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On dual B-filters and dual B-subalgebras in a topological dual B-algebra

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Katrina E. Belleza[a,*], Jimboy R. Albaracin[b]

[a] Department of Computer, Information Sciences, and Mathematics, School of Arts and Sciences, University of San Carlos, Cebu City, Philippines.

[b] Mathematics and Statistics Programs, College of Science, University of the Philippines Cebu, Cebu City, Philippines.

Abstract

This paper introduces the notion of the tdB-algebra, presents characteristics and properties of dual B-filters and dual B-subalgebras in a tdB-algebra, and introduces the uniform topology on a dual B-algebra in terms of its dual B-subalgebras. Moreover, this paper shows that a uniform dual B-structure is a tdB-algebra.

Keywords: Topological algebra, dual B-algebra, topological dual B-algebra, dual B-topological space, tdB-algebra.

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1. Introduction

Belleza and Vilela [1] in 2019 introduced and characterized the Dual B-algebra together with some of its properties and relations to other algebras. In 1999, Jun et al. [5] introduced topological BCI-algebras, provided some properties on this structure, and characterized a topological BCI-algebra in terms of neighborhoods. In 2017, Mehrshad and Golzarpoor [6] provided some properties of uniform topology and topological BE-algebras. Moreover, Gonzales [3] in 2019 introduced the notion of topological B-algebras and investigated its properties based on neighborhoods and filterbase.

The aforementioned research works have paved way to investigating dual B-filters and its relevance in the concept of a uniform dual B-topology. Specifically, this paper introduces the notion of the tdB-algebra, dual B-subalgebras, and dual B-filters, provides some properties of dual B-filters and characterizes uniform dual B-topology in a dual B-algebra.

2. Preliminaries

Definition 2.1 ([1])**.** A dual B-algebra X^D is a triple $(X^D, \circ, 1)$ where X^D is a non-empty set with a binary operation "$\circ$" and a constant 1 satisfying the following axioms for all x, y, z in X^D

(DB1) $x \circ x = 1$;

(DB2) $1 \circ x = x$;

*Corresponding author
Email addresses: kebelleza@usc.edu.ph (Katrina E. Belleza), jralbaracin@up.edu.ph (Jimboy R. Albaracin)

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(DB3) $x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z$.

Example 2.2 ([1]). Let $X^D = \{1, a, b, c\}$ with binary operation $\circ$ as defined in the table below.

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

Then $(X^D, \circ, 1)$ is a dual B-algebra.

Theorem 2.3 ([1]). *Let $X = (X, \circ, 1)$ be any algebra of type (2,0). Then X is a dual B-algebra if and only if*

(i) $x \circ x = 1$;

(ii) $x = (x \circ 1) \circ 1$;

(iii) $(x \circ y) \circ (x \circ z) = y \circ z$.

Lemma 2.4 ([1]). *Let $(X^D, \circ, 1)$ be a dual B-algebra. Then for any $x, y \in X^D$, (i) $x \circ y = 1$ implies $x = y$.*

Definition 2.5 ([2]). Let X be a set. A topology (or topological structure) in X is a family τ of subsets of X that satisfies the following:

(i) Each union of members of τ is also a member of τ;

(ii) Each finite intersection of members of τ is also a member of τ; and

(iii) $\varnothing$ and X are members of τ.

A couple (X, τ) consisting of a set X and a topology τ in X is called a topological space. We also say "τ is the topology of the space X". The members of τ are called open sets of (X, τ). By a neighborhood of an element x in X (denoted as $U(x)$) is meant any open set (that is, member of τ) containing x. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is called continuous if the inverse image of each open set in Y is open in X (that is, if f^{-1} maps τ_Y into τ_X).

Definition 2.6 ([2]). Let $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be any family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology for Y_α. The Cartersian product topology in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets $< U_\beta >= \rho_\beta^{-1}(U_\beta)$, where $\rho : \prod_\alpha Y_\alpha \to Y_\alpha$, U_β ranges over all members of τ_β and β over all elements of $\mathcal{A}$.

Theorem 2.7 ([2]). *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \to Y$ a map. Then f is continuous if and only if for each $x \in X$ and each neighborhood $W\big(f(x)\big)$ in Y, there exists a neighborhood $V(x)$ in X such that $f\big(V(x)\big) \subseteq W\big(f(x)\big)$.*

3. Introduction to topological dual B-algebra

Henceforth, the notation X^D will mean the triple $(X^D, \circ, 1)$.

Definition 3.1. Let X^D be a dual B-algebra. A topology τ on X^D is called a dual B-topology and the couple (X^D, τ) is called a dual B-topological space.

Example 3.2. Consider the set $X^D = \{1, a, b, c, d, e\}$ and binary operation $\circ$ as defined in the table below.

$\circ$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	b	1	a	d	e	c
b	a	b	1	e	c	d
c	c	d	e	1	a	b
d	d	e	c	b	1	a
e	e	c	d	a	b	1

By routine calculations, $X^D = (X^D; \circ, 1)$ is a dual B-algebra. Let $\tau = \{X^D, \varnothing, \{1\}, \{1, a, b\}, \{1, c, d, e\}\}$. Then τ is a dual B-topology on X^D.

Definition 3.3. The triple $(X^D, \circ, \tau)$ is called a topological dual B-algebra (or tdB-algebra) if τ is a dual B-topology and the binary operation $\circ : X^D \times X^D \to X^D$ is continuous where the topology on $X^D \times X^D$ is the Cartesian product topology.

Remark 3.4. Let X^D be a dual B-algebra and nonempty $A, B \subseteq X^D$. Then

$$\circ(A \times B) = A \circ B,$$

where $A \circ B = \{a \circ b | a \in A, b \in B\}$.

Theorem 3.5. *Let X^D be a dual B-algebra and τ a dual B-topology. Then $(X^D, \circ, \tau)$ is a tdB-algebra if and only if for all $x, y \in X^D$ and $U(x \circ y)$, there exists $U(x)$ and $U(y)$ such that $U(x) \circ U(y) \subseteq U(x \circ y)$.*

Proof. Suppose $(X^D, \circ, \tau)$ is a tdB-algebra and let $x, y \in X^D$ and $U(x \circ y)$ a neighborhood of $x \circ y$. Since $\circ$ is continuous, $\circ^{-1}\big(U(x \circ y)\big)$ is a neighborhood of $(x, y) \in X^D \times X^D$. By the Cartesian product topology, there exists $U(x)$ and $U(y)$ such that $U(x) \times U(y) \subseteq \circ^{-1}\big(U(x \circ y)\big)$. By Remark 3.4, it follows that

$$U(x) \circ U(y) = \circ\big(U(x) \times U(y)\big) \subseteq \circ \circ^{-1} \big(U(x \circ y)\big).$$

Hence, there exists $U(x)$ and $U(y)$ such that $U(x) \circ U(y) \subseteq U(x \circ y)$. Conversely, let $(x, y) \in X^D \times X^D$ and $U(x \circ y) \subseteq X^D$. Then there exists $U(x)$ and $U(y)$ such that $U(x) \circ U(y) \subseteq U(x \circ y)$. Note that $U(x) \times U(y)$ is a neighborhood of (x, y) in $X^D \times X^D$. By Remark 3.4, $\circ\big(U(x) \times U(y)\big) = U(x) \circ U(y)$. It follows that $\circ\big(U(x) \times U(y)\big) \subseteq U(x \circ y)$. By Theorem 2.7, $\circ$ is continuous. Therefore, $(X^D, \circ, \tau)$ is a tdB-algebra. □

Example 3.6. Consider the dual B-algebra $X^D = \{1, a, b, c\}$ from Example 2.2 and let $\tau = \{X^D, \varnothing, \{1, a\}, \{b, c\}\}$. Then τ is a dual B-topology. Through routine calculations and with Theorem 3.5, $(X^D, \circ, \tau)$ is a tdB-algebra.

Example 3.7. Consider the dual B-topological space (X^D, τ) from Example 3.2. Note that for $a \in X^D$, the neighborhoods of a are X^D and $\{1, a, b\}$. Hence, for $a \circ a = 1 \in \{1\}$, we have

$$X^D \circ X^D = X^D \circ \{1, a, b\} = \{1, a, b\} \circ X^D = X^D \nsubseteq \{1\},$$

and $\{1, a, b\} \circ \{1, a, b\} = \{1, a, b\} \nsubseteq \{1\}$. It follows that $(X^D, \circ, \tau)$ is not a tdB-algebra.

4. Dual B-filters and dual B-subalgebras

Definition 4.1. Let X^D be a dual B-algebra and S a nonempty subset of X^D. Then S is called a dual B-subalgebra of X^D if S itself is a dual B-algebra with binary operation of X^D on S.

Remark 4.2. If S is a dual B-subalgebra of X^D, then $1 \in S$.

Theorem 4.3. *Let S be a nonempty subset of a dual B-algebra X^D. Then S is a dual B-subalgebra if and only if for any $x, y \in S$, $x \circ y \in S$.*

Proof. Let S be a nonempty subset of a dual B-algebra X^D with $x \circ y \in S$ for any $x, y \in S$. Note that S satisfies (DB1), (DB2), (DB3) with $1 = x \circ x \in S$. Hence, S is itself a dual B-algebra. The converse follows immediately by definition of a binary operator. □

Example 4.4. Consder the dual B-algebra of Example 3.2. Note that $\{1, a, b\}, \{1, c\}, \{1, d\}$, and $\{1, e\}$ are dual B-subalgebras while $\{1, a\}$ is not a dual B-subalgebra since $a \circ 1 = b \notin \{1, a\}$.

Remark 4.5. Not every dual B-subalgebra of a dual B-algebra X^D is either an open or closed set in a dual B-topological space. This remark is illustrated in the next example.

Example 4.6. Consider the dual B-topological space (X^D, τ) from Example 3.2 and dual B-subalgebra $S_1 = \{1, c\}$ from Example 4.4. Note that $S_1 \notin \tau$ implying that S_1 is not an open set. Morevoer,

$$X^D \backslash S_1 = \{a, b, d, e\} \notin \tau,$$

implying that S_1 is not a closed set.

The next result is a characterization of an open subset in a tdB-algebra. Since dual B-subalgebras are also subsets of dual B-algebras, the theorem is followed by a corollary indicating that the characterization may also be applied to dual B-subalgebras.

Theorem 4.7. *Let $(X^D, \circ, \tau)$ be a tdB-algebra, $1 \in \bigcap_{U \in \tau, U \neq \varnothing} U$, and $S \subseteq X^D$ such that $1 \in S$. Then a subset S of X^D is open if and only if 1 is an interior point of S.*

Proof. Suppose S is open in X^D. Then Int(S)$= S$. Since $1 \in S$, 1 is an interior point of S. Conversely, suppose 1 is an interior point of S. Then there exists $U(1)$ such that $U(1) \subseteq S$. Let $x \in S$. By (DB1), $x \circ x = 1 \in U(1)$. By Theorem 3.5, there exists $U(x)$ such that $U(x) \circ U(x) \subseteq U(1)$. It remains to show that $U(x) \subseteq S$. By hypothesis, $1 \in U(x)$. Since $x \in U(x)$, $x = 1 \circ x \in U(x) \circ U(x)$. Thus, $U(x) \subseteq U(x) \circ U(x)$. Hence, $x \in U(x) \subseteq U(x) \circ U(x) \subseteq U(1) \subseteq S$. Therefore, S is open. □

Corollary 4.8. *Let $(X^D, \circ, \tau)$ be a tdB-algebra and $1 \in \bigcap_{U \in \tau, U \neq \varnothing} U$. Then a dual B-subalgebra S of X^D is open if and only if 1 is an interior point of S.*

Lemma 4.9. *Let $(X^D, \circ, \tau)$ be a tdB-algebra and $S_1 \subseteq X^D$ where $1 \in S_1$ with the property that if $1 \in U$, then $S_1 \subseteq U$ for all $U \in \tau$. Then for any $x \in S_1$ with $U(x)$, $S_1 \subseteq U(x)$.*

Proof. Suppose $x \in S_1$ with $U(x)$. By (DB2), $1 \circ x = x \in U(x)$. By Theorem 3.5, there exists $U(1)$ such that $U(1) \circ U(x) \subseteq U(x)$. By (DB1) and the hypothesis, $1 = x \circ x \in S_1 \circ U(x) \subseteq U(1) \circ U(x) \subseteq U(x)$. This implies that $U(x)$ is an open set containing 1. Therefore, $S_1 \subseteq U(x)$. □

Theorem 4.10. *Let $(X^D, \circ, \tau)$ be a tdB-algebra such that $1 \in \bigcap_{U \in \tau, U \neq \varnothing} U$ and S a closed dual B-subalgebra of X^D. Then S is open in X^D.*

Proof. Suppose S is a closed dual B-subalgebra of X^D. Assume on the contrary that S is not an open set in X^D. By Corollary 4.8, 1 is not an interior point of S. This implies that for all $U(1)$, $U(1) \nsubseteq S$. Let S_1 be open with property defined in Lemma 4.9. Then $S_1 \nsubseteq S$. Hence, $(X \backslash S) \bigcap S_1 \neq \varnothing$ and so there exists $z \in (X \backslash S) \bigcap S_1$. Hence, $(X \backslash S) \bigcap S_1$ is an open set containing z. By Lemma 4.9, $S_1 \subseteq (X \backslash S) \bigcap S_1$ which is a contradiction. □

Definition 4.11. Let X^D be a dual B-algebra. A subset F of X^D is called a dual B-filter if it satisfies the following axioms: For all x, y in X

(dF1) $1 \in F$;

(dF2) $(x \circ y) \in F$ and $x \in F$ imply $y \in F$.

Example 4.12. Suppose X^D is a dual B-algebra. Then X^D and $\{1\}$ are dual B-filters of X^D called the trivial dual B-filters of X^D.

Example 4.13. Consider the dual B-algebra $X^D = \{1, a, b, c, d, e\}$ from Example 3.2. The sets $\{1\}$, $F_1 = \{1, c\}$, $F_2 = \{1, d\}$, $F_3 = \{1, e\}$, and $F_4 = \{1, a, b\}$ are dual B-filters of X^D while $A = \{1, a, e\}$ is not a dual B-filter since $e \circ c = a \in A$ where $e \in A$ but $c \notin A$.

Proposition 4.14. *If F is a dual B-filter of a dual B-algebra X^D. Then F is a dual B-subalgebra of X^D.*

Proof. Suppose F is a dual B-filter of X^D and let $x, y \in F$. Since $1 \in F$ and F is a dual B-filter, $1 \circ (x \circ y)$ implies that $x \circ y \in F$. Therefore, F is a dual B-subalgebra. □

Remark 4.15. Not every open subset of a tdB-algebra X^D is a dual B-filter of X^D. This is illustrated in the next example.

Example 4.16. Consider the tdB-algebra from Example 3.6. Note that $\{b, c\} \in \tau$ but $\{b, c\}$ is not a dual B-filter since $1 \notin \{b, c\}$.

From this observation, we have the next theorem as a characterization of when an open set is a dual B-filter in a tdB-algebra provided that 1 is an element of every nonempty open set in the dual b-topology.

Theorem 4.17. *Let $(X^D, \circ, \tau)$ be a tdB-algebra and F an open subset of X^D. If $1 \in \bigcap_{U \in \tau, U \neq \varnothing} U$, then F is a dual B-filter of X^D.*

Proof. Suppose $x \circ y \in F$ and $x \in F$ for any $x, y \in X^D$. Since F is open, there exists $U(x), U(y)$ such that $U(x) \circ U(y) \subseteq F$ by Theorem 3.5. By (DB2), $y = 1 \circ y \in U(x) \circ U(y) \subseteq F$. Therefore, F is a dual B-filter of X^D. □

In Example 4.6, it is illustrated that a dual B-subalgebra may not be a close nor an open set in a dual B-topological space. However if a dual B-subalgebra S is a dual B-filter in a tdB-algebra X^D that is open, it is also a closed dual B-filter in X^D. This is formally stated in the next theorem.

Theorem 4.18. *Suppose $(X^D, \circ, \tau)$ is a tdB-algebra and F a dual B-filter of X^D. If F is open in X^D, then F is closed in X^D.*

Proof. Suppose F is a dual B-filter such that F is open in X^D. Let $x \in X^D \backslash F$. Since F is a dual B-filter of X^D and by (DB1), $x \circ x = 1 \in F$. By Theorem 3.5, there exists $U(x)$ such that $U(x) \circ U(x) \subseteq F$. We will show that $U(x) \subseteq X^D \backslash F$. Assume on the contrary that $U(x) \nsubseteq X^D \backslash F$. Then $U(x) \cap F \neq \varnothing$. Consequently, there exists $y \in U(x) \cap F$. Note that for all $z \in U(x)$, $y \circ z \in U(x) \circ U(x) \subseteq F$. Since F is a dual B-filter, it follows that $z \in F$. Hence, $U(x) \subseteq F$. This implies that $x \in F$, a contradiction. Thus, $X^D \backslash F$ is open. Therefore, F is closed in X^D. □

The next coroallry follows from Proposition 4.14, Theorem 4.10 and Theorem 4.18.

Corollary 4.19. *Let $(X^D, \circ, \tau)$ be a tdB-algebra such that $1 \in \bigcap_{U \in \tau, U \neq \varnothing} U$. Then a dual B-filter F is a closed subset of X^D if and only if F is an open subset of X^D.*

Definition 4.20. Let X^D be a dual B-algebra and N a nonempty subset of X^D. Then N is a *normal subset* of X^D if for any $x \circ y, a \circ b \in N$, $(a \circ x) \circ (b \circ y) \in N$.

Proposition 4.21. *Let X^D be a dual B-algebra and N_1, N_2 are nonempty subsets of X^D. Then the following holds:*

(i) *If N_1 and N_2 are dual B-subalgebras of X^D, then $N_1 \cap N_2$ is also a dual B-subalgebra.*

(ii) *If* N_1 *and* N_2 *are normal subsets of* X^D*, then* $N_1 \cap N_2$ *is also a normal subset of* X^D*.*

Proof. Suppose X^D is a dual B-algebra, N_1, N_2 are nonempty subsets of X^D, and $x, y, a, b \in X^D$.

(i) Let N_1 and N_2 be dual B-subalgebras. Then for any $x, y \in N_1 \cap N_2$, $x, y \in N_1$ and $x, y \in N_2$ implies that $x \circ y \in N_1, N_2$. Hence, $x \circ y \in N_1 \cap N_2$. It follows that $N_1 \cap N_2$ is a dual B-subalgebra.

(ii) Let N_1 and N_2 be normal subsets of X^D. Suppose $x \circ y, a \circ b \in N_1 \cap N_2$. Then $x \circ y, a \circ b \in N_1$ and $x \circ y, a \circ b \in N_2$. This implies that $(b \circ y) \circ (a \circ x) \in N_1, N_2$. Hence, $(a \circ x) \circ (b \circ y) \in N_1 \cap N_2$. It follows that $N_1 \cap N_2$ is a normal subset of X^D. □

Remark 4.22. The set $\{1\} \subseteq X^D$ is a normal dual B-subalgebra of X^D by Lemma 2.4 (i) and (DB1).

Example 4.23. Consider the dual B-subalgebra $\{1, a, b\}$ and $\{1, c\}$ from Example 4.4. Then $\{1, a, b\}$ is a normal dual B-subalgebra while $\{1, c\}$ is not normal since $a \circ e = c \circ 1 = c \in \{1, c\}$ but $(c \circ a) \circ (1 \circ e) = d \circ e = a \notin \{1, c\}$. Similarly, $\{1, d\}$ and $\{1, e\}$ are not normal dual B-subalgebras.

Suppose $(X^D, \circ, 1)$ is a dual B-algebra and S a normal dual B-subalgebra of X^D. Let "$\cong^S$" be a relation defined by $x \cong^S y$ if and only if $x \circ y, y \circ x \in S$.

Theorem 4.24. *Let* $(X^D, \circ, 1)$ *be a dual* B*-algebra and* S *a normal dual* B*-subalgebra of* X^D*. The relation defined by* $x \cong^S y$ *if and only if* $x \circ y, y \circ x \in S$ *is a congruence relation on* X^D *for any* $x, y \in X^D$*.*

Proof. Since $x \circ x = 1 \in S$, it follows that $x \cong^S x$ and thus, reflexive. Suppose $x \cong^S y$. Then

$$x \circ y, y \circ x \in S.$$

Hence, $y \cong^S x$ implying that the relation is symmetric. Supose $x \cong^S y$ and $y \cong^S z$. Then

$$x \circ y, y \circ x, y \circ z, z \circ y \in S.$$

By Theorem 2.3 and since S is a dual B-subalgebra, $x \circ z = (y \circ x) \circ (y \circ z) \in S$ and $z \circ x = (y \circ z) \circ (y \circ x) \in S$. Consequently, $x \cong^S z$ implying that the relation is transitive. Hence, the relation is an equivalence relation on X^D. Suppose $x \cong^S y$ and $a \cong^S b$. Then $x \circ y, y \circ x, a \circ b, b \circ a \in S$. Since S is normal, this implies that $(x \circ a) \circ (y \circ b), (y \circ b) \circ (x \circ a) \in S$ that is, $x \circ a \cong^S y \circ b$. Therefore, the relation "$\cong^S$" is a congruence relation on X^D. □

Suppose X^D is a dual B-algebra and S a normal dual B-subalgebra of X^D. By Theorem 4.24, we may now define the set $S_x = \{y | y \cong^S x\}$ to denote the equivalence class of x for any $x \in X^D$ and let $X^D/S = \{S_x | x \in X^D\}$ to be the set of all equivalence classes of x for any $x \in X^D$.

Remark 4.25. For any $S_x \in X^D$, $S_1 \circ S_x = S_{1 \circ x} = S_x$ by (DB2) *where* $S_1 \in X^D/S$.

Theorem 4.26. *Suppose* X^D *is a dual* B*-algebra and* S *a normal dual* B*-subalgebra. Let* X^D/S *to be the set as defined above. Then* X^D/S *is a dual* B*-algebra with binary operation given by* $S_x \circ S_y = S_{x \circ y}$ *for any* $x, y \in X^D$ *and with identity element* S_1*.*

Proof. Note that there exists $S_1 \in X^D/S$ such that Remark 4.25 holds. That is, S_1 is the identity element of X^D/S. For any $S_x, S_y \in X^D/S$, $S_x \circ S_y = S_{x \circ y} \in X^D/S$ since $x \circ y \in X^D$. Thus, X^D/S is a dual B-algebra by Theorem 4.3 □

Theorem 4.27. *Let* $\mathcal{S}$ *be a family of normal dual* B*-subalgebras closed under finite intersections in a dual* B*-algebra* X^D*. Then there is a topology* $\tau = \{U \subseteq X^D | \forall x \in U, \exists S \in \mathcal{S}$ *such that* $S_x \subseteq U\}$ *such that* $(X^D, \circ, \tau)$ *is a* tdB*-algebra.*

Proof. Note that for all $x \in X^D$, there exists $S \in \mathcal{S}$ such that $S_x \subseteq X^D$. This implies that $X^D \in \tau$. Vacuosuly, $\varnothing \in \tau$. Let $y \in U_1 \cap U_2$, where $U_1, U_2 \in \tau$. Then $y \in U_1$ and $y \in U_2$ which imply that there exists $S_1, S_2 \in \mathcal{S}$ such that $S_{1_y} \subseteq U_1$ and $S_{2_y} \subseteq U_2$. Let $S = S_1 \cap S_2$. By Proposition 4.21, S is a normal dual B-subalgebra of X^D. Moreover, $S \in \mathcal{S}$ by the hypothesis. We will show that $S_y \subseteq S_{1_y}$ and $S_y \subseteq S_{2_y}$. Suppose $x \in S_y$. Then $y \cong^S x$ which implies that $y \circ x \in S \subseteq S_1$ that is, $y \circ x \in S_1$. Hence, $x \in S_{1_y}$. It follows that $S_y \subseteq S_{1_y}$. Similarly, $S_y \subseteq S_{2_y}$. This implies that $S_y \subseteq U_1$ and $S_y \subseteq U_2$ or $S_y \subseteq U_1 \cap U_2$. Consequently, $U_1 \cap U_2 \in \tau$. Let $y \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ where $U_\alpha \in \tau$ for all $\alpha \in \mathcal{A}$. Then $y \in U_\beta$ for some $\beta \in \mathcal{A}$. This implies that there exists $S_\beta \in \mathcal{S}$ such that $S_\beta \subseteq U_\beta \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Hence, $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \tau$. Thus, τ is a dual B-topology. We will show that for any $S \in \mathcal{S}$ and $x \in X^D$, $S_x \in \tau$. Let $y \in S_x$. Then $y \cong^S x$. Now suppose $z \in S_y$. Then $z \cong^S y$. By transitivity, $z \cong^S x$ or $z \in S_x$ that is, $S_y \subseteq S_x$. This implies that $S_x \in \tau$. Lastly, suppose $x, y \in X^D$ and $U \in \tau$ such that $x \circ y \in U$. Then there exists $S \in \mathcal{S}$ such that $S_{x \circ y} \subseteq U$. Note that S_x and S_y are open sets containing x and y, respectively and $S_x \circ S_y = S_{x \circ y}$. Hence, there exists neighborhoods S_x and S_y such that $S_x \circ S_y \subseteq U$. Therefore, $(X^D, \circ, \tau)$ is a tdB-algebra. □

5. The uniform dual B-topology

Throughout this section, all dual B-filters of a dual B-algebra X^D are normal dual B-filters of X^D. The following definitions are parallel to that of [4, pp. 340–341].

Suppose X^D is a dual B-algebra and $U, V \subseteq X^D \times X^D$, consider the following notations:

(i) $U^{-1} = \{(y, x) | (x, y) \in U\}$;
(ii) $U * V = \{(x, z) | (x, y) \in V \text{ and } (y, z) \in U, \exists y \in X\}$;
(iii) $U[x] = \{y | (x, y) \in U\}$;
(iv) $\Delta = \{(x, x) | x \in X^D\}$.

Suppose Ω is an arbitrary family of dual B-filters F in a dual B-algebra X^D and $A \subseteq X^D$. Let us define the following notations:

(i) $U_F = \{(x, y) \in X \times X | x \cong^F y\}$;
(ii) $\mathcal{K}^* = \{U_F : F \in \Omega\}$;
(iii) $\mathcal{K} = \{U \subseteq X \times X | U_F \subseteq U, \exists U_F \in \mathcal{K}^*\}$;
(iv) $U_F[A] = \bigcup_{a \in A} U_F[a]$.

Remark 5.1. $\mathcal{K}^* \subseteq \mathcal{K}$.

Definition 5.2. By a uniformity on a dual B-algebra X^D, we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X^D \times X^D$ which satisfies the following conditions for any $U, V \in \mathcal{K}$:

(i) $\Delta \subseteq U$;
(ii) $U^{-1} \in \mathcal{K}$;
(iii) There exists $W \in \mathcal{K}$ such that $W \circ W \subseteq U$;
(iv) $U \cap V \in \mathcal{K}$;
(v) If $U \subseteq W \subseteq X^D \times X^D$, then $W \in \mathcal{K}$.

The pair $(X^D, \mathcal{K})$ is called a uniform dual B-structure.

Example 5.3. Consider the dual B-algebra $X^D = \{1, a, b, c, d, e\}$ from Example 3.2. Then the dual B-filters of X^D are that from Example 4.13 including the trivial dual B-filter $\{1\}$. Note that by Proposition 4.14 and Remark 4.22, $\{1\}$ is a normal dual B-filter. Then $\{X^D, \{1\}, F\}$ is a set containing all normal dual B-filters of X^D where $F = \{1, a, b\}$. By routine calculations, $(X^D, \mathcal{K})$ is a uniform dual B-structure where $\mathcal{K}^* = \{U_{X^D}, U_{\{1\}}, U_F\}$, $U_{X^D}[1] = U_{X^D}[a] = U_{X^D}[b] = U_{X^D}[c] = U_{X^D}[d] = U_{X^D}[e] = U_{\{1\}}[1] = \{1, a, b, c, d, e\} = X^D$, $U_{\{1\}}[1] = \{1\}, U_{\{1\}}[a] = \{a\}, U_{\{1\}}[b] = \{b\}, U_{\{1\}}[c] = \{c\}, U_{\{1\}}[d] = \{d\}, U_{\{1\}}[e] = \{e\}$, $U_F[1] = U_F[a] = U_F[b] = \{1, a, b\}, U_F[c] = U_F[d], = U_F[e] = \{c, d, e\}$.

Remark 5.4. $(X^D, \mathcal{K}^*)$ is not a uniform dual B-structure as shown in the next example.

Example 5.5. Consider the dual B-algebra $X^D = \{1, a, b, c, d, e\}$ from Example 3.2 and the dual B-filter $F = \{1, a, b\}$ in Example 4.13. As mentioned in Example 5.3, $\{X^D, \{1\}, F\}$ is a set containing all normal dual B-filters of X^D. Hence, $\mathcal{K}^* = \{U_{X^D}, U_{\{1\}}, U_F\}$ where

$$\begin{aligned}
U_{X^D} &= \{(x, y) \in X^D \times X^D | x \circ y, y \circ x \in X^D\} = X^D \times X^D,\\
U_{\{1\}} &= \{(x, y) \in X^D \times X^D | x \circ y, y \circ x \in \{1\}\} = \{(x, x) \in X^D \times X^D\},\\
U_F &= \{(1, 1), (a, a), (b, b), (c, c), (d, d), (e, e), (1, b), (b, 1),\\
&\quad (a, 1), (1, a), (b, a), (a, b), (c, e), (e, c), (d, c), (c, d), (e, d), (d, e)\}.
\end{aligned}$$

Let $M = F \cup \{1, e\} = \{1, a, b, e\}$. Then $U_M = U_F \cup \{(1, e), (e, 1), (a, d), (d, a), (b, c), (c, b)\}$. Note that

$$U_F \subseteq U_M \subseteq X^D \times X^D.$$

Moreover, $M \notin \Omega$ since $d \circ a = e, a \in M$ but $d \notin M$. Hence, $U_M \notin \mathcal{K}^*$. This implies that $\mathcal{K}^*$ does not satisfy condition (v) of Definition 5.2.

In view of Remark 5.1, the next theorem states that the pair $(X^D, \mathcal{K})$ is a uniform dual B-structure. Moreover, the pair $(X^D, \mathcal{K})$ from Example 5.5 is a uniform dual B-structure.

Theorem 5.6. *Let Ω be an arbitrary family of dual B-filters closed under finite intersections in a dual B-algebra X^D. Then $(X^D, \mathcal{K})$ is a uniform dual B-structure.*

Proof. Suppose Ω is an arbitrary family of dual B-filters in a dual B-algebra X^D and $U, V \in \mathcal{K}$. Then there exist $U_F, U_J \in \mathcal{K}^*$ such that $U_F \subseteq U$ and $U_J \subseteq V$, respectively.

(i) Let $(x, x) \in \Delta$. Since $x \cong^F x$, it follows that $(x, x) \in U_F$. Hence, $(x, x) \in U$ so that $\Delta \subseteq U$.

(ii) Let $(x, y) \in U_F$. Then $x \cong^F y$ and $y \cong^F x$. This implies that $(y, x) \in U_F$. Hence, $(y, x) \in U$. It follows that $(x, y) \in U^{-1}$ with $U_F \subseteq U^{-1}$ so that $U^{-1} \in \mathcal{K}$.

(iii) Consider $U_F \in \mathcal{K}$ and $(x, z) \in U_F \circ U_F$. Then there exists $y \in X$ such that (x, y), $(y, z) \in U_F$. This implies that $x \cong^F y$ and $y \cong^F z$. Hence, $x \cong^F z$. It follows that $(x, z) \in U_F \subseteq U$ so that $U_F \circ U_F \subseteq U$. (iv) Let $U_F, U_J \in \mathcal{K}^*$.

Claim: $U_F \cap U_J = U_{F\cap J} \in \mathcal{K}^*$.

Let $(x, y) \in U_F \cap U_J$. Then $x \cong^F y$ and $x \cong^J y$. These imply that $x \circ y, y \circ x \in F, J$. Hence, $x \circ y, y \circ x \in F \cap J$ implying that $x \cong^{F\cap J} y$ and so $(x, y) \in U_{F\cap J}$. Therefore, $U_F \cap U_J \subseteq U_{F\cap J}$. The converse is similar. This proves the claim.

Let $(x, y) \in U_{F\cap J} = U_F \cap U_J$. Then $x \cong^F y$ and $x \cong^J y$. Thus, $(x, y) \in U_F$ and $(x, y) \in U_J$. Hence, $(x, y) \in U$ and $(x, y) \in V$. Therefore, $(x, y) \in U \cap V$ so that $U_{F\cap J} \subseteq U \cap V$. Consequently, $U \cap V \in \mathcal{K}$. It remains to show that $\mathcal{K}$ satisfies condition (v). Let $U \in \mathcal{K}$ such that $U \subseteq V \subseteq X^D \times X^D$. Then there exists $U_F \in \mathcal{K}^*$ such that $U_F \subseteq U \subseteq V$. Hence, $V \in \mathcal{K}$. Therefore, $\mathcal{K}$ satisfies condition (v) and $(X^D, \mathcal{K})$ is a uniform dual B-structure. □

Definition 5.7. Let $(X^D, \mathcal{K})$ be a uniform dual B-structure. If τ is a dual B-topology on X^D, then τ is called a *uniform dual B-topology* and the pair (X^D, τ) is called a uniform dual B-topological space.

Example 5.8. Consider the uniform dual B-structure $(X^D, \mathcal{K})$ in Example 5.5. Then the family

$$\tau = \{X^D, \varnothing, \{1, a, b\}, \{c, d, e\}\},$$

is a uniform dual B-topology on X^D. Thus, (X^D, τ) is a uniform dual B-topological space.

Theorem 5.9. *Suppose $(X^D, \mathcal{K})$ is a uniform dual B-structure. Then $\tau = \{G \subseteq X | \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$ is a uniform dual B-topology on X^D.*

Proof. Suppose $(X^D, \mathcal{K})$ is a uniform dual B-structure. Note that for all $x \in X^D$ and $U \in \mathcal{K}$, $U[x] \subseteq X^D$. Hence, $X^D \in \tau$. Vacuously, $\varnothing \in \tau$. Let $x \in \bigcup_{G_i \in \tau, i \in \mathcal{A}} G_i$. Then there exists $j \in \mathcal{A}$ such that $x \in G_j$. Since $G_j \in \tau$, there exists $U_j \in \mathcal{K}$ such that $U_j[x] \subseteq G_j$. This implies that $U_j[x] \subseteq \bigcup_{G_i \in \tau, i \in \mathcal{A}} G_i$. Hence, $\bigcup_{G_i \in \tau, i \in \mathcal{A}} G_i \in \tau$. Suppose $G, H \in \tau$ such that $x \in G \cap H$. Then there exist $U, V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W = U \cap V$. By Definition 5.2 (iv), $W \in \mathcal{K}$.

Claim: $W[x] \subseteq U[x] \cap V[x]$.

Let $y \in W[x]$. Then $(x, y) \in U$ and $(x, y) \in V$. This implies that $y \in U[x]$ and $y \in V[x]$. Hence, $W[x] \subseteq U[x] \cap V[x]$. This proves the claim. By the claim, $W[x] \subseteq U[x] \subseteq G$ and $W[x] \subseteq V[x] \subseteq H$. Hence, $W[x] \subseteq G \cap H$. This implies that $G \cap H \in \tau$. Therefore, τ is a dual B-topology on X^D. □

The next remark follows from Definition 5.7 and Theorem 5.9.

Remark 5.10. Suppose X^D is a dual B-topological space.

(i) Then (X^D, τ) in Theorem 5.9 is a uniform dual B-topological space.

(ii) For any $U_F \in \mathcal{K}^*$ and $x \in X^D$, $x \in U_F[x]$ and $U_F[x] \in \tau$, that is, $U_F[x]$ is a neighborhood of x.

Lemma 5.11. *Suppose X^D is a dual B-algebra such that $U \subseteq V$ for any $U, V \in \mathcal{K}$. Then $U[x] \subseteq V[x]$ for all $x \in X^D$.*

Proof. Let $U \subseteq V$ for any $U, V \in \mathcal{K}$ and $x \in X$. Suppose $a \in U[x]$. Then $(x, a) \in U \subseteq V$. This implies that $(x, a) \in V$. Therefore, $a \in V[x]$. □

Theorem 5.12. *Suppose X^D is a uniform dual B-structure. Then X^D is a* tdB*-algebra.*

Proof. Let $(X^D, \mathcal{K})$ be a uniform dual B-structure. By Theorem 5.9 and Remark 5.10 (i), there is a uniform dual B-topology $\tau = \{G \subseteq X | \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$. Suppose $x \circ y \in U(x \circ y)$ where $x, y \in X^D$. By Theorem 5.9, there exists $W \in \mathcal{K}$ such that $W[x \circ y] \subseteq U(x \circ y)$. Then there exists $W_F \in \mathcal{K}^*$ such that $W_F \subseteq W$ for some dual B-filter F of X^D. By Lemma 5.11, $W_F[x \circ y] \subseteq W[x \circ y]$. Note that $W_F[x]$ and $W_F[y]$ are open neighborhoods of x and y, respectively by Remark 5.10 (ii).

Claim: $W_F[x] \circ W_F[y] \subseteq W_F[x \circ y]$.

Suppose $a \circ b \in W_F[x] \circ W_F[y]$. Then $(x, a), (y, b) \in W_F$. This implies that $x \cong^F a$ and $y \cong^F b$. Hence, $x \circ y \cong^F a \circ b$. It follows that $a \circ b \in W_F[x \circ y]$. This proves the claim.
Hence, $W_F[x] \circ W_F[y] \subseteq U(x \circ y)$. By Theorem 3.5, X^D is a tdB-algebra. □

The converse of Theorem 5.12 follows directly from Definition 5.7 provided that the dual B-topology is a uniform dual B-topology. This is formally stated in the next corollary.

Corollary 5.13. *Suppose X^D is a* tdB*-algebra. If τ is a uniform dual B-topology, then X^D is a uniform dual B-topological space.*

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References

[1] K. E. Belleza, J. P. Vilela, *The Dual B-Algebra*, Eur. J. Pure Appl. Math., **12** (2019), 1497–1507. 1, 2.1, 2.2, 2.3, 2.4

[2] J. Dugunji, *Topology*, Allyn and Bacon Inc., Atlantic Avenue, Boston (1966). 2.5, 2.6, 2.7

[3] N. C. Gonzaga, *Analyzing Some Structural Properties of Topological B-Algebras*, Int. J. Math. Math. Sci., **2019** (2019), 7 pages. 1

[4] K. D. Joshi, *Introduction to General Topology*, John Wiley & Sons, Inc., New York, (1983). 5

[5] Y. B. Jun, X. L. Xin, D. S. Lee, *On Topological BCI-Algebras*, Inform. Sci., **116** (1999), 253–261. 1

[6] S. Mehrshad, J. Golzarpoor, *On Topological BE-Algebras*, Math. Morav., **21** (2017), 1–13. 1