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Some properties of (r, s)-generalized fuzzy semi-closed sets and some applications

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Abstract

In this work, the notion of (r, s)-generalized fuzzy semi-closed sets is introduced and some properties are given. Also, we show that every (r, s)-generalized fuzzy closed set by Abbas (2006) is (r, s)-generalized fuzzy semi-closed set, but the converse need not be true. After that, the generalized forms of fuzzy continuous mappings between double fuzzy topological spaces (X, τ_1, τ_1^*) and (Y, τ_2, τ_2^*) are introduced and studied. Some interesting relationship between these mappings and other mappings introduced previously are investigated with the help of examples. In the end, the notion of (r, s)-fuzzy GS-connected sets is introduced and studied with help of (r, s)-generalized fuzzy semi-closed sets.

Keywords: Double fuzzy topological space, (r, s)-generalized fuzzy semi-closed set, (r, s)-fuzzy GS-connected set, DFGS-continuity, DFGS-irresolute.

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1. Introduction

Zadeh [21] introduced the basic idea of a fuzzy set as an extension of classical set theory. The theory of fuzzy sets provides a framework for mathematical modeling of those real world situations, which involve an element of imprecision, uncertainty, or vagueness in their description. This theory has found wide applications in engineering, information sciences, etc.; for details the reader is referred to [15, 23]. The notion of an intuitionistic fuzzy set which is a generalization of fuzzy sets was introduced by Atanassov [5, 6]. Çoker [8, 9] introduced the idea of fuzzy topological space in Chang's sense [7] of intuitionistic fuzzy sets. After that Samanta and Mondal [18, 19] gave the definition of an intuitionistic fuzzy topological space as a generalization of fuzzy topological space in Šostak's sense [20]. The concept of (r, s)-fuzzy semi-closed sets was introduced and investigated by Lee [14]. The name (intuitionistic) was replaced with the name (double) by Garcia and Rodabaugh [11]. Zahran et al. [22] have introduced (r, s)-semi generalized fuzzy closed sets in double fuzzy topological spaces in Šostak's sense and discussed some of their properties; for more details the reader is referred to [2–4, 16, 17].

The paper is organized in the following way. In Section 3, the notion of (r, s)-generalized fuzzy semi-closed sets is introduced in double fuzzy topological space (X, τ, τ^*) based on the sense of Šostak and

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some characterizations are given. Also, we show that every (r, s)-generalized fuzzy closed set [1] is (r, s)-generalized fuzzy semi-closed set, but the converse need not be true. In Section 4, the generalized forms of fuzzy continuous mappings between double fuzzy topological spaces are introduced and studied. The relationship between these mappings and other mappings introduced previously are investigated with the help of examples. In Section 5, the notion of (r, s)-fuzzy GS-connected sets is introduced and studied with help of (r, s)-generalized fuzzy semi-closed sets.

2. Preliminary assertions

In this section, we present the basic definitions and results which we need next sections. Throughout this paper, nonempty sets will be denoted by X, Y etc., $I = [0, 1]$, $I_\circ = (0, 1]$ and $I_1 = [0, 1)$. For $\alpha \in I$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t for $t \in I_\circ$ is an element of I^X such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denote $\lambda \overline{q} \mu$. Notions and notations not described in this paper are standard and usual.

Proposition 2.1 ([12]). *Let X be a nonempty set and $\lambda, \mu \in I^X$. Then:*

(1) $\lambda q \mu$ *iff there exists* $x_t \in \lambda$ *such that* $x_t q \mu$;
(2) *if* $\lambda q \mu$, *then* $\lambda \wedge \mu \neq \underline{0}$;
(3) $\lambda \overline{q} \mu$ *iff* $\lambda \leqslant \underline{1} - \mu$;
(4) $\lambda \leqslant \mu$ *iff* $x_t \in \lambda$ *implies* $x_t \in \mu$ *iff* $x_t q \lambda$ *implies* $x_t q \mu$ *iff* $x_t \overline{q} \mu$ *implies* $x_t \overline{q} \lambda$;
(5) $x_t \overline{q} \bigvee_{i \in \Gamma} \mu_i$ *iff there exists* $i_0 \in \Gamma$ *such that* $x_t \overline{q} \mu_{i_0}$.

Definition 2.2 ([19, 22]). A double fuzzy topology on X is a pair (τ, τ^*) of the mappings $\tau, \tau^* : I^X \to I$, which satisfies the following conditions:

(1) $\tau(\lambda) + \tau^*(\lambda) \leqslant 1, \forall \lambda \in I^X$;
(2) $\tau(\lambda_1 \wedge \lambda_2) \geqslant \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^*(\lambda_1 \wedge \lambda_2) \leqslant \tau^*(\lambda_1) \vee \tau^*(\lambda_2), \forall \lambda_1, \lambda_2 \in I^X$;
(3) $\tau(\bigvee_{i \in \Gamma} \lambda_i) \geqslant \bigwedge_{i \in \Gamma} \tau(\lambda_i)$ and $\tau^*(\bigvee_{i \in \Gamma} \lambda_i) \leqslant \bigvee_{i \in \Gamma} \tau^*(\lambda_i), \forall \{\lambda_i\}_{i \in \Gamma} \subset I^X$.

The triplet (X, τ, τ^*) is called a double fuzzy topological space (dfts, for short). $\tau(\lambda)$ and $\tau^*(\lambda)$ may be interpreted as a gradation of openness and a gradation of nonopenness for $\lambda \in I^X$, respectively.

Theorem 2.3 ([10, 13]). *Let (X, τ, τ^*) be a dfts. Then for each $\lambda \in I^X$, $r \in I_\circ$ and $s \in I_1$ we define an operator $C_{\tau,\tau^*} : I^X \times I_\circ \times I_1 \to I^X$ as follows: $C_{\tau,\tau^*}(\lambda, r, s) = \bigwedge\{\mu \in I^X : \lambda \leqslant \mu, \tau(\underline{1} - \mu) \geqslant r, \tau^*(\underline{1} - \mu) \leqslant s\}$. For λ, $\mu \in I^X$ and $r, r_1 \in I_\circ$, $s, s_1 \in I_1$ the operator C_{τ,τ^*} satisfies the following statements:*

(C1) $C_{\tau,\tau^*}(\underline{0}, r, s) = \underline{0}$;
(C2) $\lambda \leqslant C_{\tau,\tau^*}(\lambda, r, s)$;
(C3) $C_{\tau,\tau^*}(\lambda, r, s) \vee C_{\tau,\tau^*}(\mu, r, s) = C_{\tau,\tau^*}(\lambda \vee \mu, r, s)$;
(C4) $C_{\tau,\tau^*}(\lambda, r, s) \leqslant C_{\tau,\tau^*}(\lambda, r_1, s_1)$ *if* $r \leqslant r_1$ *and* $s \geqslant s_1$;
(C5) $C_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, r, s), r, s) = C_{\tau,\tau^*}(\lambda, r, s)$.

Theorem 2.4 ([10, 13]). *Let (X, τ, τ^*) be a dfts. Then for each $\lambda \in I^X$, $r \in I_\circ$ and $s \in I_1$ we define an operator $I_{\tau,\tau^*} : I^X \times I_\circ \times I_1 \to I^X$ as follows: $I_{\tau,\tau^*}(\lambda, r, s) = \bigvee\{\mu \in I^X : \mu \leqslant \lambda, \tau(\mu) \geqslant r, \tau^*(\mu) \leqslant s\}$. Also, the operator I_{τ,τ^*} satisfies the following: $I_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - C_{\tau,\tau^*}(\lambda, r, s)$ and $C_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - I_{\tau,\tau^*}(\lambda, r, s)$.*

Definition 2.5 ([14]). Let (X, τ, τ^*) be a dfts, $\lambda, \mu \in I^X$ and $r \in I_\circ$, $s \in I_1$.

(1) λ is called (r,s)-fuzzy semi-closed set ((r,s)-fsc, for short) if $I_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,r,s),r,s) \leqslant \lambda$. The (r,s)-fuzzy semi-closure of λ, denoted by $SC_{\tau,\tau^*}(\lambda,r,s)$ is defined by : $SC_{\tau,\tau^*}(\lambda,r,s) = \bigwedge\{\mu \in I^X : \mu \geqslant \lambda, \mu \text{ is } (r,s)\text{-fsc}\}$.
(2) λ is called (r,s)-fuzzy semi-open ((r,s)-fso, for short) if $\lambda \leqslant C_{\tau,\tau^*}(I_{\tau,\tau^*}(\lambda,r,s),r,s)$. The (r,s)-fuzzy semi-interior of λ, denoted by $SI_{\tau,\tau^*}(\lambda,r,s)$ is defined by : $SI_{\tau,\tau^*}(\lambda,r,s) = \bigvee\{\mu \in I^X : \mu \leqslant \lambda, \mu \text{ is } (r,s)\text{-fso}\}$.

Definition 2.6 ([1, 22]). Let (X,τ,τ^*) be a dfts, $\lambda,\mu \in I^X$ and $r \in I_\circ$, $s \in I_1$.

(1) λ is called (r,s)-generalized fuzzy closed set ((r,s)-gfc, for short) if $C_{\tau,\tau^*}(\lambda,r,s) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and $\tau(\mu) \geqslant r$, $\tau^*(\mu) \leqslant s$.
(2) λ is called (r,s)-semi generalized fuzzy closed set ((r,s)-sgfc, for short) if $SC_{\tau,\tau^*}(\lambda,r,s) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and μ is (r,s)-fso.
The complement of (r,s)-gfc (resp. (r,s)-sgfc) is (r,s)-gfo (resp. (r,s)-sgfo).

Theorem 2.7 ([1]). *Let (X,τ,τ^*) be a dfts. For each $\lambda,\mu \in I^X$ and $r \in I_\circ$, $s \in I_1$, we define an (r,s)-generalized fuzzy closure $GC_{\tau,\tau^*} : I^X \times I_\circ \times I_1 \to I^X$ as follows: $GC_{\tau,\tau^*}(\lambda,r,s) = \bigwedge\{\mu \in I^X : \mu \geqslant \lambda, \mu \text{ is } (r,s)\text{-gfc}\}$. For $\lambda,\mu \in I^X$ and $r \in I_\circ$, $s \in I_1$, the operator GC_{τ,τ^*} satisfies the following properties.*

(1) $GC_{\tau,\tau^*}(\underline{0},r,s) = \underline{0}$.
(2) $\lambda \leqslant GC_{\tau,\tau^*}(\lambda,r,s)$.
(3) $GC_{\tau,\tau^*}(\lambda,r,s) \vee GC_{\tau,\tau^*}(\mu,r,s) = GC_{\tau,\tau^*}(\lambda \vee \mu,r,s)$.
(4) $GC_{\tau,\tau^*}(GC_{\tau,\tau^*}(\lambda,r,s),r,s) = GC_{\tau,\tau^*}(\lambda,r,s)$.
(5) *If λ is (r,s)-gfc, then $GC_{\tau,\tau^*}(\lambda,r,s) = \lambda$.*
(6) $GC_{\tau,\tau^*}(\lambda,r,s) \leqslant C_{\tau,\tau^*}(\lambda,r,s)$.
(7) $C_{\tau,\tau^*}(GC_{\tau,\tau^*}(\lambda,r,s),r,s) = GC_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda,r,s),r,s) = C_{\tau,\tau^*}(\lambda,r,s)$.

Definition 2.8 ([1]). Let (X,τ,τ^*) be a dfts, $\lambda,\mu \in I^X$ and $r \in I_\circ$, $s \in I_1$. Then, two fuzzy sets λ and μ are said to be (r,s)-fuzzy separated iff $\lambda\overline{q}C_{\tau,\tau^*}(\mu,r,s)$ and $\mu\overline{q}C_{\tau,\tau^*}(\lambda,r,s)$. Also, a fuzzy set which cannot be expressed as the union of two (r,s)-fuzzy separated sets is said to be (r,s)-fuzzy connected set.

Definition 2.9 ([1]). A dfts (X,τ,τ^*) is called $DFT_{\frac{1}{2}}$ if $\tau(\underline{1}-\lambda) \geqslant r$, $\tau^*(\underline{1}-\lambda) \leqslant s$ for each $\lambda \in I^X$ is (r,s)-gfc set and $r \in I_\circ$, $s \in I_1$.

Definition 2.10 ([19, 22]). Let $f : (X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ be a mapping and $\lambda \in I^X$, $\mu \in I^Y$. Then f is called:

(1) DF-continuous if $\tau_1(f^{-1}(\mu)) \geqslant \tau_2(\mu)$ and $\tau_1^*(f^{-1}(\mu)) \leqslant \tau_2^*(\mu)$;
(2) DF-open if $\tau_2(f(\lambda)) \geqslant \tau_1(\lambda)$ and $\tau_2^*(f(\lambda)) \leqslant \tau_1^*(\lambda)$;
(3) DF-closed if $\tau_2(\underline{1}-f(\lambda)) \geqslant \tau_1(\underline{1}-\lambda)$ and $\tau_2^*(\underline{1}-f(\lambda)) \leqslant \tau_1^*(\underline{1}-\lambda)$.

Definition 2.11 ([1, 14, 22]). Let $f : (X,\tau_1,\tau_1^*) \to (Y,\tau_2,\tau_2^*)$ be a mapping and $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_\circ$, $s \in I_1$. Then f is called:

(1) DFS-continuous (resp. DFG-continuous and DFSG-continuous) if $f^{-1}(\mu)$ is (r,s)-fso (resp. (r,s)-gfo and (r,s)-sgfo) and $\tau_2(\mu) \geqslant r$, $\tau_2^*(\mu) \leqslant s$;
(2) DFS-open (resp. DFG-open and DFSG-open) if $f(\lambda)$ is (r,s)-fso (resp. (r,s)-gfo and (r,s)-sgfo) and $\tau_1(\lambda) \geqslant r$, $\tau_1^*(\lambda) \leqslant s$;
(3) DFS-closed (resp. DFG-closed and DFSG-closed) if $f(\lambda)$ is (r,s)-fsc (resp. (r,s)-gfc and (r,s)-sgfc) and $\tau_1(\underline{1}-\lambda) \geqslant r$, $\tau_1^*(\underline{1}-\lambda) \leqslant s$.

3. On (r,s)-generalized fuzzy semi-closed sets

In this section, the notion of (r,s)-generalized fuzzy semi-closed sets is introduced in double fuzzy topological space in Šostak sense and some properties are given.

Definition 3.1. Let (X,τ,τ^*) be an dfts, $\lambda,\mu \in I^X$ and $r \in I_\circ$, $s \in I_1$.

(1) λ is called (r,s)-generalized fuzzy semi-closed set ((r,s)-gfsc, for short) if $SC_{\tau,\tau^*}(\lambda,r,s) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and $\tau(\mu) \geqslant r$, $\tau^*(\mu) \leqslant s$.

(2) λ is called (r,s)-generalized fuzzy semi-open set ((r,s)-gfso, for short) if $\underline{1}-\lambda$ is an (r,s)-gfsc set.

The following implications hold:

$$\begin{array}{ccc} (r,s)-\mathrm{gfc} & \not\leftrightarrow & (r,s)-\mathrm{fsc} \\ \downarrow & & \downarrow \\ (r,s)-\mathrm{gfsc} & \longleftarrow & (r,s)-\mathrm{sgfc} \end{array}$$

In general the converses are not true.

Problem 3.2. Define the double fuzzy topology (τ,τ^*) on $X=\{a,b\}$ by:

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda)=\begin{cases} 0, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.4}, \\ 1, & \text{otherwise}. \end{cases}$$

Then, $\underline{0.35}$ is $(\frac{1}{2},\frac{1}{2})$-gfsc but it is neither $(\frac{1}{2},\frac{1}{2})$-gfc nor $(\frac{1}{2},\frac{1}{2})$-fsc.

Problem 3.3. Define the double fuzzy topology (τ,τ^*) on $X=\{a,b\}$ by:

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{3}, & \text{if } \lambda \in \{\underline{0.1},\underline{0.3}\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda)=\begin{cases} 0, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{3}, & \text{if } \lambda \in \{\underline{0.1},\underline{0.3}\}, \\ 1, & \text{otherwise}. \end{cases}$$

Then, $\underline{0.2}$ is $(\frac{1}{3},\frac{1}{3})$-gfsc but it is not $(\frac{1}{3},\frac{1}{3})$-sgfc.

Remark 3.4. The intersection of two (r,s)-gfsc sets is not (r,s)-gfsc set, in general, as shown by Problem 3.5. Also, the union of two (r,s)-gfsc sets is not (r,s)-gfsc set, in general, as shown by Problem 3.6.

Problem 3.5. Let $X=\{a,b,c\}$ and $\mu,\mu_1,\mu_2 \in I^X$ defined as follows: $\mu=\{\frac{a}{1.0},\frac{b}{0.0},\frac{c}{0.0}\}$, $\mu_1=\{\frac{a}{1.0},\frac{b}{1.0},\frac{c}{0.0}\}$, and $\mu_2=\{\frac{a}{1.0},\frac{b}{0.0},\frac{c}{1.0}\}$. Define the double fuzzy topology (τ,τ^*) on X by:

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda)=\begin{cases} 0, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 1, & \text{otherwise}. \end{cases}$$

Then, μ_1 and μ_2 are $(\frac{1}{2},\frac{1}{2})$-gfsc sets, but $\mu=\mu_1 \wedge \mu_2$ is not $(\frac{1}{2},\frac{1}{2})$-gfsc.

Problem 3.6. Let $X=\{a,b,c\}$ and $\mu,\nu,\rho \in I^X$ defined as follows: $\mu=\{\frac{a}{1.0},\frac{b}{0.0},\frac{c}{0.0}\}$, $\nu=\{\frac{a}{0.0},\frac{b}{1.0},\frac{c}{0.0}\}$, and $\rho=\{\frac{a}{1.0},\frac{b}{1.0},\frac{c}{0.0}\}$. Define the double fuzzy topology (τ,τ^*) on X by:

$$\tau(\lambda)=\begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{4}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \nu, \\ \frac{2}{3}, & \text{if } \lambda = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda)=\begin{cases} 0, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{3}{4}, & \text{if } \lambda = \mu, \\ \frac{1}{2}, & \text{if } \lambda = \nu, \\ \frac{1}{3}, & \text{if } \lambda = \rho, \\ 1, & \text{otherwise}. \end{cases}$$

Then, μ and ν are $(\frac{1}{4},\frac{3}{4})$-gfsc sets, but $\rho=\mu \vee \nu$ is not $(\frac{1}{4},\frac{3}{4})$-gfsc.

Theorem 3.7. *Let* (X, τ, τ^*) *be an dfts. For each* $\lambda, \mu \in I^X$ *and* $r \in I_\circ$, $s \in I_1$, *we define an* (r, s)*-generalized fuzzy semi-closure* $GSC_{\tau,\tau^*} : I^X \times I_\circ \times I_1 \to I^X$ *as follows:*

$$GSC_{\tau,\tau^*}(\lambda, r, s) = \bigwedge\{\mu \in I^X : \mu \geqslant \lambda, \mu \text{ is } (r, s)\text{-gfsc}\}.$$

For $\lambda, \mu \in I^X$ *and* $r \in I_\circ$, $s \in I_1$, *the operator* GSC_{τ,τ^*} *satisfies the following properties:*

(1) $GSC_{\tau,\tau^*}(\underline{0}, r, s) = \underline{0}$;
(2) $\lambda \leqslant GSC_{\tau,\tau^*}(\lambda, r, s)$;
(3) $GSC_{\tau,\tau^*}(\lambda, r, s) \vee GSC_{\tau,\tau^*}(\mu, r, s) \leqslant GSC_{\tau,\tau^*}(\lambda \vee \mu, r, s)$;
(4) $GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) = GSC_{\tau,\tau^*}(\lambda, r, s)$;
(5) *If* λ *is* (r, s)-gfsc, *then* $GSC_{\tau,\tau^*}(\lambda, r, s) = \lambda$;
(6) $GSC_{\tau,\tau^*}(\lambda, r, s) \leqslant GC_{\tau,\tau^*}(\lambda, r, s) \leqslant C_{\tau,\tau^*}(\lambda, r, s)$;
(7) $C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) = GSC_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, r, s), r, s) = C_{\tau,\tau^*}(\lambda, r, s)$.

Proof. (1), (2), and (5) are easily proved from the definition of GSC_{τ,τ^*}.

(3) Since $\lambda, \mu \leqslant \lambda \vee \mu$, we have

$$GSC_{\tau,\tau^*}(\lambda, r, s) \vee GSC_{\tau,\tau^*}(\mu, r, s) \leqslant GSC_{\tau,\tau^*}(\lambda \vee \mu, r, s).$$

(4) From (2), we only show $GSC_{\tau,\tau^*}(\lambda, r, s) \geqslant GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s)$.
Suppose $GSC_{\tau,\tau^*}(\lambda, r, s) \ngeqslant GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s)$. There exist $x \in X$ and $t \in (0, 1)$ such that

$$GSC_{\tau,\tau^*}(\lambda, r, s)(x) < t < GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s)(x). \tag{B}$$

Since $GSC_{\tau,\tau^*}(\lambda, r, s)(x) < t$, there exists (r, s)-gfsc set λ_1 with $\lambda \leqslant \lambda_1$ such that $GSC_{\tau,\tau^*}(\lambda, r, s)(x) \leqslant \lambda_1(x) < t$. Since $\lambda \leqslant \lambda_1$, we have $GSC_{\tau,\tau^*}(\lambda, r, s) \leqslant \lambda_1$. Again, $GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) \leqslant \lambda_1$. Hence

$$GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s)(x) \leqslant \lambda_1(x) < t.$$

It is a contradiction for (B). Thus,

$$GSC_{\tau,\tau^*}(\lambda, r, s) \geqslant GSC_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s).$$

(6) Since $C_{\tau,\tau^*}(\lambda, r, s)$ is (r, s)-gfsc, it is easily proved.

(7) Trivially, $GSC_{\tau,\tau^*}(C_{\tau,\tau^*}(\lambda, r, s), r, s) = C_{\tau,\tau^*}(\lambda, r, s)$. We only show that

$$C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) = C_{\tau,\tau^*}(\lambda, r, s).$$

Since $\lambda \leqslant GSC_{\tau,\tau^*}(\lambda, r, s)$, we have $C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) \geqslant C_{\tau,\tau^*}(\lambda, r, s)$.
Suppose $C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) \nleqslant C_{\tau,\tau^*}(\lambda, r, s)$. There exist $x \in X$ and $t \in (0, 1)$ such that

$$C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s)(x) > t > C_{\tau,\tau^*}(\lambda, r, s)(x).$$

Since $C_{\tau,\tau^*}(\lambda, r, s)(x) < t$, by the definition of C_{τ,τ^*}, there exists $\rho \in I^X$ with $\lambda \leqslant \rho$, $\tau(\underline{1} - \rho) \geqslant r$ and $\tau^*(\underline{1} - \rho) \leqslant s$ such that

$$C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s)(x) > t > \rho(x) \geqslant C_{\tau,\tau^*}(\lambda, r, s)(x).$$

On the other hand, since $\rho = C_{\tau,\tau^*}(\rho, r, s)$ is (r, s)-gfsc, $\lambda \leqslant \rho$ implies

$$GSC_{\tau,\tau^*}(\lambda, r, s) \leqslant GSC_{\tau,\tau^*}(\rho, r, s) = GSC_{\tau,\tau^*}(C_{\tau,\tau^*}(\rho, r, s), r, s) = C_{\tau,\tau^*}(\rho, r, s) = \rho.$$

Thus, $C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) \leqslant \rho$. It is a contradiction. Hence

$$C_{\tau,\tau^*}(GSC_{\tau,\tau^*}(\lambda, r, s), r, s) \leqslant C_{\tau,\tau^*}(\lambda, r, s).$$

□

Theorem 3.8. *Let* (X, τ, τ^*) *be an dfts. For each* $\lambda, \mu \in I^X$ *and* $r \in I_\circ$, $s \in I_1$, *we define an* (r, s)*-generalized fuzzy semi-interior* $GSI_{\tau,\tau^*} : I^X \times I_\circ \times I_1 \to I^X$ *as follows:* $GSI_{\tau,\tau^*}(\lambda, r, s) = \bigvee\{\mu \in I^X : \mu \leqslant \lambda, \mu$ *is* (r, s)-gfso$\}$. *Then* $GSI_{\tau,\tau^*}(\underline{1} - \lambda, r, s) = \underline{1} - GSC_{\tau,\tau^*}(\lambda, r, s)$.

Proof. For each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$ we have

$$
\begin{aligned}
GSI_{\tau,\tau^*}(\underline{1} - \lambda, r, s) &= \bigvee\{\mu \in I^X : \mu \leqslant \underline{1} - \lambda, \mu \text{ is (r,s)-gfso}\} \\
&= \underline{1} - \bigwedge\{\underline{1} - \mu \in I^X : \underline{1} - \mu \geqslant \lambda, \underline{1} - \mu \text{ is (r,s)-gfsc}\} \\
&= \underline{1} - GSC_{\tau,\tau^*}(\lambda, r, s).
\end{aligned}
$$

□

4. Double fuzzy generalized semi-continuous mappings

In this section, the generalized forms of fuzzy continuous mappings between double fuzzy topological spaces are introduced and studied. Moreover, the relationship between these mappings and other mappings introduced previously are investigated with the help of examples.

Definition 4.1. Let $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ be a mapping and $r \in I_\circ$, $s \in I_1$. Then f is called:

(1) double fuzzy generalized semi-continuous (DFGS-continuous, for short) if $f^{-1}(\mu)$ is (r, s)-gfsc for each $\mu \in I^Y$, $\tau_2(\underline{1} - \mu) \geqslant r$ and $\tau_2^*(\underline{1} - \mu) \leqslant s$;
(2) double fuzzy generalized semi-open (DFGS-open, for short) if $f(\lambda)$ is (r, s)-gfso for each $\lambda \in I^X$, $\tau_1(\lambda) \geqslant r$ and $\tau_1^*(\lambda) \leqslant s$;
(3) double fuzzy generalized semi-closed (DFGS-closed, for short) if $f(\lambda)$ is (r, s)-gfsc for each $\lambda \in I^X$, $\tau_1(\underline{1} - \lambda) \geqslant r$ and $\tau_1^*(\underline{1} - \lambda) \leqslant s$.

The following implications hold:

$$
\begin{array}{ccc}
 & \text{DF-continuity} & \\
\swarrow & & \searrow \\
\text{DFG-continuity} & & \text{DFS-continuity} \\
\downarrow & & \downarrow \\
\text{DFGS-continuity} & \longleftarrow & \text{DFSG-continuity}
\end{array}
$$

In general the converses are not true.

Problem 4.2. Let $X = \{a, b\}$. Define $\tau, \tau^*, \eta, \eta^* : I^X \longrightarrow I$ as follows:

$$
\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.4}, \\ 0, & \text{otherwise,} \end{cases} \qquad
\tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.4}, \\ 1, & \text{otherwise,} \end{cases}
$$

$$
\eta(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.65}, \\ 0, & \text{otherwise,} \end{cases} \qquad
\eta^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.65}, \\ 1, & \text{otherwise.} \end{cases}
$$

The identity mapping $id_X : (X, \tau, \tau^*) \to (X, \eta, \eta^*)$ is DFGS-continuous but it is neither DFG-continuous nor DFS-continuous.

Problem 4.3. Let $X=\{a,b\}$. Define $\tau,\tau^*,\eta,\eta^*: I^X \longrightarrow I$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \lambda\in\{\underline{0.1},\underline{0.3}\},\\ 0, & \text{otherwise,}\end{cases}\qquad \tau^*(\lambda)=\begin{cases}0, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \lambda\in\{\underline{0.1},\underline{0.3}\},\\ 1, & \text{otherwise,}\end{cases}$$

$$\eta(\lambda)=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \lambda=\underline{0.8},\\ 0, & \text{otherwise,}\end{cases}\qquad \eta^*(\lambda)=\begin{cases}0, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \lambda=\underline{0.8},\\ 1, & \text{otherwise.}\end{cases}$$

The identity mapping $id_X:(X,\tau,\tau^*)\to(X,\eta,\eta^*)$ is DFGS-continuous but it is not DFSG-continuous.

Theorem 4.4. *Let $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$ be DFGS-continuous mapping. The following statements hold for each $\lambda\in I^X$, $\mu\in I^Y$ and $r\in I_\circ$, $s\in I_1$.*

(1) $f(GSC_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant C_{\tau_2,\tau_2^*}(f(\lambda),r,s)$.

(2) $GSC_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s)\leqslant f^{-1}(C_{\tau_2,\tau_2^*}(\mu,r,s))$.

(3) $f^{-1}(I_{\tau_2,\tau_2^*}(\mu,r,s))\leqslant GSI_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s)$.

Proof.

(1) For each $\lambda\in I^X$, $\tau_2(\underline{1}-C_{\tau_2,\tau_2^*}(f(\lambda),r,s))\geqslant r$, $\tau_2^*(\underline{1}-C_{\tau_2,\tau_2^*}(f(\lambda),r,s))\leqslant s$. Since f is DFGS-continuous then $f^{-1}(C_{\tau_2,\tau_2^*}(f(\lambda),r,s))$ is (r,s)-gfsc set of X. Since $f(\lambda)\leqslant C_{\tau_2,\tau_2^*}(f(\lambda),r,s)$ then $\lambda\leqslant f^{-1}(f(\lambda))\leqslant f^{-1}(C_{\tau_2,\tau_2^*}(f(\lambda),r,s))$ and so $GSC_{\tau_1,\tau_1^*}(\lambda,r,s)\leqslant f^{-1}(C_{\tau_2,\tau_2^*}(f(\lambda),r,s))$. Hence

$$f(GSC_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant C_{\tau_2,\tau_2^*}(f(\lambda),r,s).$$

(2) For each $\mu\in I^Y$,$\tau_2(\underline{1}-C_{\tau_2,\tau_2^*}(\mu,r,s))\geqslant r$ and $\tau_2^*(\underline{1}-C_{\tau_2,\tau_2^*}(\mu,r,s))\leqslant s$. Since f is DFGS-continuous then $f^{-1}(C_{\tau_2,\tau_2^*}(\mu,r,s))$ is (r,s)-gfsc set of X. Since $\mu\leqslant C_{\tau_2,\tau_2^*}(\mu,r,s)$, $f^{-1}(\mu)\leqslant f^{-1}(C_{\tau_2,\tau_2^*}(\mu,r,s))$ and so

$$GSC_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s)\leqslant GSC_{\tau_1,\tau_1^*}(f^{-1}(C_{\tau_2,\tau_2^*}(\mu,r,s)),r,s)=f^{-1}(C_{\tau_2,\tau_2^*}(\mu,r,s)).$$

(3) For each $\mu\in I^Y$,$\tau_2(I_{\tau_2,\tau_2^*}(\mu,r,s))\geqslant r$ and $\tau_2^*(I_{\tau_2,\tau_2^*}(\mu,r,s))\leqslant s$. Since f is DFGS-continuous, $f^{-1}(I_{\tau_2,\tau_2^*}(\mu,r,s))$ is (r,s)-gfso set of X. Since $I_{\tau_2,\tau_2^*}(\mu,r)\leqslant\mu$ then $f^{-1}(I_{\tau_2,\tau_2^*}(\mu,r,s))\leqslant f^{-1}(\mu)$ and so

$$f^{-1}(I_{\tau_2,\tau_2^*}(\mu,r,s))=GSI_{\tau_1,\tau_1^*}(f^{-1}(I_{\tau_2,\tau_2^*}(\mu,r,s)),r,s)\leqslant GSI_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s).$$

□

Theorem 4.5. *If $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$ is DFGS-closed then for each $\lambda\in I^X$ and $r\in I_\circ$, $s\in I_1$ we have $f(C_{\tau_1,\tau_1^*}(\lambda,r,s))\geqslant GSC_{\tau_2,\tau_2^*}(f(\lambda),r,s)$.*

Proof. For each $\lambda\in I^X$, since $\lambda\leqslant C_{\tau_1,\tau_1^*}(\lambda,r,s)$, then $f(\lambda)\leqslant f(C_{\tau_1,\tau_1^*}(\lambda,r,s))$. Since $\tau_1(\underline{1}-C_{\tau_1,\tau_1^*}(\lambda,r,s))\geqslant r$, $\tau_1^*(\underline{1}-C_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant s$ and f is DFGS-closed then $f(C_{\tau_1,\tau_1^*}(\lambda,r,s)$ is (r,s)-gfsc of Y. Hence $f(C_{\tau_1,\tau_1^*}(\lambda,r,s))\geqslant GSC_{\tau_2,\tau_2^*}(f(\lambda),r,s)$. □

Theorem 4.6. *Let $f:(X,\tau_1,\tau_1^*)\to(Y,\tau_2,\tau_2^*)$ be DFGS-open mapping. The following statements hold for each $\lambda\in I^X$, $\mu\in I^Y$ and $r\in I_\circ$, $s\in I_1$.*

(1) $f(I_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant GSI_{\tau_2,\tau_2^*}(f(\lambda),r,s)$.

(2) $I_{\tau_1,\tau_1^*}(f^{-1}(\mu),r,s)\leqslant f^{-1}(GSI_{\tau_2,\tau_2^*}(\mu,r,s))$.

Proof.

(1) For each $\lambda\in I^X$, since $I_{\tau_1,\tau_1^*}(\lambda,r,s)\leqslant\lambda$ then $f(I_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant f(\lambda)$. Since $\tau_1(I_{\tau_1,\tau_1^*}(\lambda,r,s))\geqslant r$, $\tau_1^*(I_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant s$ and f is DFGS-open, then $f(I_{\tau_1,\tau_1^*}(\lambda,r,s))$ is (r,s)-gfso. Hence $f(I_{\tau_1,\tau_1^*}(\lambda,r,s))\leqslant GSI_{\tau_2,\tau_2^*}(f(\lambda),r,s)$.

(2) For all $\mu \in I^Y$, put $\lambda = f^{-1}(\mu)$. From (1),

$$f(I_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s)) \leqslant GSI_{\tau_2,\tau_2^*}(f(f^{-1}(\mu)), r, s) \leqslant GSI_{\tau_2,\tau_2^*}(\mu, r, s).$$

Hence $I_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s) \leqslant f^{-1}(f(I_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s))) \leqslant f^{-1}(GSI_{\tau_2,\tau_2^*}(\mu, r, s))$. □

Theorem 4.7. *Let* $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ *and* $g : (Y, \tau_2, \tau_2^*) \to (Z, \tau_3, \tau_3^*)$ *be mappings. Then* gof *is:*

(1) *DFGS-continuous, if* f *is DFGS-continuous and* g *is DF-continuous;*
(2) *DFGS-open, if* f *is DF-open and* g *is DFGS-open;*
(3) *DFGS-closed, if* f *is DF-closed and* g *is DFGS-closed.*

Proof.

(1) Suppose that $\mu \in I^Z$, $\tau_3(\mu) \geqslant r$ and $\tau_3^*(\mu) \leqslant s$. Since g is DF-continuous, then $\tau_2(g^{-1}(\mu)) \geqslant r$ and $\tau_2^*(g^{-1}(\mu)) \leqslant s$. Since f is DFGS-continuous, then $f^{-1}(g^{-1}(\mu))$ is (r, s)-gfso set in (X, τ_1, τ_1^*). Thus, $(gof)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$ is (r, s)-gfso and therefore gof is DFGS-continuous.

(2) Suppose that $\lambda \in I^X$, $\tau_1(\lambda) \geqslant r$ and $\tau_1^*(\lambda) \leqslant s$. Since f is DF-open, then $\tau_2(f(\lambda)) \geqslant r$ and $\tau_2^*(f(\lambda)) \leqslant s$. Since g is DFGS-open, then $g(f(\lambda))$ is (r, s)-gfso set in (Z, τ_3, τ_3^*). Thus, $(gof)(\lambda) = g(f(\lambda))$ is (r, s)-gfso and therefore (gof) is DFGS-open.

(3) Suppose that $\lambda \in I^X$, $\tau_1(\underline{1} - \lambda) \geqslant r$ and $\tau_1^*(\underline{1} - \lambda) \leqslant s$. Since f is DF-closed, then $\tau_2(\underline{1} - f(\lambda)) \geqslant r$ and $\tau_2^*(\underline{1} - f(\lambda)) \leqslant s$. Since g is DFGS-closed, then $g(f(\lambda))$ is (r, s)-gfsc set in (Z, τ_3, τ_3^*). Thus, $(gof)(\lambda) = g(f(\lambda))$ is (r, s)-gfsc and therefore (gof) is DFGS-closed. □

Definition 4.8. Let $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ be a mapping and $r \in I_\circ$, $s \in I_1$. Then f is called:

(1) double fuzzy generalized semi-irresolute (DFGS-irresolute, for short) if $f^{-1}(\mu)$ is (r, s)-gfsc for each $\mu \in I^Y$ is (r, s)-gfsc;
(2) double fuzzy generalized semi-irresolute open (DFGS-irresolute open, for short) if $f(\lambda)$ is (r, s)-gfso for each $\lambda \in I^X$ is (r, s)-gfso;
(3) double fuzzy generalized semi-irresolute closed (DFGS-irresolute closed, for short) if $f(\lambda)$ is (r, s)-gfsc for each $\lambda \in I^X$ is (r, s)-gfsc;
(4) DFGS-irresolute homeomorphism iff it is bijective and both of f and f^{-1} are DFGS-irresolute.

The following implication holds:

$$\text{DFGS-irresolute} \quad \longrightarrow \quad \text{DFGS-continuity}$$

In general the converses are not true.

Problem 4.9. Let $X = \{a, b\}$. Define $\tau, \tau^*, \eta, \eta^* : I^X \longrightarrow I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda \in \{\underline{0.1}, \underline{0.3}\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda \in \{\underline{0.1}, \underline{0.3}\}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\eta(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.1}, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.1}, \\ 1, & \text{otherwise}. \end{cases}$$

The identity mapping $id_X : (X, \tau, \tau^*) \to (X, \eta, \eta^*)$ is DFGS-continuous but it is not DFGS-irresolute.

Theorem 4.10. *Let* $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ *be DFGS-irresolute mapping. The following statements hold for each* $\lambda \in I^X$, $\mu \in I^Y$ *and* $r \in I_\circ$, $s \in I_1$*:*

(1) $f(GSC_{\tau_1,\tau_1^*}(\lambda, r, s)) \leqslant SC_{\tau_2,\tau_2^*}(f(\lambda), r, s)$;

(2) $GSC_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s) \leqslant f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s))$;
(3) $f^{-1}(SI_{\tau_2,\tau_2^*}(\mu, r, s)) \leqslant GSI_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s)$;
(4) *If* f *is bijective, then* $SI_{\tau_2,\tau_2^*}(f(\lambda), r, s)) \leqslant f(GSI_{\tau_1,\tau_1^*}(\lambda, r, s))$.

Proof.

(1) For each $\lambda \in I^X$, $SC_{\tau_2,\tau_2^*}(f(\lambda), r, s)$ is (r, s)-gfsc set of Y. Since f is DFGS-irresolute, then $f^{-1}(SC_{\tau_2,\tau_2^*}(f(\lambda), r, s))$ is (r, s)-gfsc set of X. Since

$$f(\lambda) \leqslant SC_{\tau_2,\tau_2^*}(f(\lambda), r, s),\ \ \lambda \leqslant f^{-1}(f(\lambda)) \leqslant f^{-1}(SC_{\tau_2,\tau_2^*}(f(\lambda), r, s)),\ \ GSC_{\tau_1,\tau_1^*}(\lambda, r, s) \leqslant f^{-1}(SC_{\tau_2,\tau_2^*}(f(\lambda), r, s)).$$

Hence

$$f(GSC_{\tau_1,\tau_1^*}(\lambda, r, s)) \leqslant SC_{\tau_2,\tau_2^*}(f(\lambda), r, s).$$

(2) For each $\mu \in I^Y$, $SC_{\tau_2,\tau_2^*}(\mu, r, s)$ is (r, s)-gfsc set of Y. Since f is DFGS-irresolute, $f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s))$ is (r, s)-gfsc set of X. Since $\mu \leqslant SC_{\tau_2,\tau_2^*}(\mu, r, s)$, then $f^{-1}(\mu) \leqslant f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s))$. Then,

$$GSC_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s) \leqslant GSC_{\tau_1,\tau_1^*}(f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s)), r, s) = f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s)).$$

(3) For each $\mu \in I^Y$, $SI_{\tau_2,\tau_2^*}(\mu, r, s)$ is (r, s)-gfso set of Y. Since f is DFGS-irresolute then $f^{-1}(SI_{\tau_2,\tau_2^*}(\mu, r, s))$ is (r, s)-gfso set of X. Since $SI_{\tau_2,\tau_2^*}(\mu, r, s) \leqslant \mu$ then $f^{-1}(SI_{\tau_2,\tau_2^*}(\mu, r, s)) \leqslant f^{-1}(\mu)$ and so

$$f^{-1}(SI_{\tau_2,\tau_2^*}(\mu, r, s)) = GSI_{\tau_1,\tau_1^*}(f^{-1}(SI_{\tau_2,\tau_2^*}(\mu, r, s)), r) \leqslant GSI_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s).$$

(4) Let f be DFGS-irresolute and $\lambda \in I^X$. Then $f^{-1}(SI_{\tau_2,\tau_2^*}(f(\lambda), r, s))$ is (r, s)-gfso. By (3), and the fact that f is injective, we have $f^{-1}(SI_{\tau_2,\tau_2^*}(f(\lambda), r, s)) \leqslant GSI_{\tau_1,\tau_1^*}(f^{-1}(f(\lambda)), r, s) = GSI_{\tau_1,\tau_1^*}(\lambda, r, s)$. Since f is surjective, we have

$$SI_{\tau_2,\tau_2^*}(f(\lambda), r, s) = f(f^{-1}(GSI_{\tau_1,\tau_1^*}(\lambda, r, s))) \leqslant f(GSI_{\tau_1,\tau_1^*}(\lambda, r, s)).$$

□

Theorem 4.11. *If* $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ *be DF-irresolute, DF-open and bijective mapping. Then* f *is DFGS-irresolute.*

Proof. Let $\mu \in I^Y$ be (r, s)-gfsc set. We will show that $f^{-1}(\mu)$ is (r, s)-gfsc set. Let $f^{-1}(\mu) \leqslant \nu$ with $\tau_1(\nu) \geqslant r$, $\tau_1^*(\nu) \leqslant s$. Since f is onto and DF-open, $\mu = f(f^{-1}(\mu)) \leqslant f(\nu)$ with $\tau_2(f(\nu)) \geqslant r$, $\tau_2^*(f(\nu)) \leqslant s$. Since μ is (r, s)-gfsc, $SC_{\tau_2,\tau_2^*}(\mu, r, s) \leqslant f(\nu)$. Since f is injective, $f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s)) \leqslant f^{-1}(f(\nu)) = \nu$. Since f is DF-irresolute, $f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s))$ is (r, s)-gfsc. Hence,

$$SC_{\tau_1,\tau_1^*}(f^{-1}(\mu), r, s) \leqslant SC_{\tau_1,\tau_1^*}(f^{-1}(SC_{\tau_2,\tau_2^*}(\mu, r, s)), r, s) \leqslant \nu.$$

Thus, $f^{-1}(\mu)$ is (r, s)-gfsc set. □

Theorem 4.12. *If* $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ *and* $g : (Y, \tau_2, \tau_2^*) \to (Z, \tau_3, \tau_3^*)$ *be mappings and* $r \in I_\circ$, $s \in I_1$, *then* gof *is:*

(1) *DFGS-irresolute, if* f *and* g *are both DFGS-irresolute;*
(2) *DFGS-irresolute open, if* f *and* g *are both DFGS-irresolute open;*
(3) *DFGS-irresolute closed, if* f *and* g *are both DFGS-irresolute closed;*
(4) *DFGS-continuous, if* f *is DFGS-irresolute and* g *is DFGS-continuous;*
(5) *DFGS-open, if* f *is DFGS-open and* g *is DFGS-irresolute open;*
(6) *DFGS-closed, if* f *is DFGS-closed and* g *is DFGS-irresolute closed.*

Proof. It is similarly proved as in Theorem 4.7. □

Definition 4.13. A dfts (X, τ, τ^*) is called $\mathrm{DFST}_{\frac{1}{2}}$ if $\tau(\underline{1} - \lambda) \geqslant r$, $\tau^*(\underline{1} - \lambda) \leqslant s$ for each $\lambda \in I^X$ is (r, s)-gfsc set and $r \in I_\circ$, $s \in I_1$.

It is clear that $\mathrm{DFST}_{\frac{1}{2}}$ implies that $\mathrm{DFT}_{\frac{1}{2}}$.

Theorem 4.14. *A dfts (X, τ, τ^*) is $\mathrm{DFST}_{\frac{1}{2}}$ iff $GSC_{\tau,\tau^*}(\lambda, r, s) = C_{\tau,\tau^*}(\lambda, r, s)$ for each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$.*

Proof.
($\Longrightarrow$) Let (X, τ, τ^*) be $\mathrm{DFST}_{\frac{1}{2}}$. By definitions of GSC_{τ,τ^*} and C_{τ,τ^*}, we have $GSC_{\tau,\tau^*}(\lambda, r, s) = C_{\tau,\tau^*}(\lambda, r, s)$ for each $\lambda \in I^X$ and $r \in I_\circ$, $s \in I_1$.
($\Longleftarrow$) Suppose (X, τ, τ^*) is not $\mathrm{DFST}_{\frac{1}{2}}$. There exist (r, s)-gfsc $\mu \in I^X$ and $r \in I_\circ$, $s \in I_1$ such that $\tau(\underline{1} - \mu) < r$. Hence $GSC_{\tau,\tau^*}(\mu, r, s) = \mu$ but $C_{\tau,\tau^*}(\mu, r, s) \neq \mu$. Thus, $GSC_{\tau,\tau^*}(\mu, r, s) \neq C_{\tau,\tau^*}(\mu, r, s)$ it is a contradiction. Then (X, τ, τ^*) is $\mathrm{DFST}_{\frac{1}{2}}$. □

Corollary 4.15. *Let $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ be a mapping and $r \in I_\circ$, $s \in I_1$. The following statements hold.*

(1) *If (X, τ_1, τ_1^*) is $\mathrm{DFST}_{\frac{1}{2}}$, then the concepts of DF-continuity, DFS-continuity, DFSG-continuity, DFGS-continuity, and DGF-continuity are equivalent.*
(2) *If (Y, τ_2, τ_2^*) is $\mathrm{DFST}_{\frac{1}{2}}$, then the concepts of DFSG-continuity and DFSG-irresolute are equivalent. Also, the concepts of DFGS-continuity and DFGS-irresol-ute are equivalent.*
(3) *If (X, τ_1, τ_1^*) and (X, τ_2, τ_2^*) are $\mathrm{DFST}_{\frac{1}{2}}$, then the concepts of DF-continuity, DFS-continuity, DFSG-continuity, DFGS-continuity, DFG-continuity, DFSG-irresolute, and DFGS-irresolute are equivalent.*

Corollary 4.16. *Let $f : (X, \tau_1, \tau_1^*) \to (Y, \tau_2, \tau_2^*)$ and $g : (Y, \tau_2, \tau_2^*) \to (Z, \tau_3, \tau_3^*)$ be DFGS-continuous and (Y, τ_2, τ_2^*) be $\mathrm{DFST}_{\frac{1}{2}}$. Then $gof : (X, \tau_1, \tau_1^*) \to (Z, \tau_3, \tau_3^*)$ is DFGS-continuous.*

5. (r, s)-fuzzy GS-connected sets

In this section, the notion of (r, s)-fuzzy GS-connected sets is introduced and studied with help of (r, s)-generalized fuzzy semi-closed sets.

Definition 5.1. Let (X, τ, τ^*) be an dfts, $\lambda, \mu \in I^X$ and $r \in I_\circ$, $s \in I_1$. Then, two fuzzy sets λ and μ are said to be (r, s)-fuzzy GS-separated iff $\lambda \overline{q} GSC_{\tau,\tau^*}(\mu, r, s)$ and $\mu \overline{q} GSC_{\tau,\tau^*}(\lambda, r, s)$. Also, a fuzzy set which cannot be expressed as the union of two (r, s)-fuzzy GS-separated sets is said to be (r, s)-fuzzy GS-connected set.

Remark 5.2. It is clear that:

(1) (r, s)-fuzzy separated [1] implies that (r, s)-fuzzy GS-separated;
(2) (r, s)-fuzzy GS-connected implies that (r, s)-fuzzy connected [1].

The converse is not true in general as shown by the following Problem.

Problem 5.3. Let $X = \{a, b, c\}$ and $\mu, \mu_1, \mu_2 \in I^X$ defined as follows: $\mu = \{\frac{a}{0.6}, \frac{b}{0.2}, \frac{c}{0.4}\}$, $\mu_1 = \{\frac{a}{0.0}, \frac{b}{0.6}, \frac{c}{0.0}\}$, and $\mu_2 = \{\frac{a}{0.0}, \frac{b}{0.0}, \frac{c}{0.3}\}$. Define the double fuzzy topology (τ, τ^*) on X by:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ 1, & \text{otherwise}. \end{cases}$$

Then μ_1 and μ_2 are $(\frac{1}{2}, \frac{1}{2})$-fuzzy GS-separated, but are not $(\frac{1}{2}, \frac{1}{2})$-fuzzy separated. Also $\rho = \mu_1 \vee \mu_2$ is $(\frac{1}{2}, \frac{1}{2})$-fuzzy connected, but it is not $(\frac{1}{2}, \frac{1}{2})$-fuzzy GS-separated.

Theorem 5.4. *Let (X, τ, τ^*) be an dfts. For each $\lambda, \mu \in I^X$ and $r \in I_\circ$, $s \in I_1$.*

(1) *If λ, μ are (r, s)-fuzzy GS-separated and $\nu, \eta \in I^X - \{\underline{0}\}$ such that $\nu \leqslant \lambda$ and $\eta \leqslant \mu$, then ν, η are also (r, s)-fuzzy GS-separated.*

(2) *If $\lambda \overline{q} \mu$ and either both are (r, s)-gfso or both (r, s)-gfsc, then λ and μ are (r, s)-fuzzy GS-separated.*

(3) *If $\lambda, \mu \in I^X - \{\underline{0}\}$ and there exist two an (r, s)-gfso sets ν, ω such that $\lambda \leqslant \nu$, $\mu \leqslant \omega$, $\lambda \overline{q} \omega$ and $\mu \overline{q} \nu$, then λ and μ are (r, s)-fuzzy GS-separated.*

(4) *If λ, μ are either both (r, s)-gfso or both (r, s)-gfsc, then $\lambda \wedge (\underline{1} - \mu)$ and $\mu \wedge (\underline{1} - \lambda)$ are (r, s)-fuzzy GS-separated.*

Proof. (1) and (2) are obvious.

(3) Let ν and ω be (r, s)-gfso sets such that $\lambda \leqslant \nu$, $\mu \leqslant \omega$, $\lambda \overline{q} \omega$ and $\mu \overline{q} \nu$. Then $\lambda \leqslant \underline{1} - \omega$, $\mu \leqslant \underline{1} - \nu$. Hence $GSC_{\tau,\tau^*}(\lambda, r, s) \leqslant \underline{1} - \omega$, $GSC_{\tau,\tau^*}(\mu, r, s) \leqslant \underline{1} - \nu$, which in turn imply that $GSC_{\tau,\tau^*}(\lambda, r, s) \overline{q} \mu$ and $GSC_{\tau,\tau^*}(\mu, r, s) \overline{q} \lambda$. Thus λ and μ are (r, s)-fuzzy GS-separated.

(4) Let λ and μ be (r, s)-gfso. Since $\lambda \wedge (\underline{1} - \mu) \leqslant \underline{1} - \mu$, $GSC_{\tau,\tau^*}(\lambda \wedge (\underline{1} - \mu), r, s) \leqslant \underline{1} - \mu$ and hence $GSC_{\tau,\tau^*}(\lambda \wedge (\underline{1} - \mu), r, s) \overline{q} \mu$. Then $GSC_{\tau,\tau^*}(\lambda \wedge (\underline{1} - \mu), r, s) \overline{q} (\mu \wedge (\underline{1} - \lambda))$. Again, since $\mu \wedge (\underline{1} - \lambda) \leqslant \underline{1} - \lambda$, $GSC_{\tau,\tau^*}(\mu \wedge (\underline{1} - \lambda), r, s) \leqslant \underline{1} - \lambda$ and hence $GSC_{\tau,\tau^*}(\mu \wedge (\underline{1} - \lambda), r, s) \overline{q} \lambda$. Then $GSC_{\tau,\tau^*}(\mu \wedge (\underline{1} - \lambda), r, s) \overline{q} (\lambda \wedge (\underline{1} - \mu))$. Thus $\lambda \wedge (\underline{1} - \mu)$ and $\mu \wedge (\underline{1} - \lambda)$ are (r, s)-fuzzy GS-separated.

Similarly we can prove when λ and μ are (r, s)-gfsc. □

Theorem 5.5. *Let (X, τ, τ^*) be an dfts, $\lambda \in I^X - \{\underline{0}\}$ and $r \in I_\circ$, $s \in I_1$. If λ is a (r, s)-fuzzy GS-connected set such that $\lambda \leqslant \mu \leqslant GSC_{\tau,\tau^*}(\lambda, r, s)$, then μ is also (r, s)-fuzzy GS-connected.*

Proof. Suppose that μ is not (r, s)-fuzzy GS-connected. Then there exist (r, s)-fuzzy GS-separated sets ω_1 and ω_2 in X such that $\mu = \omega_1 \vee \omega_2$. Let $\nu = \lambda \wedge \omega_1$ and $\omega = \lambda \wedge \omega_2$. Then $\lambda = \nu \vee \omega$. Since $\nu \leqslant \omega_1$ and $\omega \leqslant \omega_2$, by Theorem 5.4, ν and ω are (r, s)-fuzzy GS-separated, contradicting the (r, s)-fuzzy GS-connectedness of λ. Thus μ is (r, s)-fuzzy GS-connected. □

6. Conclusion

In this paper, we have continued to study the fuzzy sets on double fuzzy topological space (X, τ, τ^*) in Šostak sense. As a weaker form of (r, s)-generalized fuzzy closed sets by Abbas [1], the notion of (r, s)-generalized fuzzy semi-closed sets is introduced and some properties are given. After that, the generalized forms of fuzzy continuous mappings between double fuzzy topological spaces are introduced and studied. Furthermore, some relationship between these mappings and other mappings introduced previously are investigated with the help of examples. Finally, the notion of (r, s)-fuzzy GS-connected sets is introduced with help of (r, s)-generalized fuzzy semi-closed sets.

In the upcoming works, we will define some separation axioms using (r, s)-generalized fuzzy semi-closed sets. Also, we shall introduce the notion of (r, s)-strongly* generalized fuzzy semi-closed set as a stronger form of (r, s)-generalized fuzzy semi-closed set.

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