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Interpolating sesqui harmonic slant curve in S-space form

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Abstract

In this paper, we study interpolating sesqui harmonic slant curve in S-space form and thus generalizing the results of the papers [D. Fetcu, J. Korean Math. Soc., **45** (2008), 393–404], [C. Özgür, S. Güvenc, Turkish J. Math., **38** (2014), 454–461], [F. Karaca, C. Özgür, U. C. De, Int. J. Geom. Methods Mod. Phys., **17** (2020), 13 pages]. Finally we give examples in support of our results.

Keywords: Interpolating sesqui harmonic map, slant curve, S-space form.

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1. Introduction

A map $\tilde{\varphi}$ between two Riemannian manifolds (M, g_1) and (N, g_2) is called harmonic if the divergence of its differential vanishes. The harmonic map equation is given by

$$\tau(\tilde{\varphi}) = \operatorname{trace}(\nabla d\tilde{\varphi}) = 0. \tag{1.1}$$

Eells and Sampson gave the natural generalization of the harmonic map as biharmonic map which is critical point of bienergy functional [5]

$$E_2(\tilde{\varphi}) = \frac{1}{2}\int_M |\tau(\tilde{\varphi})|^2 dv_g.$$

The Euler-Lagrange equation for biharmonic maps is defined by Jiang [10]

$$\tau_2(\tilde{\varphi}) = \operatorname{trace}(\nabla^N\nabla^N - \nabla^N_{\nabla})(\tau(\tilde{\varphi})) - \operatorname{trace}(R^N(d\tilde{\varphi}, \tau(\phi))d\tilde{\varphi}) = 0,$$

where $\tau_2(\tilde{\varphi})$ is called bitension of $\tilde{\varphi}$.

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As a generalization of biharmonic map, Branding defined interpolating sesqui-harmonic map as a critical point of $E_{\delta_1,\delta_2}(\tilde{\varphi})$ [1]

$$E_{\delta_1,\delta_2}(\tilde{\varphi}) = \delta_1 \int_M |d(\tilde{\varphi})|^2 dv_g + \delta_2 \int_M |\tau(\tilde{\varphi})|^2 dv_g,$$

where $\delta_1, \delta_2 \in R$. In string theory of Physics the above functional is used and known as bosonic string with extrinsic curvature term [15]. The equation for interpolating sesqui harmonic map is given by

$$\tau_{\delta_1,\delta_2}(\tilde{\varphi}) = \delta_2 \tau_2(\tilde{\varphi}) - \delta_1 \tau(\tilde{\varphi}) = 0.$$

In [1], Branding studied interpolating sesqui-harmonic curves in 3-dimensional sphere. Cho et al. classified biharmonic curves in 3-dimensional Sasakian space form and as a generalization of Legendre curve the notion of slant curve in Sasakian 3-manifolds is defined by [4] and [3], respectively. Calin and Crasmareanu studied slant curve in 3-dimensional normal almost contact manifolds [2]. Güvenc and Özgür studied slant curves in S-manifolds [9]. Biharmonic Legendre curve in Sasakian space form has been studied by Fetcu and Oniciuc [7]. In 2014, Özgür and Güvenc generalized their results in S-space form [13] and generalized Sasakian space form [14]. In [12] Luo and Ou studied some properties of Bi-f-harmonic and f-biharmonic maps. Further Güvenc Özgür [8] characterizes the f-biharmonic Legendre curves in Sasakian space form. Recently, Karaca et al. [11] consider interpolating sesqui harmonic Legendre curves in Sasakian space form which generalized some results of [7].

It is noted that interpolating sesqui-harmonic slant curve is

(1) Interpolating sesqui harmonic Legendre curve in Sasakian space form if $s = 1$ and $\theta = \frac{\pi}{2}$;

(2) Biharmonic Legendre curve in S-space form if $\theta = \frac{\pi}{2}$ and $\delta_2 = 1, \delta_1 = 0$;

(3) Biharmonic Legendre curve in Sasakian-space form if $\theta = \frac{\pi}{2}$ and $\delta_2 = 1, \delta_1 = 0$, $s = 1$.

In this paper we discuss interpolating sesqui harmonic slant curve in S-space form and thus generalizing the results of the papers [6, 11, 13]. In the last section we give examples in support of our results.

2. Preliminaries

Let $(\overline{\mathcal{M}}^{(2n+s)}, g)$ be a $(2n+s)$-dimensional Riemannian manifolds. $\overline{\mathcal{M}}^{(2n+s)}$ is called S-manifold if there exist a ϕ-structure (where rank ϕ=2n) and structure vector fields $\xi_1 \cdots \xi_s$ and their dual forms $\eta_1 \cdots \eta_s$ such that

$$\phi\xi_\alpha = 0, \eta_\alpha \circ \phi = 0, \phi^2 = -I + \sum_\alpha \xi_\alpha \otimes \eta_\alpha,$$

$$g(X,Y) = g(\phi X, \phi Y) + \sum_\alpha \eta_\alpha(X)\eta_\alpha(Y), \tag{2.1}$$

$$\eta_\alpha(X) = g(X,\xi), \qquad d\eta_\alpha(X,Y) = g(X,\phi Y). \tag{2.2}$$

The Riemannian connection $\overline{\nabla}$ of g on an S-manifold $\overline{\mathcal{M}}^{(2n+s)}$ satisfies

$$(\overline{\nabla}_X \phi)Y = \sum_{\alpha=1}^{s} \{g(\phi X, \phi Y)\xi_\alpha + \eta_\alpha(Y)\phi^2 X\},$$

and

$$\overline{\nabla}_X \xi_\alpha = -\phi X,$$

for any $X, Y \in T\overline{\mathcal{M}}$ and any $\alpha = 1, \cdots, s$.

The sectional curvature of two planes spanned by X and ϕX, where X is a unit orthogonal to $\xi_1 \cdots \xi_s$ called ϕ-sectional curvature. An S-manifold of constant ϕ-sectional curvature c is called an S-space form denoted by $\overline{\mathcal{M}}(c)$. Then curvature tensor field of S-space form $\overline{\mathcal{M}}(c)$ is given by [13, 16]

$$\begin{aligned} R^{\overline{\mathcal{M}}}(X,Y)Z = \sum_{\alpha,\beta} &\{\eta_\alpha(X)\eta_\beta(Y)\phi^2 Y - \eta_\alpha(Y)\eta_\beta(Z)\phi^2 X \\ &- g(\phi X,\phi Z)\eta_\alpha(Y)\xi_\beta + g(\phi Y,\phi Z)\eta_\alpha(X)\xi_\beta\} \\ &+ \frac{(c+3s)}{4}\{-g(\phi Y, fZ)\phi^2 X + g(\phi X,\phi Z)\phi^2 Y\} \\ &+ \frac{(c-s)}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)Z\}, \end{aligned} \tag{2.3}$$

for all $X, Y, Z \in T\overline{\mathcal{M}}$. If $s = 1$, then $\overline{\mathcal{M}}$ is known as Sasakian space form.

Definition 2.1 ([13]). If $\tilde{\varphi}$ is a unit speed curve in an S-manifold then it is called slant curve if there exists a constant angle θ called the contact angle of $\tilde{\varphi}$ such that $\eta_\alpha(X) = \cos(\theta)$, for all $\alpha = \{1, \cdots, s\}$. For $\theta = \frac{\pi}{2}$ slant curve becomes Legendre curve.

Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a unit speed curve in an n-dimensional Riemannian manifold $(\overline{\mathcal{M}}, g)$. If $\{E_1, E_2, \cdots, E_r\}$ is a set of orthonormal vectors then the curve $\tilde{\varphi}$ is called Frenet curve of osculating order $r, 1 \leqslant r \leqslant n$ such that [13]

$$\begin{cases} T = E_1 = \tilde{\varphi}', \\ \nabla_T E_1 = k_1 E_2, \\ \nabla_T E_i = -k_{i-1}E_{i-1} + k_i E_{i+1}, \quad \text{for } 2 \leqslant i \leqslant n-1, \\ \nabla_T E_r = -k_{r-1}E_{r-1}, \end{cases} \tag{2.4}$$

where $k_i, 1 \leqslant i \leqslant r-1$ are curvature functions of $\tilde{\varphi}$.

(1) A Frenet curve of osculating order $r = 1$ is a geodesic.

(2) A Frenet curve of osculating order $r = 2$ with k_1 non zero positive constant is a circle.

(3) A Frenet curve of osculating order $r \geqslant 3$ with $k_1 \cdots k_{r-1}$ non zero positive constant is a helix of order r. A helix of order 3 is simply called helix [13].

3. Interpolating sesqui-harmonic slant curves in S-space form

A curve $\tilde{\varphi}$ is called Interpolating sesqui harmonic if and only if the following equation satisfied [1]:

$$\tau_{\delta_1,\delta_2}(\tilde{\varphi}) \equiv \delta_2(\nabla_T\nabla_T\nabla_T T) - \delta_2 R^{\overline{\mathcal{M}}}(T, \nabla_T T)T - \delta_1 \nabla_T T = 0, \tag{3.1}$$

where δ_1, $\delta_2 \in R$.

Now for Interpolating sesqui harmonic slant curve in S-space form we may state the following theorem.

Theorem 3.1. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form $\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha = \{1 \cdots s\}$ and $p = \min\{r, 4\}$. Then $\tilde{\varphi}$ is interpolating sesqui harmonic if and only if there exists δ_1, δ_2 such that*

1. $c = s$ *or* $\phi T \perp E_2$ *or* $\phi T \in \{E_2, \cdots, E_n\}$;
2. *first* p *of the following equations are satisfied*

$$\begin{cases} \delta_2 k_1 k_1' = 0, \\ \delta_2[k_1'' - k_1^3 - k_1 k_2^2 + s^2 k_1 \cos^2(\theta) + k_1(\frac{c+3s}{4})(1 - s\cos^2(\theta)) + 3k_1 \frac{(c-s)}{4} g(\phi T, E_2)^2] = \delta_1 k_1, \\ \delta_2[2k_1' k_2 + k_1 k_2' + 3\frac{(c-s)}{4} k_1 g(\phi T, E_2) g(\phi T, E_3)] = 0, \\ \delta_2[k_1 k_2 k_3 + 3\frac{(c-s)}{4} k_1 g(\phi T, E_2) g(\phi T, E_4)] = 0. \end{cases} \tag{3.2}$$

Proof. Making use of (1.1) and (2.4), we get

$$\nabla_T E_1 = k_1 E_2 = \tau(\tilde{\varphi}), \tag{3.3}$$

which gives

$$\nabla_T \nabla_T T = -k_1^2 E_1 + k_1' E_2 + k_1 k_2 E_3,$$

and

$$\begin{aligned} \nabla_T \nabla_T \nabla_T T = &-3k_1 k_1' E_1 + (k_1'' - k_1^3 - k_1 k_2^2) E_2 + (2k_1' k_2 \\ &+ k_1 k_2') E_3 + (k_1 k_2 k_3) E_4. \end{aligned}$$

Moreover by virtue of (2.3) it yields

$$\begin{aligned} R(T, \nabla_T T)T = &-s^2 \cos^2(\theta) k_1 E_2 + \frac{(c+3s)}{4} s(\cos^2(\theta) - 1) k_1 E_2 \\ &+ \frac{(c-s)}{4}(-3k_1 g(\phi T, E_2)\phi T. \end{aligned} \tag{3.4}$$

Thus it follows from (3.3), (3.4) and (3.1) that

$$\begin{aligned} \tau_{\delta_1,\delta_2}(\tilde{\varphi}) = &-3\delta_2 k_1 k_1' E_1 + [\delta_2(k_1'' - k_1^3 - k_1 k_2^2) + s^2 \cos^2(\theta) k_1 \\ &+ k_1 \frac{(c+3s)}{4}(1 - s\cos^2(\theta)) - \delta_1 k_1] E_2 + \delta_2(2k_1' k_2 + k_1 k_2') E_3 \\ &+ (\delta_2 k_1 k_2 k_3) E_4 + 3\frac{(c-s)}{4} k_1 g(\phi T, E_2)\phi T, \end{aligned}$$

and by taking the inner product with E_1, E_2, E_3 and E_4 we get the desired result. □

Next, we discuss four different cases to investigate and simplify the result of Theorem 3.1. In each case we take $\frac{\delta_1}{\delta_2} \neq 0$.

Case 1: c = s.

Proposition 3.2. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha = \{1 \cdots s\}$ *such that $c = s$ and $p = \min\{r, 4\}$. Then $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if*

$$\begin{cases} k_1 = \text{constant} > 0, \\ k_1^2 + k_2^2 = s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}, \\ k_2 = \text{constant}, \quad k_2 k_3 = 0. \end{cases} \tag{3.5}$$

Proof. For $c = s$ and making use of (3.2) we find

$$\begin{cases} k_1 k_1' = 0, \\ (k_1'' - k_1^3 - k_1 k_2^2) + s^2 \cos^2(\theta) k_1 + k_1(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2} k_1 = 0, \\ 2k_1' k_2 + k_1 k_2' = 0, \\ k_1 k_2 k_3 = 0. \end{cases} \tag{3.6}$$

By using $k_1 = \text{constant} > 0$ in last three equations of (3.6) we get the result. □

Now using proposition (3.2) we have the following theorem.

Theorem 3.3. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha = \{1 \cdots s\}$ *such that* $c = s$ *and* $p = \min\{r, 4\}$. *Then*

1. $\tilde{\varphi}$ *is a geodesic, or*
2. $\tilde{\varphi}$ *is interpolating sesqui-harmonic if and only if it is a circle with*

$$k_1 = \sqrt{s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}},$$

3. $\tilde{\varphi}$ *is interpolating sesqui-harmonic if and only if it is a helix with*

$$k_1^2 + k_2^2 = s^2 \cos^2(\theta) + s(1 - \cos^2(\theta)s) - \frac{\delta_1}{\delta_2}.$$

Proof. If $\tilde{\varphi}$ is of osculating order $r = 2$ with $\frac{\delta_1}{\delta_2} \neq 0$, then $k_2 = 0$ and thus (3.5) yields

$$k_1 = \sqrt{s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}}, \quad \text{where} \quad \frac{\delta_1}{\delta_2} < s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)).$$

Moreover $\tilde{\varphi}$ is osculating order $r = 3$, then $k_3 = 0$ therefore by (3.5) we have,

$$k_1^2 + k_2^2 = s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}, \quad \text{where} \quad \frac{\delta_1}{\delta_2} < s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)).$$

In each case $\tilde{\varphi}$ satisfies Theorem 3.1. If $s^2 \cos^2(\theta) + s(1 - s\cos^2(\theta)) = \frac{\delta_1}{\delta_2}$, then $\tilde{\varphi}$ is geodesic. □

In particular for a interpolating sesqui harmonic Legendre curve in Sasakian space form, that is, $s = 1$ and $\theta = \frac{\pi}{2}$, we have [11, Theorem (3)]. Further for biharmonic Legendre curve in S-space form, that is, $\theta = \frac{\pi}{2}$, $\delta_1 = 0$ and $\delta_2 = 1$, from Theorem 3.3 we have

Corollary 3.4 ([13])**.** *Let φ be a Legendre frenet curve in an S-space form $\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1, \cdots, s\}$, $c = s$ and $2m + s > 3$. Then φ is proper biharmonic if and only if either φ is a circle with $k_1 = \sqrt{s}$ or a helix with $k_1^2 + k_2^2 = s$.*

Case 2: $c \neq s$ and $\phi T \perp E_2$. Then from Theorem 3.1 we have

Proposition 3.5. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha = \{1 \cdots s\}$ *such that* $c \neq s$, $\phi T \perp E_2$ *and* $p = \min\{r, 4\}$. *Then $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if*

$$\begin{cases} k_1 = \text{constant} > 0, \\ k_1^2 + k_2^2 = s^2 \cos^2(\theta) + \frac{(c+3s)}{4}(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}, \\ k_2 = \text{constant}, \\ k_2 k_3 = 0. \end{cases}$$

Next, we have

Theorem 3.6. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha = \{1 \cdots s\}$ *such that* $c \neq s$ *and* $\phi T \perp E_2$*. Then we have*

1. *if* $c \leqslant 4(\frac{\delta_1}{\delta_2} - s^2\cos^2(\theta))\frac{1}{1-s\cos^2(\theta)} - 3s$ *such that* $1 - s\cos^2(\theta) \neq 0$ *, then* $\tilde{\varphi}$ *is interpolating sesqui-harmonic if and only if it is geodesic;*
2. *if* $c > 4(\frac{\delta_1}{\delta_2} - s^2\cos^2(\theta))\frac{1}{1-s\cos^2(\theta)} - 3s$ *such that* $1 - s\cos^2(\theta) \neq 0$*, then* $\tilde{\varphi}$ *is interpolating sesqui-harmonic if and only if either*
   (a) $\tilde{\varphi}$ *is of osculating order* $r = 2, n \geqslant 2$ *and it is circle with*
   $$k_1^2 = s^2\cos^2(\theta) + \frac{c+3s}{4}(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2},$$
   (b) $\tilde{\varphi}$ *is of osculating order* $r = 3, n \geqslant 3$ *and it helix with*
   $$k_1^2 + k_2^2 = s^2\cos^2(\theta) + \frac{c+3s}{4}(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}.$$

Proof. If $\phi T \perp E_2$, then we have $g(\phi T, E_2) = 0$ by Proposition 3.5. If we take

$$c \leqslant 4(\frac{\delta_1}{\delta_2} - s^2\cos^2(\theta))\frac{1}{1 - s\cos^2(\theta)} - 3s,$$

such that $1 - s\cos^2(\theta) \neq 0$, then it can be easy seen that $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if it is a geodesic. Making use of Proposition 3.5 with

$$c > 4(\frac{\delta_1}{\delta_2} - s^2\cos^2(\theta))\frac{1}{1 - s\cos^2(\theta)} - 3s,$$

such that $1 - \cos^2(\theta) \neq 0$ and $\tilde{\varphi}$ is of osculating order $r = 2$, $n \geqslant 2$, then it is a circle with

$$k_1^2 = s^2\cos^2(\theta) + \frac{(c+3s)}{4}(1 - \cos^2(\theta)) - \frac{\delta_1}{\delta_2},$$

is a non-zero positive constant. if $\tilde{\varphi}$ is of osculating order $r = 3$, $n \geqslant 2$, then it is helix with

$$k_1^2 + k_2^2 = s^2\cos^2(\theta) + \frac{(c+3s)}{4}(1 - \cos^2(\theta)) - \frac{\delta_1}{\delta_2}.$$

Conversely, if $\tilde{\varphi}$ is circle with $k_1^2 = s^2\cos^2(\theta) + \frac{(c+3s)}{4}(1 - \cos^2(\theta)) - \frac{\delta_1}{\delta_2}$ or helix with

$$k_1^2 + k_2^2 = s^2\cos^2(\theta) + \frac{c+3s}{4}(1 - \cos^2(\theta) - \frac{\delta_1}{\delta_2},$$

then $\tilde{\varphi}$ satisfies Theorem 3.1 and this completes the proof. □

In particular for a Legendre curve in Sasakian space form, that is, $s = 1$ and $\theta = \frac{\pi}{2}$ we have [11, Theorem (7)]. Further for biharmonic Legendre curve in S-space form, that is, $\theta = \frac{\pi}{2}$, $\delta_1 = 0$ and $\delta_2 = 1$ from Theorem 3.3, we have

Corollary 3.7 ([13])**.** *Let $\tilde{\varphi}$ be a Legendre Frenet curve in an S-space form*

$$(\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha \in \{1, \cdots, s\}, c \neq s$ *and* $\phi T \perp E_2$*. Then* $\tilde{\varphi}$ *is proper biharmonic if and only if either*

1. *$n \geqslant 2$ and $\tilde{\varphi}$ is a circle with $k_1 = \frac{1}{2}\sqrt{c+3}$, where $c > -3s$ and $\{T = E_1, E_2, \phi T, \nabla_T \phi T, \xi_1, \cdots, \xi_s\}$ is linearly independent, or*
2. *$n \geqslant 3$ and $\tilde{\varphi}$ is a helix with $k_1^2 + k_2^2 = c + 3$, where $c > -3s$ and $\{T = E_1, E_2, \phi T, \nabla_T \phi T, \xi_1, \cdots, \xi_s\}$ is linearly independent.*

If $c \leqslant -3s$, then $\tilde{\varphi}$ is biharmonic if and only if it is a geodesic.

Case 3: $c \neq s$ and $\phi T \| E_2$.

Proposition 3.8. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha = \{1 \cdots s\}$ such that $c \neq s$ and $\phi T \| E_2$. Then $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if

$$\begin{cases} k_1 = \text{constant} > 0, \\ k_1^2 + k_2^2 = s^2\cos^2(\theta) + c(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2}, \\ k_2 = \text{constant}, \\ k_2 k_3 = 0. \end{cases}$$

Proof. For $c \neq s$, Using (3.2) and Definition 2.1 we have,

$$g(\phi T, \phi T) = 1 - s\cos^2(\theta).$$

So for unit vector E_2 we write $E_2 = \pm\frac{1}{\sqrt{1-s\cos^2(\theta)}}\phi T$. Therefore we have $g(\phi T, E_2) = \pm\sqrt{1 - s\cos^2(\theta)}$, $g(\phi T, E_3) = 0$ and $g(\phi T, E_4) = 0$. Using these relations in Theorem 3.1 we obtain the results. □

Theorem 3.9. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

$\alpha = \{1 \cdots s\}$ such that $c \neq s$ and $\phi T \| E_2$ with the Frenet frame $\{T, \phi T, \frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s}\xi_\alpha\}$. Then

1. *if $c \leqslant s + \frac{\delta_1}{\delta_2(1-s\cos^2(\theta))}$ such that $1 - s\cos^2(\theta) \neq 0$, then $\tilde{\varphi}$ is interpolating sesqui-harmonic with $\frac{\delta_1}{\delta_2} \neq 0$ if and only if it is geodesic;*
2. *if $c > s + \frac{\delta_1}{\delta_2(1-s\cos^2(\theta))}$ such that $1 - s\cos^2(\theta) \neq 0$, then $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if $\tilde{\varphi}$ is of osculating order $r = 3, n \geqslant 3$ and it helix with*

$$k_1^2 = s^2\cos^2(\theta) + c(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2} - s \quad \textit{and} \quad k_2 = \sqrt{s}.$$

Proof. If $\phi T \| E_2$, then we have $g(\phi T, E_2) = \sqrt{1 - s\cos^2(\theta)}$. By Proposition 3.8, if we take

$$c \leqslant s + \frac{\delta_1}{\delta_2(1 - s\cos^2(\theta))},$$

such that $1 - s\cos^2(\theta) \neq 0$, then it is easy to see that $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if it is a geodesic.

If $c > s + \frac{\delta_1}{\delta_2(1-s\cos^2(\theta))}$ such that $1 - s\cos^2(\theta) \neq 0$, and if $\tilde{\varphi}$ is of osculating order $r = 3$, $n \geqslant 3$, then it is helix with $k_1^2 = s^2\cos^2(\theta) + c(1 - s\cos^2(\theta)) - \frac{\delta_1}{\delta_2} - s$ and $k_2 = \sqrt{s}$. Conversely, if $\tilde{\varphi}$ is helix with $k_1^2 + k_2^2 = s^2\cos^2(\theta) + c(1 - \cos^2(\theta)) - \frac{\delta_1}{\delta_2}$ then $\tilde{\varphi}$ satisfies Theorem 3.1. □

In particular for a Legendre curve in Sasakian space form that is $s = 1$ and $\theta = \frac{\pi}{2}$. Thus, we have [11, Theorem (10)]. Further for biharmonic Legendre curve in S-space form, that is, $\theta = \frac{\pi}{2}$, $\delta_1 = 0$ and $\delta_2 = 1$ from Theorem 3.3, we have

Corollary 3.10 ([13])**.** *Let $\tilde{\varphi}$ be a Frenet curve in an S-space form $(\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g)$ $\alpha \in \{1, \cdots, s\}, c \neq s$ and $\phi T \| E_2$. Then*

$$\{T, \phi T, \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha\},$$

is the Frenet frame of $\tilde{\varphi}$ and $\tilde{\varphi}$ is proper biharmonic if and only if it is helix with $k_1 = \sqrt{c-s}$ and $k_2 = \sqrt{s}$, where $c > s$. If $c \leqslant s$, then $\tilde{\varphi}$ is biharmonic if and only if it is a geodesic.

Case 4: $c \neq s$ and $g(\phi T, E_2) \neq 0, -1, 1$.

Proposition 3.11. *Let $\tilde{\varphi} : I \to \overline{\mathcal{M}}(c)$ be a slant curve of osculating order r in S-space form*

$$\overline{\mathcal{M}}(c) = (\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g),$$

such that $4 \leqslant r \leqslant 2n+1, n \geqslant 2$ and $\phi T \in span\{E_2, \cdots, E_p\}$. Then $\tilde{\varphi}$ is interpolating sesqui-harmonic if and only if

$$\begin{cases} k_1 = \text{constant} > 0, \\ k_1^2 + k_2^2 = s^2\cos^2(\theta) + (1 - s\cos^2(\theta))\frac{c+3s}{4} - \frac{\delta_1}{\delta_2} + \frac{3(c-s)}{4}(1 - s\cos^2(\theta))\cos^2(\theta_1), \\ k_2 k_3 = \frac{-3(c-s)}{4}(1 - s\cos^2\theta)\sin(2\theta_1. \end{cases}$$

where $\theta_1 \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$.

Proof. If $\tilde{\varphi}$ is a interpolating sesqui-harmonic frenet curve of osculating order $r \geqslant 4$ and $g(E_2, \phi T) \neq 0, 1, -1$. If θ_1 is the angle between ϕT and E_2 such that

$$g(\phi T, E_2) = \sqrt{1 - s\cos^2\theta}\cos\theta_1(t).$$

Differentiating above equation and using (2.1), (2.2) and (2.3) we get,

$$g(\phi T, E_3) = -\frac{1}{k_2}\sqrt{1 - s\cos^2\theta}\theta_1'(t)\sin\theta_1(t). \tag{3.7}$$

We can write $\varphi T_1 = g(\varphi T_1, E_2) E_2 + g(\varphi T_1, E_3) E_3 + g(\varphi T_1, E_4) E_4$. So, the equations in Theorem 3.1 become

$$\begin{cases} k_1 = \text{constant} > 0, \\ k_1^2 + k_2^2 = s^2\cos^2\theta + \frac{c+3s}{4}\left(1 - s\cos^2\theta\right) + \frac{3(c-s)}{4}(1 - s\cos^2\theta)\cos^2\theta_1(t) - \frac{\delta_1}{\delta_2}, \\ k_2 k_2' - \frac{3(c-s)}{4}(1 - s\cos^2\theta)\theta_1'\sin\theta_1\cos\theta_1 = 0, \\ k_2 k_3 + \frac{3(c-s)}{4} g(\varphi T, E_2)\, g(\varphi T, E_4) = 0. \end{cases}$$

On solving the third equation of the above system, we obtain

$$k_2^2 = -3\sqrt{1 - s\cos^2\theta}\frac{(c-s)}{4}\cos^2\theta_1 + \delta_0, \tag{3.8}$$

where δ_0 is a constant. If we write (3.8) in the second equation, we have

$$k_1^2 = s^2\cos^2\theta + \frac{c+3s}{4}\left(1 - s\cos^2\theta\right) + \frac{3(c-s)}{4}\left(1 - s\cos^2\theta + \sqrt{1 - s\cos^2\theta}\right)\cos^2\theta_1 - \frac{\delta_1}{\delta_2} + \delta_0.$$

Hence θ_1 is a constant. From (3.7), we have $g(\varphi T, E_3) = 0$ and $k_2 = \text{constant} > 0$. Next, using

$$\|\varphi T\| = \sqrt{1 - s\cos^2\theta},$$

and $\varphi T = \sqrt{1 - s\cos^2\theta}\cos\theta_1 E_2 + g(\varphi T, E_4) E_4$, we obtain $g(\varphi T, E_4) = \sqrt{1 - s\cos^2\theta}\sin\theta_1$ where $\theta_1 \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Thus we have the result. □

4. Example

In this section we discuss the two cases for interpolating sesqui harmonic slant curve in S-space form when $\phi T \perp E_2$ and $\phi T \| E_2$ separately in the following examples.

Example 4.1. Let $(\overline{\mathcal{M}}^{(2n+s)}, \phi, \xi_\alpha, \eta_\alpha, g)$ be S-space form with coordinate functions

$$\{x_1, \cdots, x_n, y_1, \cdots, y_n, z_1, \cdots, z_s\}.$$

The vector fields

$$X_i = 2\frac{\partial}{\partial y_i}, \quad X_{n+i} = \phi X_i = 2(\frac{\partial}{\partial y_i} + y_i \sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}), \quad \xi_\alpha = 2\frac{\partial}{\partial z_\alpha}, \tag{4.1}$$

form a g-orthonormal basis and the Levi-Civita connection is calculated as

$$\nabla_{X_i} X_j = \nabla_{X_{n+i}} X_{n+j} = 0, \quad \nabla_{X_i} X_{n+j} = \delta_{ij} \sum_{\alpha=1}^{s} \xi_\alpha, \quad \nabla_{X_{n+i}} X_j = -\delta_{ij} \sum_{\alpha=1}^{s} \xi_\alpha,$$
$$\nabla_{X_i} \xi_\alpha = \nabla_{\xi_\alpha} X_i = -X_{n+i}, \quad \nabla_{X_{n+i}} \xi_\alpha = \nabla_{\xi_\alpha} X_{n+i} = X_i.$$

Let $\tilde{\varphi}(t) = (\tilde{\varphi}_1(t), \tilde{\varphi}_2(t), \tilde{\varphi}_3(t), \tilde{\varphi}_4(t))$ be unit speed slant curve in $R^4(-6)$. Then for a tangent vector of the slant curve we have

$$T = \frac{1}{2}[\tilde{\varphi}_1' \frac{\partial}{\partial x_1} + \tilde{\varphi}_2' \frac{\partial}{\partial y_1} + \tilde{\varphi}_3' \frac{\partial}{\partial z_1} + \tilde{\varphi}_4' \frac{\partial}{\partial z_2}].$$

From (4.1), we find

$$X_1 = 2\frac{\partial}{\partial y_1}, \quad X_2 = \phi X_1 = 2(\frac{\partial}{\partial x_1} + y_1(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2})), \quad \xi_1 = 2\frac{\partial}{\partial z_1}, \xi_2 = 2\frac{\partial}{\partial z_2}.$$

By using these values, it follows that

$$T = \frac{1}{2}[\tilde{\varphi}_2' X_1 + \tilde{\varphi}_1' X_2 + (\tilde{\varphi}_3' - \tilde{\varphi}_1' \tilde{\varphi}_2)\xi_1 + (\tilde{\varphi}_4' - \tilde{\varphi}_1' \tilde{\varphi}_2)\xi_2]. \tag{4.2}$$

Thus for a slant curve $\eta_\alpha(T) = \cos(\theta)$, we have

$$\tilde{\varphi}_4' = \tilde{\varphi}_1' \tilde{\varphi}_2 + 2\cos(\theta), \tag{4.3}$$
$$\tilde{\varphi}_3' = \tilde{\varphi}_1' \tilde{\varphi}_2 + 2\cos(\theta), \tag{4.4}$$
$$\tilde{\varphi}_1'^2 + \tilde{\varphi}_2'^2 = 4(1 - 2\cos^2(\theta)). \tag{4.5}$$

Differentiating (4.2) and making use of (4.3) and (4.4), it yields

$$\nabla_T T = \frac{1}{2}[\tilde{\varphi}_2'' X_1 + \tilde{\varphi}_1'' X_2].$$

Then for $\theta = \frac{\pi}{3}$ in (4.5), we get $\tilde{\varphi}_1 = \sqrt{2}\sin t$ and $\tilde{\varphi}_2 = -\sqrt{2}\cos t$. Now using these values in (4.3) and (4.4), we have $\tilde{\varphi}_3 = \frac{1}{2}\sin 2t$ and $\tilde{\varphi}_4 = \frac{1}{2}\sin 2t$, respectively. Therefore, we have $\tilde{\varphi}(t) = (\sqrt{2}\sin t, -\sqrt{2}\cos t, \frac{1}{2}\sin 2t, \frac{1}{2}\sin 2t)$. Now making use of (4.6), we have

$$\nabla_T T = \frac{1}{2}[\sqrt{2}\cos t X_1 - \sqrt{2}\sin t X_2].$$

Taking the inner product of above equation with itself, we have $k_1 = \frac{1}{\sqrt{2}}$ which satisfies Theorem 3.1 for the case of osculating order 2, $\phi T \perp E_2, \delta_1 = -1, \delta_2 = 2$.

For $\cos(\theta) = \frac{\sqrt{3}}{2\sqrt{2}}$ and $\phi T || E_2$ we have the following example.

Example 4.2. The value $\cos(\theta) = \frac{\sqrt{3}}{2\sqrt{2}}$ in (4.5) implies $\tilde{\varphi}_1 = \sin t$ and $\tilde{\varphi}_2 = \cos t$. Now using these values in (4.3) and (4.4) we have $\tilde{\varphi}_3 = \frac{1}{2}(t + \sqrt{6}t + \frac{\sin 2t}{2})$ and $\tilde{\varphi}_4 = \frac{1}{2}(t + \sqrt{6}t + \frac{\sin 2t}{2})$, respectively. Therefore, we get $\tilde{\varphi}(t) = (\sin t, \cos t, \frac{1}{2}(t + \sqrt{6}t + \frac{\sin 2t}{2}), \frac{1}{2}(t + \sqrt{6}t + \frac{\sin 2t}{2}))$, which by making use of (4.6), gives

$$\nabla_T T = \frac{1}{2}[\cos t X_1 - \sin t X_2].$$

Then by taking the inner product of above equation with itself we find $k_1 = \frac{1}{2}$ which satisfies Theorem 3.1 for the case of osculating order 2, $\phi T || E_2, \delta_1 = -19, \delta_2 = 4$.

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References

[1] V. Branding, *On interpolating sesqui-harmonic maps between Riemannian manifolds*, J. Geom. Anal., **30** (2019), 248–273. 1, 3
[2] C. Călin, M. Crasmareanu, *Slant curves in 3-dimensional normal almost contact geometry*, Mediterr. J. Math., **10** (2013), 1067–1077. 1
[3] J. T. Cho, J.-I. Inoguchi, J.-E. Lee, *Slant curves in Sasakian 3-manifolds*, Bull. Austral. Math. Soc., **74** (2006), 359–367. 1
[4] J. T. Cho, J.-I. Inoguchi, J.-E. Lee, *Biharmonic curves in 3-dimensional Sasakian space forms*, Ann. Mat. Pura Appl. (4), **186** (2007), 685–701. 1
[5] J. Eells, J. H.Sampson, *Harmonic mappings of Riemannian manifolds*, Amer J. Math., **86** (1964), 109–160. 1
[6] D. Fetcu, *Biharmonic Legendre curves in Sasakian space forms*, J. Korean Math. Soc., **45** (2008), 393–404. 1
[7] D. Fetcu, C. Oniciuc, *Explicit formulas for biharmonic submanifolds in Sasakian space form*, Pacific J. Math., **240** (2009), 85–107. 1
[8] S. Güvenc, C. Özgür, *On the characterizations of f-biharmonic Legendre curves in Sasakian space forms*, Filomat, **31** (2017), 639–648. 1
[9] S. Güvenc, C. Özgür, *On slant curves in S-manifolds*, Commun. Korean Math. Soc., **33** (2018), 293–303. 1
[10] G. Y. Jiang, *2-Harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A, **7** (1986), 389–402. 1
[11] F. Karaca, C. Özgür, U. C. De, *On interpolating sesqui harmonic legendre curves in Sasakian space forms*, Int. J. Geom. Methods Mod. Phys., **17** (2020), 13 pages. 1, 3, 3, 3
[12] Y. Lou, Y.-L. Ou, *Some remarks on Bi-f harmonic maps and f-biharmonic maps*, Results Math., **74** (2019), 19 pages. 1
[13] C. Özgür, S. Güvenc, *On biharmonic Legendre curves in S-space forms*, Turkish J. Math., **38** (2014), 454–461. 1, 2, 2.1, 2, 2, 3.4, 3.7, 3.10
[14] C. Özgür, S. Güvenc, *On some classes of biharmonic Legendre curves in generalized Sasakian space forms*, Collect. Math., **65** (2014), 203–218. 1
[15] A. Polyako, *Fine structure of strings*, Nuclear Phys. B, **268** (1986), 406–412. 1
[16] K. Yano, M. Kon, *Structures on manifolds*, World Scientific Publishing Co., Singapore, (1984). 2