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Stability and existence results for a system of fractional differential equations via Atangana-Baleanu derivative with ϕ_p-Laplacian operator

Tariq Q. S. Abdullah[a,b], Haijun Xiao[a], Gang Huang[a,*], Wadhah Al-Sadi[a]

[a] *School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, P. R. China.*

[b] *Department of Mathematics, Faculty of Applied Sciences, Thamar University, Dhamar, Yemen.*

Abstract

This study focused on the existence and uniqueness(EU) and stability of the solution for a system of fractional differential equations(FDEs) via Atangana-Baleanu derivative in the sense of Caputo (ABC) with ϕ_p-Laplacian operator. Green function $\mathcal{G}^{\eth}(t,s)$, $m < \eth < m+1$, $m \geqslant 4$ used for converting the suggested problem to an integral equation. Guo-Krasnoselskii theorem used for proving the EU of solution for the suggested problem. The stability of the solution was derived by Hyers-Ulam stability method(HUS). One illustrative example is used for manifesting the results.

Keywords: ABC-Derivative, Riemann-Liuoville fractional derivative, Green function, existence and uniqueness, Hyers-Ulam stability.

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1. Introduction

Fractional calculus is a mathematical part that aims to model problems that have integration and differentiation with fractional order. It has been exploited in different aspects of sciences and engineering to accurately demonstrate the dynamics of many systems by using the differential equations of fractional order. For example, the researchers in [8, 12, 15] used FDEs for studying the dynamics of viruses transmissions and disease treatment and control. In [3, 4], the researchers used numerical methods to analyse some differential equations with fractional order, such as nonlinear Schrödinger equations in multi-dimentional space. Another studies exploited FDEs for solving some time and time-delay problems, nonlinear dynamical systems, and heat transfer equations; for details, see [10, 16, 26, 30]. The usual used fractional derivatives are Riemann-Liouville (RL) fractional derivative [18], Caputo derivative [13], and Atangana-Baleanu (AB) derivative [9].

Some researchers recently applied FDEs with singularity by using some mathematical theories and techniques. For example, Zhang [31] proved some theorems that are related to EU of nonlinear FDEs.

*Corresponding author

Email addresses: tariq.almoqri@tu.edu.ye (Tariq Q. S. Abdullah), xiaohj@cug.edu.cn (Haijun Xiao), huanggang@cug.edu.cn (Gang Huang), waddah27@gmail.com (Wadhah Al-Sadi)

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The existence and HUS of positive solutions for nonlinear FDEs with a singularity are studied in [7, 27]. In [5, 6, 28], the authors studied the positive solution's uniqueness and stability for singular and hybrid FDEs with integral boundary value conditions. In the past decades, all of the fractional operators had kernel singularities that involve their kernels, which cause many troubles, especially when those operators are used for modeling some complex phenomena. For getting more efficient models, Caputo et al. [14] introduced a new fractional derivative called Caputo-Fibroize derivative (CF), which is efficiently used for FDEs without singular kernel. In 2016, Atangana and Baleanu [9] introduced a new fractional derivative with the nonlocal and nonsingular kernel, which was applied for heat transfer modelling. In [11], the authors established collocation methods for FDEs with the nonsingular kernel. Many studies were conducted on the EU of FDEs with ϕ_p-Laplacian operator. For more details, the readers can see [19, 24, 25].

In 2018, Jarad et al. [20] discussed the EU of the following equation

$$\begin{cases} {}_{a}^{ABC}\mathcal{D}^{\Upsilon}\mathcal{Q}(t) = \mathcal{J}(t,\mathcal{Q}(t)), \\ \mathcal{Q}(a) = \mathcal{Q}(0), \end{cases}$$

where ${}_{a}^{ABC}\mathcal{D}^{\Upsilon}$ is ABC derivative with fractional order, and $\Upsilon \in (0,1)$, ${}_{a}^{ABC}\mathcal{D}^{\Upsilon}\mathcal{Q}(t)$, $\mathcal{J}(t,\mathcal{Q}(t)) \in C[a,b]$.

Khan et al. [23] have discussed the EU and stability of the following non-linear ABC derivative with ϕ_p operator with singularity

$$\begin{cases} {}_{0}^{ABC}\mathcal{D}^{\alpha}(\phi_p({}_{0}^{ABC}\mathcal{D}^{\beta}\mathcal{Q}(t))) = -\mathcal{J}(t,\mathcal{Q}(t)), \\ \phi_p({}_{0}^{ABC}\mathcal{D}^{\beta}\mathcal{Q}(t))|_{t=0} = 0, \quad \mathcal{Q}(1) = 0, \end{cases} \tag{1.1}$$

where ${}_{0}^{ABC}\mathcal{D}^{\alpha}$ and ${}_{0}^{ABC}\mathcal{D}^{\beta}$ are ABC-derivative with order α and β respectively, such that $0 < \alpha, \beta \leqslant 1$, and $\mathcal{J}(t,\mathcal{Q}(t)) \in C[a,b]$.

In [22], the authors studied the existence and stability for a system of FDEs with singularity based on ABC derivative with ϕ_p-Laplacian operator given by:

$$\begin{cases} {}_{0}^{ABC}\mathcal{D}^{\alpha}(\phi_p({}_{0}^{C}\mathcal{D}^{\beta}\mathcal{Q}(t))) = -\mathcal{J}(t,\mathcal{Q}(t)), \\ \phi_p({}_{0}^{C}\mathcal{D}^{\beta}\mathcal{Q}(t))|_{t=0} = \mathcal{Q}^{(k)}(0) = \mathcal{Q}^{(m)}(1) = 0, \quad \mathcal{Q}(1) = \mathcal{Q}^{(m)}(0), \end{cases} \tag{1.2}$$

where $k = 1,2,\dots,m-1$, ${}_{0}^{ABC}\mathcal{D}^{\alpha}$ is ABC derivative with order α such that $0 < \alpha \leqslant 1$, ${}_{0}^{C}\mathcal{D}^{\beta}$ is the Caputo derivative with the order β such that $m < \beta \leqslant m+1$, and $\mathcal{J}(t,\mathcal{Q}(t)) \in C[a,b]$. The studies [22, 23] established the existence and stability of the FDEs mentioned in the Equations (1.1) and (1.2) for the value of $0 < \alpha \leqslant 1$. In [21], the existence and stability of the positive solution were investigated for the FDEs with the nonlinear singular ϕ_p-Laplacian operator that has the form

$$\begin{cases} \mathcal{D}^{\Upsilon}(\phi_p(\mathcal{D}^{\eth}\mathcal{Q}(t)) + \mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t)) = 0, \\ (\phi_p(\mathcal{D}^{\eth}\mathcal{Q}(t)))^{(k)}|_{t=0} = 0, k = 1,2,\dots,m-1, \\ \mathcal{Q}''(1) = \mathcal{Q}^{(k)}(0) = 0, \end{cases}$$

where $\mathcal{D}^{\Upsilon}$ and $\mathcal{D}^{\eth}$ are the Caputo derivative of the fractional order $m-1 < \Upsilon, \eth \leqslant m$, $m \geqslant 4$, and $\phi_p \in C[0,1]$. No one studied the existence and stability of the FDEs in the domain of $\alpha \in (m, m+1]$, $m \geqslant 4$ for nonlinear singular FDEs with ABC derivative. In this study, Guo-Krasnoselskii Theorem is used for studying the EU of solution and HUS-Theorem for studying the stability of solution for the following

$$\begin{cases} {}_{0}^{ABC}\mathcal{D}^{\Upsilon}(\phi_p({}_{0}\mathcal{D}^{\eth}\mathcal{Q}(t))) = -\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t)), \\ {}_{0}\mathcal{J}^{k-\eth}(\mathcal{Q}(t))|_{t=0} = 0, \quad k = 1,3,4,\dots,m, \\ (\phi_p(\mathcal{D}^{\eth}\mathcal{Q}(t)))^{(k)}|_{t=0} = 0, \quad k = 1,2,\dots,m, \\ {}_{0}\mathcal{D}^{\eth-2}(\mathcal{Q}(t))|_{t=1} = 0, \end{cases} \tag{1.3}$$

where $m < \Upsilon, \eth < m+1$, $m \geqslant 4$, $\mathcal{J}(t, \mathcal{Q}(t)) \in C[0,1]$ are continuous functions, ${}_{0}^{ABC}\mathcal{D}^{\Upsilon}$ is ABC derivative of order $\Upsilon \in (m, m+1], m \geqslant 4$, and ${}_{0}\mathcal{D}^{\eth}$ is RL derivative with fractional order $\eth \in (m, m+1], m \geqslant 4$. The $\phi_p = |r|^{p-2}r$ is the p-Laplacian operator provided with $1/p + 1/q = 1$, and $\phi_p^{-1} = \phi_q$. The positive solution $\mathcal{Q}(t)$ of the Equation (1.3) is the solution in which $\mathcal{Q}(t) > 0$, $t \in (0,1]$.

2. Auxiliary results

For studying the existence and stability of the suggested problem (1.3), we will investigate some preliminary concepts about ABC and RL fractional derivatives and Guo-Krasnoselskii Theorem. The study will be achieved by converting the problem in Equation (1.3) to the equivalent form of the AB-fractional integral ${}_{0}^{AB}I$ and RL-fractional integral.

Definition 2.1 ([22])**.** The RL fractional integral of a function $\mathcal{Q}(t) : (0, \infty) \to R$ of order $\Upsilon > 0$ is given by

$$ {}_{0}I^{\Upsilon}\mathcal{Q}(t) = \frac{1}{\Gamma(\Upsilon)} \int_{0}^{t} (t-s)^{\Upsilon-1}\mathcal{Q}(s)ds, $$

where $\Gamma(\Upsilon)$ is gamma function given by

$$ \Gamma(\Upsilon) = \int_{0}^{\infty} e^{-s}s^{\Upsilon-1}ds, \quad Re(\Upsilon) > 0. $$

Definition 2.2 ([2])**.** By assuming that the function $\mathcal{Q}(t) : (0, \infty) \to R$ is continuous function, the RL fractional derivative of $\mathcal{Q}(t)$ is given by

$$ {}_{0}\mathcal{D}^{\Upsilon}\mathcal{Q}(t) = \frac{1}{\Gamma(\mathcal{L}-\Upsilon)} (\frac{1}{dt})^{\mathcal{L}} \int_{0}^{t} (t-s)^{\mathcal{L}-\Upsilon-1}\mathcal{Q}(s)ds = (\frac{d}{dt})^{\mathcal{L}} I^{\mathcal{L}-\Upsilon}\mathcal{Q}(t), $$

provided the existence of the derivative, $\mathcal{L} = [\Upsilon] + 1$, where $[\Upsilon]$ is noted for the integer part of Υ. For $\rho \in [0, \Upsilon)$,

$$ \mathcal{D}^{\rho} I^{\Upsilon}\mathcal{Q}(t) = I^{\Upsilon-\rho}\mathcal{Q}(t). $$

Definition 2.3 ([9])**.** For $0 < \alpha \leqslant 1$, AB-fractional integral of the function $\mathcal{Q}(t) \in \mathcal{H}^{*}(a,b), a < b$ is defined by

$$ {}_{a}^{AB}I^{\alpha}\mathcal{Q}(t) = \frac{1-\alpha}{\mathbb{B}(\alpha)}\mathcal{Q}(t) + \frac{\alpha}{\mathbb{B}(\alpha)}({}_{a}I^{\alpha}\mathcal{Q}(t)), $$

where $\mathbb{B}(\alpha) > 0$ is a normalization function satisfies the condition $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Definition 2.4 ([1, 9])**.** For $m < \Upsilon \leqslant m+1$, and f is a function in which $\mathcal{Q}^{(m)} \in \mathcal{H}^{1}(a,b)$, by setting $\alpha = \Upsilon - m$, then $\alpha \in (0,1]$ and the ABC-derivative of order Υ defines as following

$$ {}_{a}^{ABC}\mathcal{D}^{\Upsilon}\mathcal{Q}(t) = \quad {}_{a}^{ABC}\mathcal{D}^{\alpha}[\mathcal{Q}(t)]^{(m)}. $$

The AB-integral of fractional order Υ is given by:

$$ {}_{a}^{AB}I^{\Upsilon}\mathcal{Q}(t) ={}_{a} I^{n}({}_{a}^{AB}I^{\alpha}[\mathcal{Q}(t)]). $$

Lemma 2.5 ([1, 9])**.** *The ABC derivative of the function $\mathcal{Q} \in H^{*}(a,b)$, where $b > a, \alpha \in [0,1]$ is given as follows*

$$ {}_{a}^{ABC}\mathcal{D}_{t}^{\alpha}\mathcal{Q}(t) = \frac{\mathbb{B}(\alpha)}{1-\alpha} \int_{a}^{t} \mathcal{Q}'(s)\mathbb{E}_{\alpha}(-\alpha\frac{(t-s)^{\alpha}}{1-\alpha})ds, \quad \alpha < t, $$

where $\mathbb{E}_{\alpha}(G)$ is Mittag-Leffler function with one parameter which is defined by

$$ \mathbb{E}_{\alpha}(G) = \sum_{l=0}^{\infty} \frac{G^{l}}{\Gamma(\alpha l + 1)}, \qquad G \in \mathbb{C}; \quad Re(\alpha) > 0. $$

Lemma 2.6 ([1]). *For $\mathcal{Q}(t)$ defined on $[a,b]$, and $m < \Upsilon \leqslant m+1, m \in \mathbb{N}$, we have*

$$ {}^{AB}_{a}I^{\Upsilon}({}^{ABC}_{a}\mathcal{D}^{\alpha}_{t}\mathcal{Q}(t)) = \mathcal{Q}(t) - \sum_{k=0}^{m} \frac{\mathcal{Q}^{(m)}(a)}{k!}(t-a)^k. $$

Lemma 2.7 ([7, 29]). *For a function $\mathcal{Q}(t) \in C^{(m-1)}, m < \Upsilon \leqslant m+1, m \in \mathbb{N}$, and ${}_0\mathcal{D}^{\Upsilon}$ is RL fractional derivative, we have*

$$ {}_0I^{\Upsilon}({}_0\mathcal{D}^{\Upsilon}\mathcal{Q}(t)) = \mathcal{Q}(t) + a_1t^{\Upsilon-1} + a_2t^{\Upsilon-2} + a_3t^{\Upsilon-3} + \cdots + a_mt^{\Upsilon-m}, $$

where $a_i \in \mathbb{R}, \quad i = 1,2,3,\ldots,m.$

Lemma 2.8 ([7, 29]). *For $\Upsilon, \rho > 0$, the following statements are satisfying*

$$ \mathcal{D}^{\rho}\tau^{\Upsilon} = \frac{\Gamma(\Upsilon+1)}{\Gamma(1+\Upsilon-\rho)}\tau^{\Upsilon-\rho}, \qquad I^{\rho}\tau^{\Upsilon} = \frac{\Gamma(\Upsilon+1)}{\Gamma(1+\Upsilon+\rho)}\tau^{\Upsilon+\rho}. $$

Lemma 2.9 ([17]). *For a nonlinear operator ϕ_p,*

i. *for $1 < p \leqslant 2$, $\alpha_1, \alpha_2 > 0$ and $|\alpha_1|, |\alpha_2| \geqslant \rho_1 > 0$, then $|\phi_p(\alpha_1) - \phi_p(\alpha_2)| \leqslant (p-1)\rho_1^{(p-2)}|\alpha_1 - \alpha_2|$;*
ii. *if $p > 2$, and $|\alpha_1|, |\alpha_2| \leqslant \rho_2$, then $|\phi_p(\alpha_1) - \phi_p(\alpha_2)| \leqslant (p-1)\rho_2^{(p-2)}|\alpha_1 - \alpha_2|$.*

Theorem 2.10 (Guo-Krasnoselskii Theorem [17]). *Let C^* be a cone in a Banach space $\mathfrak{D}$, and $\mathfrak{U}_1, \mathfrak{U}_2$ are two bounded subsets of $\mathfrak{D}$ such that $0 \in \mathfrak{U}_1, \overline{\mathfrak{U}_1} \subset \mathfrak{U}_2$, and suppose that $\mathfrak{A} : C^* \cap \overline{\mathfrak{U}_2} \setminus \mathfrak{U}_1 \to C^*$ is a continuous operator satisfying*

i. *$\|\mathfrak{A}T\| \leqslant \|T\|$ if $T \in C^* \cap \partial\mathfrak{U}_1$, and $\|\mathfrak{A}T\| \geqslant \|T\|$ if $T \in C^* \cap \partial\mathfrak{U}_2$; or*
ii. *$\|\mathfrak{A}T\| \geqslant \|T\|$ if $T \in C^* \cap \partial\mathfrak{U}_1$, and $\|\mathfrak{A}T\| \leqslant \|T\|$ if $T \in C^* \cap \partial\mathfrak{U}_2$.*

Then $\mathfrak{A}$ has a fixed point at $C^* \cap \overline{\mathfrak{U}_2} \setminus \mathfrak{U}_1$.

3. Main results

Theorem 3.1. *Let $\mathcal{J}(t, \mathcal{Q}(t)) \in C[0,1]$ be an integrable function holding (1.3), then, for $m < \Upsilon, \eth \leqslant m+1, m \geqslant 4$, the unique solution for (1.3) is given by*

$$ \mathcal{Q}(t) = \int_0^1 \mathcal{G}^{\eth}(s,t)\phi_p({}^{AB}_0I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds, $$

where

$$ \mathcal{G}^{\eth}(t,s) = \begin{cases} -\frac{1}{\Gamma(\eth)}(t-s)^{(\eth-1)} + t^{(\eth-2)}\frac{1}{2\Gamma(\eth-1)}(1-s), & s \leqslant t \leqslant 1, \\ t^{(\eth-2)}\frac{1}{2\Gamma(\eth-1)}(1-s), & t \leqslant s \leqslant 1. \end{cases} \tag{3.1} $$

Proof. If we apply AB-fractional integral operator ${}^{AB}_0I^{\Upsilon}$ on (1.3) and use Lemma 2.6, we get

$$ \phi_p[{}_0\mathcal{D}^{\eth}\mathcal{Q}(t)] - \sum_{k=0}^{m} \frac{(\phi_p[{}_0\mathcal{D}^{\eth}\mathcal{Q}(0)])^{(k)}}{k!}t^k = -{}^{AB}_0I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))], \qquad k = 0,1,2,\ldots,m. $$

By using the condition $(\phi_p(\mathcal{D}^{\eth}\mathcal{Q}(t)))^{(k)}|_{t=0} = 0$, $k = 0,1,2,\ldots,m$, we have

$$ \phi_p[{}_0\mathcal{D}^{\eth}\mathcal{Q}(t)] = -{}^{AB}_0I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))]. $$

By applying the operator $\phi_p^{-1} = \phi_q$, we have

$$ {}_0\mathcal{D}^{\eth}\mathcal{Q}(t) = -\phi_q({}^{AB}_0I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))]). $$

Applying RL fractional integral, we have

$$\mathcal{Q}(t) = -{}_0I^{\eth}[\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])] + a_1t^{\eth-1} + a_2t^{\eth-2} + a_3t^{\eth-3} + \cdots + a_mt^{\eth-m}. \tag{3.2}$$

By using the condition ${}_0I^{k-\eth}(\mathcal{Q}(t))|_{t=0} = 0$, $k = 1,3,4,\ldots,m$, the constants $a_1 = a_3 = a_4 = \cdots = a_m = 0$. By substituting the values of the constants in (3.2), we get

$$\mathcal{Q}(t) = -{}_0I^{\eth}[\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])] + a_2t^{\eth-2}.$$

By using the condition ${}_0\mathcal{D}^{\eth-2}(\mathcal{Q}(t))|_{t=1} = 0$, and Lemma 2.8, we have

$$a_2 = \frac{1}{\Gamma(\eth-1)} \quad {}_0I^{2}[\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])]|_{t=1}.$$

So, the solution of problem (1.3) will become

$$\begin{aligned}
\mathcal{Q}(t) &= -{}_0I^{\eth}[\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])] + t^{\eth-2}\frac{1}{\Gamma(\eth-1)} \quad {}_0I^{2}[\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])]|_{t=1}\\
&= -\frac{1}{\Gamma(\eth)}\int_0^t (t-s)^{\eth-1}\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds\\
&\quad + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}\int_0^1 (1-s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds\\
&= \int_0^1 [-\frac{1}{\Gamma(\eth)}(t-s)^{\eth-1} + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)]\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds\\
&= \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds,
\end{aligned}$$

where $\mathcal{G}^{\eth}(t,s)$ is the green function defined in (3.1). □

Lemma 3.2. *The green function $\mathcal{G}^{\eth}(t,s)$ defined in (3.1) holds the following assumptions:*

$\mathcal{M}_1:$ $\mathcal{G}^{\eth}(t,s) > 0$ *for all* $s,t \in (0,1]$;
$\mathcal{M}_2:$ $\mathcal{G}^{\eth}(t,s)$ *is increasing function such that* $\max_{t\in[0,1]}\mathcal{G}^{\eth}(t,s) = \mathcal{G}^{\eth}(1,s)$;
$\mathcal{M}_3:$ $\mathcal{G}^{\eth}(t,s) \geqslant t^{\eth-1}\max_{t\in[0,1]}\mathcal{G}^{\eth}(t,s)$, *where* $s,t \in (0,1]$.

Proof. For proving $\mathcal{M}_1$, we consider two cases.

Case 1: If $s \leqslant t$,

$$\begin{aligned}
\mathcal{G}^{\eth}(t,s) &= -\frac{1}{\Gamma(\eth)}(t-s)^{\eth-1} + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)\\
&= -\frac{t^{\eth-1}}{(\eth-1)\Gamma(\eth-1)}(1-\frac{s}{t})^{\eth-1} + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)\\
&\geqslant -\frac{t^{\eth-1}}{(\eth-1)\Gamma(\eth-1)}(1-s)^{\eth-1} + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)\\
&\geqslant -\frac{t^{\eth-1}}{(\eth-1)\Gamma(\eth-1)}(1-s)^{\eth-1} + t^{\eth-1}\frac{1}{2\Gamma(\eth-1)}(1-s)^{\eth-1}\\
&= -\frac{t^{\eth-1}}{\Gamma(\eth-1)}(1-s)^{\eth-1}(\frac{-1}{\eth-1} + \frac{1}{2}) \geqslant 0.
\end{aligned}$$

It implies that

$$\mathcal{G}^{\eth}(t,s) \geqslant 0, \qquad s \leqslant t \leqslant 1. \tag{3.3}$$

Case 2: If $s \geqslant t$, $\mathcal{G}^{\eth}(t,s) = t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s) \geqslant 0$. It implies that

$$\mathcal{G}^{\eth}(t,s) \geqslant 0, \qquad t \leqslant s \leqslant 1. \tag{3.4}$$

From (3.3) and (3.4), it is shown that

$$\mathcal{G}^{\eth}(t,s) \geqslant 0, \qquad s,t \in (0,1]. \tag{3.5}$$

$\mathcal{M}_2$ can be proved by considering two cases.

Case 1: If $s \leqslant t$, we have

$$\begin{aligned}\frac{\partial \mathcal{G}^{\eth}(t,s)}{\partial t} &= -\frac{\eth-1}{\Gamma(\eth)}(t-s)^{\eth-2} + (\eth-2)t^{\eth-3}\frac{1}{2\Gamma(\eth-2)}(1-s)\\ &= -\frac{t^{\eth-2}}{\Gamma(\eth-1)}(1-\frac{s}{t})^{\eth-2} + t^{\eth-3}\frac{1}{2\Gamma(\eth-2)}(1-s)\\ &\geqslant -\frac{t^{\eth-2}}{(\eth-2)\Gamma(\eth-2)}(1-s)^{\eth-2} + t^{\eth-2}\frac{1}{2\Gamma(\eth-2)}(1-s)^{\eth-2}\\ &= \frac{t^{\eth-2}}{\Gamma(\eth-2)}(1-s)^{\eth-2}(\frac{-1}{(\eth-2)}+\frac{1}{2}) \geqslant 0.\end{aligned} \tag{3.6}$$

Case 2: For $s \geqslant t$, we have $\eth-2, t^{\eth-3}, \Gamma(\eth-1), (1-s) \geqslant 0$, so

$$\frac{\partial \mathcal{G}^{\eth}(t,s)}{\partial t} = (\eth-2)t^{\eth-3}\frac{1}{2\Gamma(\eth-1)}(1-s) \geqslant 0.$$

From (3.5) and (3.6), it is clear that $\frac{\partial \mathcal{G}^{\eth}(t,s)}{\partial t} \geqslant 0$, which shows $\mathcal{G}^{\eth}(t,s)$ is increasing function for $s,t \in (0,1)$, and we get

$$\max_{t\in[0,1]} \mathcal{G}^{\eth}(t,s) = -\frac{1}{\Gamma(\eth)}(t-s)^{(\eth-1)} + \frac{1}{2\Gamma(\eth-1)}(1-s) = \mathcal{G}^{\eth}(1,s), \qquad s \leqslant t \leqslant 1.$$

Similarly for $t \leqslant s \leqslant 1$, we have

$$\max_{t\in[0,1]} \mathcal{G}^{\eth}(t,s) = \frac{1}{2\Gamma(\eth-1)}(1-s) = \mathcal{G}^{\eth}(1,s).$$

For $\mathcal{M}_3$, we have two cases.

Case 1: For $s \leqslant t$,

$$\begin{aligned}\mathcal{G}^{\eth}(t,s) &= -\frac{1}{\Gamma(\eth)}(t-s)^{\eth-1} + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)\\ &= -\frac{t^{\eth-1}}{\Gamma(\eth)}(1-\frac{s}{t})^{\eth-1} + t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)\\ &\geqslant t^{\eth-1}[-\frac{1}{\Gamma(\eth)}(1-s)^{\eth-1} + t^{-1}\frac{1}{2\Gamma(\eth-1)}(1-s)]\\ &\geqslant t^{\eth-1}[-\frac{1}{\Gamma(\eth)}(1-s)^{\eth-1} + \frac{1}{2\Gamma(\eth-1)}(1-s)] = t^{\eth-1}\max_{t\in[0,1]}\mathcal{G}^{\eth}(t,s) = t^{\eth-1}\mathcal{G}^{\eth}(1,s).\end{aligned} \tag{3.7}$$

Case 2: For $t \leqslant s$, we have

$$\begin{aligned}\mathcal{G}^{\eth}(t,s) &= t^{\eth-2}\frac{1}{2\Gamma(\eth-1)}(1-s)\\ &= t^{\eth-1}[t^{-1}\frac{1}{2\Gamma(\eth-1)}(1-s)]\\ &\geqslant t^{\eth-1}[\frac{1}{2\Gamma(\eth-1)}(1-s)] = t^{\eth-1}\max_{t\in[0,1]}\mathcal{G}^{\eth}(t,s) = t^{\eth-1}\mathcal{G}^{\eth}(1,s).\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we conclude that $\mathcal{G}^{\eth}(t,s) = t^{\eth-1}\max_{t\in[0,1]}\mathcal{G}^{\eth}(t,s) = t^{\eth-1}\mathcal{G}^{\eth}(1,s)$. □

4. Existence solutions

Let $\mathcal{P} \in C[0,1]$ is a Banach space with the norm $\|x\| = \max_{t\in[0,1]}\{x(t) : x \in \mathcal{P}\}$ and let P be a cone consisting of positive functions in the space y such that $P = \{x \in y : x(t) \geqslant t^{\epsilon}\|x\|, t \in [0,1]\}$. Consider $B(r) = \{x \in P : \|x\| < r\}$ and $\partial B(r) = \{x \in P : \|x\| = r\}$, and by using Theorem 3.1, the alternative form of the solution that is shown in Theorem 3.1 is

$$\mathcal{Q}(t) = \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds. \tag{4.1}$$

Define $\mathcal{T} : P\setminus\{0\} \to \mathcal{P}$ by

$$\mathcal{T}\mathcal{Q}(t) = \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds.$$

With the help of Theorem 3.1, the fixed point of the operator $\mathcal{T}$ is equivalent to the solution of equation (1.3), and it is defined as

$$\mathcal{T}\mathcal{Q}(t) = \mathcal{Q}(t). \tag{4.2}$$

The following are satisfied:

$\mathcal{N}_1$: $\mathcal{J} : (0,1)\times(0,\infty) \to [0,\infty)$ is continuous and $\|\mathcal{J}(t,\mathcal{Q}(t))\| \leqslant \mathfrak{Y} < \infty$;
$\mathcal{N}_2$: $\mathfrak{U} : (0,1)\times[0,\infty) \to [0,\infty)$ is non-vanishing and continuous on (0,1) such that

$$\|\mathfrak{U}\| = \max_{t\in[0,1]} |\mathfrak{U}(t)| < \infty;$$

$\mathcal{N}_3$: for a positive number d_1, d_2, and $k \in [0,1]$, the function $\mathcal{J}$ fulfills

$$|\mathcal{J}(t,\mathcal{Q})| \leqslant \phi_q(d_1|\mathcal{Q}(t)|^k + d_2);$$

$\mathcal{N}_4$: for all $u, v \in \mathcal{P}$, there exist positive constant a_1, such that

$$|\mathcal{J}(t,u) - \mathcal{J}(t,v)| \leqslant a_1|u(t) - v(t)|.$$

Theorem 4.1. *If the conditions $\mathcal{N}_1$-$\mathcal{N}_3$ are hold, $\mathcal{T}$ is an operator with completely continuous property.*

Proof. For any $x \in \overline{\mathcal{P}(r_2)}\setminus\mathcal{P}(r_1)$, from (4.2) and Lemma 3.2, we have

$$\mathcal{T}\mathcal{Q}(t) = \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \tag{4.3}$$

and

$$\mathcal{T}\mathcal{Q}(t) = \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \geqslant t^{\eth-1}\int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds. \tag{4.4}$$

With the help of the (4.3) and (4.4), we obtain

$$\mathcal{T}\mathcal{Q}(t) \geqslant t^{\eth-1}\|\mathcal{T}\mathcal{Q}(t)\|, t \in [0,1].$$

This implies $\overline{\mathcal{P}(r_2)}\setminus\mathcal{P}(r_1) \to P$. For proving the continuity of $\mathcal{T}$, we need to show $\|\mathcal{T}\mathcal{Q}_m - \mathcal{T}\mathcal{Q}\| \to 0$ as $m \to \infty$.

$$\begin{aligned}\|\mathcal{T}\mathcal{Q}_m - \mathcal{T}\mathcal{Q}\| &= |\int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}_m(t))])ds - \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds| \\ &\leqslant \int_0^1 |\mathcal{G}^{\eth}(t,s)||(\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}_m(t))]) - \phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))]))|ds.\end{aligned} \tag{4.5}$$

Because of (4.5) and the continuity of $\mathcal{J}$, we have $\|\mathcal{T}\mathcal{Q}_m - \mathcal{T}\mathcal{Q}\|$ as $m \to \infty$. This shows that $\mathcal{T}$ is continuous.

In the next step, we need to show the uniform boundedness of $\mathcal{T}$. Making use of (4.1), and pre-assumption $\mathcal{N}_1$, we obtain

$$
\begin{aligned}
|\mathcal{T}\mathcal{Q}| &= |\int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])|ds \\
&\leqslant |\int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])|ds \\
&\leqslant \int_0^1 |\mathcal{G}^{\eth}(1,s)|\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \\
&\leqslant \int_0^1 |\mathcal{G}^{\eth}(1,s)|\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2)) \\
&\quad + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon)}\int_0^s (s-w)^{\Upsilon-1}\|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2))dw)ds \\
&\leqslant (-\frac{1}{\Gamma(\eth+1)} + \frac{1}{4\Gamma(\eth-1)})\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2)) \\
&\quad + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+2)}\|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2)(0+1)) \\
&\leqslant (\frac{1}{\Gamma(\eth+1)} + \frac{1}{4\Gamma(\eth-1)})\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\mathfrak{M} + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+2)}\mathfrak{M}) < \infty,
\end{aligned}
\tag{4.6}
$$

where $\mathfrak{M} = \|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2))$. Equation (4.6) shows that the operator $\overline{\mathcal{P}(r_2)} \setminus \mathcal{P}(r_1)$ is a uniformly bounded operator.

Regarding the operator $\mathcal{T}$, its equicontinuity can be prove by using the assumption $\mathcal{N}_3$, Theorem 2.10, and (4.1), which gives us the following: For $t_1, t_2 \in [0,1]$, we have

$$
\begin{aligned}
|\mathcal{T}\mathcal{Q}(t_2) - \mathcal{T}\mathcal{Q}(t_1)| &= |\int_0^1 \mathcal{G}^{\eth}(t_2,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \\
&\quad - \int_0^1 \mathcal{G}^{\eth}(t_1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds| \\
&\leqslant \int_0^1 (|\mathcal{G}^{\eth}(t_2,s)| - |\mathcal{G}^{\eth}(t_1,s)|)\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2)) \\
&\quad + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+1)}\int_0^s (s-w)^{\Upsilon-1}\|\mathfrak{U}\|(\phi_p(d_1\|\mathcal{Q}\|^k + d_2))dw)ds \\
&\leqslant (|\mathcal{G}^{\eth}(t_2,s)| - |\mathcal{G}^{\eth}(t_1,s)|)\phi_q[\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\mathfrak{M} + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+2)}\mathfrak{M}(0+1)] \\
&\leqslant (\frac{|t_2^{\eth} - t_1^{\eth}|}{\Gamma(\eth+1)} + \frac{|t_2^{\eth-2} - t_1^{\eth-2}|}{4\Gamma(\eth-1)})\phi_q[\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\mathfrak{M} + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+2)}\mathfrak{M}].
\end{aligned}
$$

As $t_2 \to t_1, |\mathcal{T}\mathcal{Q}(t_2) - \mathcal{T}\mathcal{Q}(t_1)| \to 0$, so $\mathcal{T}$: $\overline{\mathcal{P}(r_2)}\backslash\mathcal{P}(r_1)$ is equicontinuous. By using Arzela-Ascoli Theorem, $\mathcal{T}$: $\overline{\mathcal{P}(r_2)}\backslash\mathcal{P}(r_1)$ is compact. It indicates $\mathcal{T}$ is compact in $\overline{\mathcal{P}(r_2)}\backslash\mathcal{P}(r_1)$. Then $\mathcal{T}$: $\overline{\mathcal{P}(r_2)}\backslash\mathcal{P}(r_1) \to P$ is completely continuous. □

For controlling the growth of nonlinear function $\mathcal{J}(t,\mathcal{Q}(t))$, the hight functions are defined as follows:

$$
\begin{cases}
\phi_{max}(t,r) = \max\{\mathcal{J}(t,\mathcal{Q}(t))\} : t^{\Upsilon-1}r \leqslant \mathcal{Q} \leqslant r, \quad r > 0, \quad t \in (0,1), \\
\phi_{min}(t,r) = \min\{\mathcal{J}(t,\mathcal{Q}(t))\} : t^{\Upsilon-1}r \leqslant \mathcal{Q} \leqslant r, \quad r > 0, \quad t \in (0,1).
\end{cases}
\tag{4.7}
$$

Theorem 4.2. *Assume that $\mathcal{N}_1$-$\mathcal{N}_3$ are hold, and $\mu_1, \mu_2 \in \mathbb{R}$ with one of the next conditions is fulfilled:*

$\mathcal{H}_1$:

$$\mu_1 \leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\phi_{min}\mathcal{J}(t,\mu_1)])ds < \infty,$$

and

$$\int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\phi_{max}\mathcal{J}(t,\mu_2)])ds < \mu_2;$$

$\mathcal{H}_2$:

$$\int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\phi_{max}\mathcal{J}(t,\mu_1)])ds < \mu_1,$$

and

$$\mu_2 \leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\phi_{min}\mathcal{J}(t,\mu_2)])ds < \infty.$$

Then, the suggested problem (1.3) *has an increasing solution* $\mathcal{Q} \in P$, *such that* $\mu_1 \leqslant\|\mathcal{Q}\| \leqslant \mu_2$.

Proof. Without loss of generalization, we consider the case $\mathcal{H}_1$. By assuming $\mathcal{Q} \in \partial\mathcal{P}\mu_1$, we have $\|\mathcal{Q}\| = \mu_1$, and $t^{\eth-1}\mu_1 \leqslant \mathcal{Q}(t) \leqslant \mu_1, t \in [0,1]$. By using (4.7), it shows that $\phi_{min}(t,\mathcal{Q}) \leqslant \mathcal{J}(t,\mathcal{Q}(t))$, which gives

$$\begin{aligned}
\|\mathcal{T}\mathcal{Q}(t)\| &= \max_{t\in[0,1]} \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \\
&\geqslant t^{\eth-1} \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \\
&\geqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\phi_{min}\mathcal{J}(t,\mu_1)])ds \geqslant \mu_1 =\|\mathcal{Q}\|.
\end{aligned}$$

If $\mathcal{Q} \in \partial\mathcal{P}\mu_2$, we have $\|\mathcal{Q}\| = \mu_2$, and $t^{\eth-1}\mu_2 \leqslant \mathcal{Q}(t) \leqslant \mu_2, t \in [0,1]$. By using (4.7), it shows that $\phi_{max}(t,\mathcal{Q}) \geqslant \mathcal{J}(t,\mathcal{Q}(t))$, which gives

$$\begin{aligned}
\|\mathcal{T}\mathcal{Q}(t)\| &= \min_{t\in[0,1]} \int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \\
&\leqslant t^{\eth-1} \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds \\
&\leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\phi_{max}\mathcal{J}(t,\mu_2)])ds \leqslant \mu_2 =\|\mathcal{Q}\|.
\end{aligned}$$

By exploiting Lemma 2.9, the operator $\mathcal{T}$ has a fixed point, assume $\mathcal{Q}^* \in \overline{\mathcal{P}(\mu_2)}\diagdown\mathcal{P}(\mu_1)$. Then, $\mu_1 \leqslant\|\mathcal{Q}^*\| \leqslant \mu_2$. By applying Theorem 3.1 and Lemma 3.2, we obtain $\mathcal{Q}^*(t) \geqslant t^{\eth-1}\|\mathcal{Q}^*| \geqslant \mu_1 t^{\eth-1}, t \in [0,1]$. This intend that $\mathcal{Q}^*$ is a positive solution. Again Theorem 3.1 and Lemma 3.2 were applied, and it gave

$$\frac{\partial}{\partial t}\mathcal{Q}^*(t) = \frac{\partial}{\partial t}(\mathcal{T}\mathcal{Q}(t)) = \int_0^1 \frac{\partial}{\partial t}\mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}^*(t))])ds.$$

This indicates that the solution $\mathcal{Q}^*(t)$ is increasing and positive. □

5. Stability analysis

In this part, we use HUS for our proposed ABC derivative problem with nonlinear operator ϕ_p (1.3). For this task, we follow many related studies [7, 21, 22].

Definition 5.1. Equation (4.1) satisfies HUS if this condition holds: For each positive number γ, there exists a positive constant ρ such that, if

$$|\mathcal{Q}(t)-\int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds| \leqslant \gamma,$$

there is a $\mathcal{Z}(t)$ satisfying

$$\mathcal{Z}(t)=\int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds, \tag{5.1}$$

which implies $|\mathcal{Q}(t)-\mathcal{Z}(t)| \leqslant \gamma\rho$.

In the next theorem, we introduce and prove the stability of our proposed problem (1.3).

Theorem 5.2. *By assuming that the assumptions $\mathcal{N}_1, \mathcal{N}_2$, and $\mathcal{N}_4$ are satisfied, the problem (1.3) is HU stable.*

Proof. Let $\mathcal{Q}(t)$ is an exact solution of (4.1) and $\mathcal{Z}(t)$ is an approximate solution and holds (5.1), we get

$$\begin{aligned}|\mathcal{Q}(t)-\mathcal{Z}(t)| &= |\int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Q}(t))])ds-\int_0^1 \mathcal{G}^{\eth}(t,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(t)\mathcal{J}(t,\mathcal{Z}(t))])ds\\ &\leqslant (p-1)\rho^{p-2}\|\mathfrak{U}\|^{q-1}(\int_0^1 \mathcal{G}^{\eth}(t,s)[\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\|\mathcal{J}(t,\mathcal{Q}(t))-\mathcal{J}(t,\mathcal{Z}(t))\|\\ &\quad+\frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon)}\int_0^s (s-w)^{\Upsilon-1}\|\mathcal{J}(w,\mathcal{Q}(w))-\mathcal{J}(w,\mathcal{Z}(w))\|dw]ds)\\ &\leqslant (p-1)\rho^{p-2}\|\mathfrak{U}\|^{q-1}(\frac{1}{\Gamma(\eth+1)}+\frac{1}{4\Gamma(\eth-1)})[\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}+\frac{\Upsilon(1-0)}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+2)}]^{q-1}\rho\|\mathcal{Q}-\mathcal{Z}\|,\end{aligned}$$

where $\gamma=(p-1)\rho^{p-2}\|\mathfrak{U}\|^{q-1}(\frac{1}{\Gamma(\eth+1)}+\frac{1}{4\Gamma(\eth-1)})[\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}+\frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon+2)}]^{q-1}\rho$, therefore, the fractional integral operator in (4.2) satisfies HUS. Based on HUS method, the solution of the proposed problem (1.3) is stable. □

6. Illustrative example

In this Section, we investigate one example for the application of our results that have proved in Sections 4 and 5.

Example 6.1. For $t\in[0,1]$, $\mathcal{J}(t,\mathcal{Q}(t))=\mathcal{Q}^5(t)+\frac{1}{3\mathcal{Q}^{\frac{2}{5}}(t)}$, $p=3, q=1.5$, $\eth=\Upsilon=5.5$, $\mathfrak{U}(t)=\frac{t+1}{\sqrt{1-t}}$, we consider the following example for problem (1.3):

$$\begin{cases} {}_0^{ABC}\mathcal{D}^{5.5}(\phi_3({}_0\mathcal{D}^{5.5}\mathcal{Q}(t)))+\frac{t+1}{\sqrt{1-t}}(\mathcal{Q}^5(t)+\frac{1}{3\mathcal{Q}^{\frac{2}{5}}(t)})=0,\\ (\phi_3({}_0\mathcal{D}^{5.5}\mathcal{Q}(t)))^{(k)}|_{t=0}=0,\ k=1,2,3,4,5,\\ {}_0I^{k-5.5}(\mathcal{Q}(t))|_{t=0}=0,\ k=1,3,4,5,\ {}_0\mathcal{D}^{3.5}(\mathcal{Q}(t))|_{t=1}=0.\end{cases} \tag{6.1}$$

It is clear that the function $\mathfrak{U}\in C((0,1),[0,\infty)), \mathcal{J}(t,\mathcal{Q}(t))\in C((0,1)\times(0,\infty),[0,\infty))$. We consider the following height functions

$$\begin{aligned}\phi_{max}(t,r)&=\max\{\mathcal{J}(t,\mathcal{Q}(t)):t^{\frac{9}{2}}r\leqslant\mathcal{Q}\leqslant r\}\leqslant t^{\frac{45}{2}}r^5+\frac{1}{3r^{\frac{2}{5}}},\\ \phi_{min}(t,r)&=\min\{\mathcal{J}(t,\mathcal{Q}(t)):t^{\frac{9}{2}}r\leqslant\mathcal{Q}\leqslant r\}\geqslant r^5+\frac{1}{3t^{\frac{9}{5}}r^{\frac{2}{5}}}.\end{aligned} \tag{6.2}$$

Then, for $t \in [0,1]$, we have

$$\begin{aligned}
&\int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(w)\phi_{\max}\mathcal{J}(w,r)]dw)ds \\
&\leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(w)\phi_{\max}\mathcal{J}(w,\mu_1)]dw)ds \\
&\leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\frac{t+1}{\sqrt{1-t}}(t^{\frac{45}{2}}r^5+\frac{1}{3r^{\frac{2}{5}}}) \\
&\quad + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon)}\int_0^s (s-w)^{\Upsilon-1}\times\frac{w+1}{\sqrt{1-w}}(w^{\frac{45}{2}}r^5+\frac{1}{3r^{\frac{2}{5}}})dw)ds \\
&\leqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\frac{s+1}{\sqrt{1-s}}(s^{\frac{45}{2}}+\frac{1}{3})+\frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon)} \\
&\quad \times\int_0^s (s-w)^{\Upsilon-1}\frac{w+1}{\sqrt{1-w}}(w^{\frac{45}{2}}+\frac{1}{3})dw)ds \leqslant 0.04958 \leqslant 1,
\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}
&\int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(w)\phi_{\min}\mathcal{J}(w,r)]dw)ds \\
&\geqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q({}_0^{AB}I^{\Upsilon}[\mathfrak{U}(w)\phi_{\min}\mathcal{J}(w,\mu_2)]dw)ds \\
&\geqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\frac{t+1}{\sqrt{1-t}}(r^5+\frac{1}{3t^{\frac{9}{5}}r^{\frac{2}{5}}}) \\
&\quad + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon)}\int_0^s (s-w)^{\Upsilon-1}\frac{w+1}{\sqrt{1-w}}(r^5+\frac{1}{3w^{\frac{9}{5}}r^{\frac{2}{5}}})dw)ds \\
&\geqslant \int_0^1 \mathcal{G}^{\eth}(1,s)\phi_q(\frac{1-\Upsilon}{\mathbb{B}(\Upsilon)}\frac{s+1}{\sqrt{1-s}}(\frac{1}{1000^5}+\frac{1}{3s^{\frac{9}{5}}1000^{\frac{2}{5}}}) \\
&\quad + \frac{\Upsilon}{\mathbb{B}(\Upsilon)\Gamma(\Upsilon)}\int_0^s (s-w)^{\Upsilon-1}\frac{w+1}{\sqrt{1-w}}(\frac{1}{1000^5}+\frac{1}{3w^{\frac{9}{5}}1000^{\frac{2}{5}}})dw)ds \\
&\geqslant 0.02043 \geqslant \frac{1}{1000}.
\end{aligned} \tag{6.4}$$

By the assistance of Theorem 4.2, the problem (6.1) has a unique solution which satisfies $\frac{1}{1000} \leqslant \|\Omega\| \leqslant 1$.

7. Conclusion

In this study, we have considered a system of fractional differential equations shown in the problem (1.3) with singularity based on ABC-fractional derivative under boundary conditions with ϕ_p- Laplacian operator. The proposed problem (1.3) has converted into an integral equation through Green function. Then, we examined Green function's behavior for being decreasing or increasing, and negative or positive function. The EU of the solution were evaluated via Guo-Krasnoselskiis fixed point theorem. HU-stability was used for ensuring the stability of the solution. An illustrative example was investigated to show the application of this study, and we used the Mathematica programming for calculating the numerical values in (6.3) and (6.4). Future studies can be done by considering that system under various fractional derivatives, such as Caputo-Fabrizio fractional derivative.

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References

[1] T. Abdeljawad, *A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel*, J. Inequal. Appl., **2017** (2017), 11 pages. 2.4, 2.5, 2.6

[2] B. Ahmed, J. J. Nieto, *Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions*, Bound. Value Probl., **2011** (2011), 9 pages. 2.2

[3] M. Almushaira, F. Liu, *Fourth-order time-stepping compact finite difference method for multi-dimensional space-fractional coupled nonlinear Schrödinger equations*, SN Partial Differ. Equ. Appl., **1** (2020), 29 pages. 1

[4] M. Almushaira, F. Liu, *A Fast Conservative Scheme for the Space Fractional Nonlinear Schrödinger Equation with Wave Operator*, J. Math. Study, **54** (2021), 407–426. 1

[5] W. Al-Sadi, A. Alkhazan, T. Q. S. Abdullah, M. Al-Soswa, *Stability and existence the solution for a coupled system of hybrid fractional differential equation with uniqueness*, Arab. J. Basic Appl. Sci., **28** (2021), 340–350. 1

[6] W. Al-Sadi, M. Hussein, T. Q. S. Abdullah, *Existence and stability criterion for the results of fractional order Φp-Laplacian operator boundary value problem*, Comput. Methods Differ. Equ., **9** (2021), 1042–1058. 1

[7] W. Al-Sadi, H. Zhenyou, A. Alkhazzan, *Existence and stability of a positive solution for nonlinear hybrid fractional differential equations with singularity*, J. Taibah Univ. Sci., **13** (2019), 951–960. 1, 2.7, 2.8, 5

[8] A. Atangana, *Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination?*, Chaos Solitons Fractals, **136** (2020), 38 pages. 1

[9] A. Atangana, D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, arXiv, **2016** (2016), 8 pages. 1, 2.3, 2.4, 2.5

[10] K. Balachandran, J. Y. Park, J. J. Trujillo *Controllability of nonlinear fractional dynamical systems*, Nonlinear Anal., **75** (2012), 1919–1926. 1

[11] D. Baleanu, B. Shiri, *Collocation methods for fractional differential equations involving non-singular kernel*, Chaos Solitons Fractals, **116** (2018), 136–145. 1

[12] A. Boukhouima, K. Hattaf, E. M. Lotfi, M. Mahrouf, D. F. M. Torres, N. Yousfi, *Lyapunov functions for fractional-order systems in biology: Methods and applications*, Chaos Solitons Fractals, **140** (2020), 9 pages. 1

[13] M. Caputo, *Linear modles of dissipation whose Q is almost frequenty independent–II*, J. R. Astr. Soc., **13** (1967), 529–539. 1

[14] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Differ. Appl., **1** (2015), 73–85. 1

[15] A. M. A. El-Sayed, S. Z. Rida, Y. A. Gaber, *Dynamical of curative and preventive treatments in a two-stage plant disease model of fractional order*, Chaos Solitons Fractals, **137** (2020), 11 pages. 1

[16] O. Guner, A. Bekir, *The Exp-function method for solving nonlinear space–time fractional differential equations in mathematical physics*, J. Assoc. Arab Univ. Basic Appl. Sci., **24** (2017), 277–282. 1

[17] D. J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, (2014). 2.9, 2.10

[18] R. Hilfer, *Application of fractional Calculus in Physics*, World Scientific Publishing Co., River Edge, (2000). 1

[19] H. Jafari, D. Baleanu, H. Khan, R. A. Khan, A. Khan, *Existence criterion for the solutions of fractional order* p*-Laplacian boundary value problems*, Bound. Value Probl., **2015** (2015), 10 pages. 1

[20] F. Jarad, T. Abdeljawad, Z. Hammouch, *On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative*, Chaos Solitons Fractals, **117** (2018), 16–20. 1

[21] H. Khan, W. Chen, H. G. Sun, *Analysis of positive solution and Hyers-Ulam stability for a class of singular fractional differential equations with* p*-Laplacian in Banach space*, Math. Methods Appl. Sci., **41** (2018), 3430–3440. 1, 5

[22] H. Khan, F. Jarad, T. Abdeljawad, A. Khan, *A singular ABC-fractional differential equation with* p*-Laplacian operator*, Chaos Solitons Fractals, **129** (2019), 56–61. 1, 1, 2.1, 5

[23] A. Khan, H. Khan, J. Gómez-Aguilar, T. Abdeljawad, *Existence and Hyers-Ulam stability for a nonlinear singular fractional differential equations with Mittag-Leffler kernel*, Chaos Solitons Fractals, **127** (2019), 422–427. 1, 1

[24] H. Khan, Y. J. Li, W. Chen, D. Baleanu, A. Khan, *Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with* p*-Laplacian operator*, Bound. Value Probl., **2017** (2017), 16 pages. 1

[25] Y. H. Li, *Existence of positive solutions for fractional differential equation involving integral boundary conditions with* p*-Laplacian operator*, Adv. Difference Equ., **2017** (2017), 11 pages. 1

[26] D. F. Luo, Q. x. Zhu, Z. G. Luo, *An averaging principle for stochastic fractional differential equations with time-delays*, Appl. Math. Lett., **105** (2020), 8 pages. 1

[27] J.-H. Mai, *Devaney's chaos implies existence of s–scrambled sets*, Proc. Amer. Math. Soc., **132** (2004), 2761–2767. 1

[28] D. D. Min, L. S. Liu, Y. H. Wu, *Uniqueness of positive solutions for the singular fractional differential equations involving integral boundary value conditions*, Bound. Value Probl., **2018** (2018), 18 pages. 1

[29] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Yverdon, (1993). 2.7, 2.8

[30] M. Saqib, I. Khan, S. Shafie, *Application of fractional differential equations to heat transfer in hybrid nanofluid: modeling and solution via integral transforms*, Adv. Difference Equ., **2019** (2019), 18 pages. 1

[31] S. Q. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl., **252** (2000), 804–812. 1