)$. Obviously, $\psi_0(s) \equiv 0$ is a solution of the differential equation $\zeta^{(n)}(s) - (\ell(s)-\mu(s))\,\zeta(s) = 0$ with boundary conditions $\zeta(\imath) = \zeta(\jmath) = 0$. Therefore,

$$|\zeta(s) - \psi_0(s)| \leqslant K\epsilon\phi(s)E_\nu(s).$$

Hence, the linear differential equation (1.1) has Mittag-Leffler-Hyers-Ulam-Rassias stability in the presence of boundary conditions (1.2). □

Now, we study the Mittag-Leffler-Hyers-Ulam-Rassias stability of a differential equation (1.1) with initial conditions (1.3).

Theorem 4.2. *If* $\max|(\ell(s)-\mu(s))| < \dfrac{n!}{(\jmath-\imath)^n}$ *for* $s \in [\imath, \jmath]$. *Then the differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam-Rassias stability in the presence of initial conditions* (1.3).

Proof. For each $\epsilon > 0$, there exists $\zeta \in C^n([\imath, \jmath])$ such that

$$\left|\zeta^{(n)}(s) + (\ell(s)-\mu(s))\,\zeta(s)\right| \leqslant \epsilon\phi(s)E_\nu(s),$$

where $E_\nu(s)$ is a Mittag-Leffler function. By Taylor's formula, we have

$$\zeta(s) = \zeta(\imath) + \zeta'(\imath)(s-\imath) + \frac{\zeta''(\imath)}{2!}(s-\imath)^2 + \cdots + \frac{\zeta^{(n)}(\xi)}{n!}(s-\imath)^n. \tag{4.6}$$

Using the condition (1.3), then (4.6) becomes $\zeta(s) = \dfrac{\zeta^{(n)}(\xi)}{n!}(s-\imath)^n$ and thus

$$\max|\xi(s)| \leqslant \max\left|\zeta^{(n)}(s)\right| \frac{(\jmath-\imath)^n}{n!},$$

so, we obtain

$$\begin{aligned}\max|\zeta(s)| &\leqslant \frac{(\jmath-\imath)^n}{n!}\left\{\max\left|\zeta^{(n)}(s)+(\ell(s)-\mu(s))\,\zeta(s)-(\ell(s)-\mu(s))\,\zeta(s)\right|\right\}\\ &\leqslant \frac{(\jmath-\imath)^n}{n!}\left\{\max\left|\zeta^{(n)}(s)+(\ell(s)-\mu(s))\,\zeta(s)\right|+\max|(\ell(s)-\mu(s))|\ \max|\zeta(s)|\right\}.\end{aligned}$$

Now, let us choose $\eta = \frac{(\jmath-\imath)^n}{n!}\ \max|(\ell(s)-\mu(s))|$. Then

$$\max|\zeta(s)| \leqslant \frac{(\jmath-\imath)^n}{n!}\epsilon\phi(s)E_\nu(s)+\eta\ \max|\zeta(s)|.$$

Hence, $\max|\zeta(s)| \leqslant K\ \epsilon\phi(s)E_\nu(s)$, where $K = \frac{(\jmath-\imath)^n}{n!\,(1-\eta)}$. Obviously, $\psi_0(s) \equiv 0$ is a solution of the differential equation $\zeta^{(n)}(s)-(\ell(s)-\mu(s))\,\zeta(s)=0$ with the initial conditions

$$\zeta(\imath)=\zeta'(\imath)=\zeta''(\imath)=\cdots=\zeta^{(n-1)}(\imath)=0.$$

Thus, $|\zeta(s)-\psi_0(s)| \leqslant K\epsilon\phi(s)E_\nu(s)$. Then the differential equation (1.1) has Mittag-Leffler-Hyers-Ulam-Rassias stability in the presence of initial conditions (1.3). □

If we replace $\epsilon\phi(s)E_\nu(s)$ by $\epsilon\phi(s)$ in the inequality (4.1), one can obtain the HURS. Now, we investigate the Hyers-Ulam-Rassias stability of a differential equation (1.1) with boundary conditions (1.2).

Corollary 4.3. *Let* $\max|\ell(s)-\mu(s)| < \frac{n!\,2^n}{(\jmath-\imath)^n}$ *for* $s \in [\imath,\jmath]$. *Then, the differential equation* (1.1) *has Hyers-Ulam-Rassias stability with boundary conditions* (1.2).

Proof. For every $\epsilon > 0$, there is $\zeta \in C^n(I)$ and a function $\phi(s) : I \to [0,\infty)$ satisfying

$$\left|\zeta^{(n)}(s)+(\ell(s)-\mu(s))\,\zeta(s)\right| \leqslant \epsilon\phi(s),$$

with boundary conditions $\zeta(\imath)=\zeta(\jmath)=0$. By using the same technique as applied in Theorem 4.2, we can easily reach at

$$\max|\zeta(s)| \leqslant \frac{(\jmath-\imath)^n}{2^n n!}\epsilon E_\nu(s)+\lambda\ \max|\zeta(s)|,$$

where $\lambda = \frac{(\jmath-\imath)^n}{2^n n!}\ \max|(\ell(s)-\mu(s))|$. We get $\max|\zeta(s)| \leqslant \frac{(\jmath-\imath)^n}{2^n\ n!\ (1-\lambda)}\ \epsilon\phi(s)$. taking $K = \frac{(\jmath-\imath)^n}{2^n\ n!\ (1-\lambda)}$, we have $\max|\zeta(s)| \leqslant K\epsilon\phi(s)$. Obviously, $\psi_0(s) \equiv 0$ is a solution of the linear differential equation

$$\zeta^{(n)}(s)-(\ell(s)-\mu(s))\,\zeta(s)=0,$$

with boundary conditions (1.2). Therefore, $|\zeta(s)-\psi_0(s)| \leqslant K\epsilon\phi(s)$. Hence the linear differential equation (1.1) has Ulam-Rassias stable with boundary conditions (1.2). □

Finally, we study the Hyers-Ulam-Rassias stability of a differential equation (1.1) with initial conditions (1.3).

Corollary 4.4. *If* $\max|(\ell(s)-\mu(s))| < \frac{n!}{(\jmath-\imath)^n}$ *for* $s \in [\imath,\jmath]$. *Then, the differential equation* (1.1) *has Hyers-Ulam-Rassias stability with of initial conditions* (1.3).

Proof. For every $\epsilon > 0$, there exists a function $\zeta \in C^n([\imath, \jmath])$ and $\phi : I \to [0, \infty)$ such that

$$\left|\zeta^{(n)}(s) + (\ell(s) - \mu(s))\,\zeta(s)\right| \leqslant \epsilon\phi(s),$$

with $\zeta(\imath) = \zeta'(\imath) = \zeta''(\imath) = \cdots = \zeta^{(n-1)}(\imath) = 0$. By using the same methodology as used in Theorem 4.2, we can easily reach the rest of the proof. So, we have

$$\max|\zeta(s)| \leqslant \frac{(\jmath - \imath)^n}{n!}\phi(s)E_\nu(s)\epsilon + \eta\ \max|\zeta(s)|.$$

Hence, we have $\max|\zeta(s)| \leqslant K\ \epsilon\phi(s)E_\nu(s)$, where $K = \dfrac{(\jmath - \imath)^n}{n!\ (1 - \eta)}$. Clearly, $\psi_0(s) \equiv 0$ is a solution of

$$\zeta^{(n)}(s) - (\ell(s) - \mu(s))\,\zeta(s) = 0,$$

with (1.3). Thus, $|\zeta(s) - \psi_0(s)| \leqslant K\epsilon\phi(s)$. Hence, the differential equation (1.1) is Hyers-Ulam-Rassias stable with initial conditions (1.3). □

5. Conclusion

In [3, 11, 12], they proved the Hyers-Ulam stability and Superstability of the second order differential equations using initial or boundary conditions. In this paper, we generalized the above studied results and established the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the special type general linear differential equation of higher order with initial and boundary conditions using Taylor's series formula.

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