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On some characterizations of the solutions of a hybrid non-linear functional integral inclusion and applications

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Abstract

Here, we prove two existence theorems for the solutions a hybrid nonlinear functional integral inclusion and study some properties and applications of these two theorems. A Chandrasekhar quadratic integral inclusion and a nonlinear cubic Chandrasekhar functional integral equation are studied as an application. The continuous dependence of the solutions on some functions is proved.

Keywords: Hybrid integral equations, Urysohn-Stieltjes type integral inclusion, Chandrasekhar quadratic integral inclusion, cubic Chandrasekhar integral equation.

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1. Introduction

The classical theory of the integral equations and inclusions have been generalized with the help of the Volterra-Stieltjes and Urysohn-Stieltjes integral operators (see [4, 6–8, 16–29] and reference therein). The hybrid differential and integral equations are important in many applications (see [13–15, 35] and reference therein). Some publications have investigated the quadratic and cubic Chandrasekhar integral equations (see [2, 5, 9–12, 31]).

Let us mention that these equations find numerous applications in the theories of radiative transfer, neutron transport and in the kinetic theory of gases ([3, 9–12, 31–35]).

Let $I = [0, 1]$. Consider the nonlinear hybrid integral inclusion of Urysohn-Stieltjes type

$$\frac{x(t) - \eta(t, x(t))}{\omega(t, x(t))} \in \int_0^1 \Phi\left(t, s, x(s), \int_0^1 \hbar(s, \theta, x(\theta)) d_\theta \ltimes_2 (s, \theta)\right) d_s \ltimes_1 (t, s), \ t \in I. \tag{1.1}$$

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Here we prove two existence theorems for the solution $x \in C[0,1]$ of the nonlinear hybrid Urysohn-Stieltjes integral inclusion (1.1) and continuous dependence on the functions $\ltimes_i$, $(i = 1,2)$, and on the set of selection S_Φ will be proved.

As an application of the first existence theorem, we study the existence of solutions $x \in C[0,1]$ of the nonlinear hybrid Chandrasekhar functional integral inclusion

$$\frac{x(t) - \eta(t,x(t))}{\omega(t,x(t))} \in \int_0^1 \frac{t}{t+s}\,\Phi\left(\sigma_1(s)x(s), \int_0^1 \frac{s}{s+\theta}\sigma_2(s)x(\theta)\,d\theta\right)ds, \quad t \in I \tag{1.2}$$

and for the second existence theorem, we investigate the existence of solutions $x \in C(I)$ for the nonlinear cubic Chandrasekhar integral equation

$$x(t) = \eta(t,x(t)) + \omega(t,x(t)) \int_0^1 \frac{t}{t+s}\sigma_1(s)x(s)\cdot(\int_0^1 \frac{s}{s+\theta}\sigma_2(s)x(\theta)\,d\theta)ds, \quad t \in I. \tag{1.3}$$

The paper is stated as follows. In Section 2 we present first existence theorems by establish the existence and uniqueness results for (1.1), we discuss some special case by present the existence of solutions for (1.2). There is an example included, we also prove the continuous dependence of solution $\ltimes_i$, $(i = 1,2)$ and on the set of selection S_Φ. In Section 3 we present second existence theorem and as application, we discuss the existence of solutions for (1.3), an example is given, and finally, Section 4 presents our conclusions.

2. Existence theorem I

Check out the following assumptions for the functional integral equation (1.1).

(i) Let $\Phi : I \times I \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$, satisfy the following assumptions.

(a) The set $\Phi(t,s,u,v)$ is nonempty, closed and convex for all $(t,s,u,v) \in I \times I \times \mathbb{R} \times \mathbb{R}$.
(b) $\Phi(t,s,u,v)$ is upper semicontinuous in x and y for every $t,s \in I$.
(c) $\Phi(t,s,u,v)$ is measurable in $t,s \in I$ for every $u,v \in \mathbb{R}$.
(d) There exist two continuous functions $\varpi_1, \kappa_1 : I \times I \to \mathbb{R}$, with

$$\|\Phi(t,s,u,v)\| = \sup\{|\phi| : \phi \in \Phi(t,s,u,v)\} \leqslant \varpi_1(t,s) + \kappa_1(t,s)(|u| + |v|).$$

Remark 2.1. From the assumption (i) we can deduce that (see [1, 3, 27]) there exists $\phi \in \Phi(t,s,x,y)$, such that

$$\phi(t,s,u,v) \leqslant \varpi_1(t,s) + \kappa_1(t,s)(|u| + |v|)$$

and

$$\frac{x(t) - \eta(t,x(t))}{\omega(t,x(t))} = \int_0^1 \phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\right)d_s \ltimes_1(t,s), \quad t \in I. \tag{2.1}$$

So, every solution of (2.1) is a solution of (1.1)

(ii) $\hbar : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exist two continuous functions $\varpi_2, \kappa_2 : I \times I \to \mathbb{R}$, with

$$|\hbar(t,s,u)| \leqslant \varpi_2(t,s) + \kappa_2(t,s)|u|.$$

$\kappa = \sup\{\kappa_i(t,s) : t,s \in I\}$, and $\varpi = \sup\{\varpi_i(t,s) : t,s \in I\}$, $i = 1,2$.

(iii) $\eta : I \times \mathbb{R} \to \mathbb{R}$ and $\omega : I \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exist a bounded mappings $l_i : I \to \mathbb{R}^+$, $i = 1,2$, such that

$$|\eta(t,u_1) - \eta(t,u_2)| \leqslant l_1(t)\,|u_1(t) - u_2(t)|,$$

$$|\omega(t,u_1)-\omega(t,u_2)| \leqslant l_2(t)\ |u_1(t)-u_2(t)|$$

for all $u_1, u_2 \in \mathbb{R}$ and $t \in [0,1]$.

Note

$$\eta(t,x)| \leqslant l_1|x(t)| + \mathcal{H}, \quad |\omega(t,x)| \leqslant l_2|x(t)| + G,$$

where $\mathcal{H} = \sup_{t\in I} |\eta(t,0)|$ and $G = \sup_{t\in I} |\omega(t,0)|$.

(iv) $\ltimes_i : I \times \mathbb{R} \to \mathbb{R},\ i = 1,2$ are continuous functions with

$$\mu = \max\{\sup|\ltimes_i(t,1)| + \sup|\ltimes_i(t,0)|, \text{ on } I\}.$$

(v) For all $t_1, t_2 \in I,\ t_1 < t_2$ the functions $s \to \ltimes_i(t_2,s) - \ltimes_i(t_1,s)$ are nondecreasing on I.

(vi)$\ltimes_i(0,s) = 0$, for any $s \in I$.

(vii) The following algebraic equation

$$[(1+\kappa)\ l_2\ \mu^2]\ r^2 + (\ell_1 + (1+\kappa\ \mu)\ \varpi\ \mu\ l_2 + (1+\kappa)G\ \kappa\ \mu^2 - 1)r + [\mathcal{H} + (1+\ \kappa\ \mu)G\ \varpi\ \mu] = 0,$$

has a positive root r.

The following lemma can be easily proved.

Lemma 2.2. *If the solution of the equation (2.1) exists then it can be expressed with the integral equation*

$$x(t) = \eta(t,x) + \omega(t,x)\int_0^1 \phi\left(t,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\right)d_s \ltimes_1(t,s),\ t \in I.$$

Theorem 2.3. *Assume that assumptions* (i)-(vii) *are valid. Then the functional integral equation* (2.1) *has at least one continuous solution* $x \in C(I)$ *(consequently, (1.1)).*

Proof. Let A be operator as

$$Ax(t) = \eta(t,x) + \omega(t,x)\int_0^1 \phi\left(t,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\right)d_s \ltimes_1(t,s),\ t \in I$$

and let the set Q_r be defined by

$$Q_r = \{x \in \mathbb{R}:\ \|x\| \leqslant r\} \subseteq C(I), \quad \|x\| = \sup_{t\in I}|x(t)|,$$

where r is a positive solution of

$$[(1+\kappa)\ l_2\ \mu^2]\ r^2 + (l_1 + (1+\kappa\ \mu)\ \varpi\ \mu\ l_2 + (1+\kappa)G\ \kappa\ \mu^2 - 1)r + [\mathcal{H} + (1+\ \kappa\ \mu)G\ \varpi\ \mu] = 0.$$

The set Q_r is obvious to be nonempty, closed, bounded, and convex. Set $x \in Q_r$. Then

$$\begin{aligned}
|Ax(t)| &= |\eta(t,x) + \omega(t,x)\int_0^1 \phi\left(t,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\right)d_s \ltimes_1(t,s)| \\
&\leqslant |\eta(t,x)| + |\omega(t,x)|\int_0^1 \left|\phi\left(t,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\right)\right|d_s \ltimes_1(t,s) \\
&\leqslant [l_1|x(t)| + \mathcal{H}] + [l_2|x(t)| + G]\int_0^1 \left(\varpi_1(t,s) + \kappa_1(t,s)\big(|x(t)| + \int_0^1 |\hbar(s,\theta,x(\theta))|d_\theta \ltimes_2(s,\theta)\big)\right)d_s \ltimes_1(t,s) \\
&\leqslant [l_1|x(t)| + \mathcal{H}] + [l_2|x(t)| + G] \\
&\quad \times \int_0^1 \left(\varpi_1(t,s) + \kappa_1(t,s)\big(|x(t)| + \int_0^1 (\varpi_2(s,\theta) + \kappa_2(s,\theta)|x(\theta)|d_\theta \ltimes_2(s,\theta))\big)d_s \ltimes_1(t,s)\right)
\end{aligned}$$

$$\leqslant (l_1 r + \mathcal{H}) + (l_2 r + G) \int_0^1 \big(\varpi_1(t,s) + \kappa_1(t,s)(|x(t)| + (\varpi + \kappa r)\ \mu\big) d_s \ltimes_1 (t,s)$$
$$\leqslant (l_1 r + \mathcal{H}) + (l_2 r + G)\big(\varpi + \kappa(r + (\varpi + kr)\ \mu)\big)\mu \leqslant r.$$

This shows that the the class $\{Ax\}$ and the operator $A : Q_r \to Q_r$, are uniformly bounded on Q_r.

After that, for $x \in Q_r$ and $y(s) = \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)$. Sets are defined as

$$\theta_\phi(\delta) = \sup\{|\phi(t_2,s,x,y) - \phi(t_1,s,x,y)| : t_1,t_2,s \in I,\ \ t_1 < t_2, |t_2 - t_1| < \delta,\ |x| \leqslant r,\ |y| \leqslant r\},$$
$$\theta_\eta(\delta) = \sup\{|\eta(t_2,x) - \eta(t_1,x)| : t_1,t_2 \in I,\ |t_1 - t_2| \leqslant \delta,\ |x| \leqslant r\},$$
$$\theta_\omega(\delta) = \sup\{|\omega(t_2,x) - \omega(t_1,x)| : t_1,t_2 \in I,\ |t_1 - t_2| \leqslant \delta,\ |x| \leqslant r\},$$

then based on the function $\phi : I \times I \times Q_r \times Q_r \to \mathbb{R}$ is uniform continuity, assumptions (i) and (ii), we have concluded $\theta_\phi(\delta) \to 0$, $\theta_\eta(\delta) \to 0$, and $\theta_\omega(\delta) \to 0$, as $\delta \to 0$ independent of $x, y \in Q_r$.

Let $t_2, t_1 \in I$, $|t_2 - t_1| < \delta$. Then we have

$$\begin{aligned}
&|Ax(t_2) - Ax(t_1)| \\
&= \Big|\eta(t_2,x(t_2)) + \omega(t_2,x(t_2)) \int_0^1 \phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad - \eta(t_1,x(t_1)) - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_1,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\leqslant |\eta(t_2,x(t_2)) - \eta(t_1,x(t_1))| \\
&\quad + \Big|\omega(t_2,x(t_2)) \int_0^1 \phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad + \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\quad + \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\quad - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_1,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\leqslant \big|\eta(t_2,x(t_2)) - \eta(t_2,x(t_1))\big| + \big|\eta(t_2,x(t_1)) - \eta(t_1,x(t_1))\big| \\
&\quad + \big[\big|\omega(t_2,x(t_2)) - \omega(t_2,x(t_1))\big| + \big|\omega(t_2,x(t_1)) - \omega(t_1,x(t_1))\big|\big] \\
&\quad \times \int_0^1 \Big|\phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big)\Big| \ltimes_1 (t_2,s)\Big| \\
&\quad + \big|\omega(t_1,x(t_1))\big|\Big| \int_0^1 \Big|\phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big)\Big|\Big|d_s \ltimes_1 (t_2,s) - d_s \ltimes_1 (t_1,s)\Big| \\
&\quad + \big|\omega(t_1,x(t_1))\big| \\
&\quad \times \int_0^1 \Big|\phi\Big(t_2,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta)) d_\theta \ltimes_2 (s,\theta)\Big) - \phi\Big(t_1,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big)\Big| \ltimes_1 (t_1,s) \\
&\leqslant \ell_1|x(t_2) - x(t_1)| + \theta_\eta(\delta_1) + \big[\ell_2|x(t_2) - x(t_1)| + \theta_\omega(\delta_2)\big] \int_0^1 \big(\varpi_1(t,s) + \kappa_1(t,s)(|x| + |y|)\big)\ \ltimes_1 (t_2,s) \\
&\quad + [l_2|x(t_1)| + G] \int_0^1\ \big(\varpi_1(t,s) + \kappa_1(t,s)(|x| + |y|)\big)\ \big|d_s \ltimes_1 (t_2,s) - d_s \ltimes_1 (t_1,s)\big|
\end{aligned}$$

$$+ [l_2|x(t_1)| + G]\Big| \int_0^1 \theta(\delta)\, d_s \ltimes_1 (t_1, s).$$

By the the above inequality, the class of functions $\{Ax\}$ is equicontinuous. As a result, according to the Arzela-Ascoli Theorem [33], A is compact.

For $\{x_n\} \subset Q_r, \quad x_n \to x$. We have

$$Ax_n(t) = \eta(t, x_n(t)) + \omega(t, x_n(t)) \int_0^1 \phi\left(t, s, x_n(s), \int_0^1 \hbar(s, \theta, x_n(\theta))\, d_\theta \ltimes_2 (s, \theta)\right) d_s \ltimes_1 (t, s),$$

$$\lim_{n\to\infty} Ax_n(t) = \lim_{n\to\infty} (\eta(t, x_n(t)) + \omega(t, x_n(t)) \int_0^1 \phi\left(t, s, x_n(s), \int_0^1 \hbar(s, \theta, x_n(\theta)) d_\theta \ltimes_2 (s, \theta)\right) d_s \ltimes_1 (t, s)),$$

and we can get from assumption (ii) that (see [30])

$$\begin{aligned}
&\lim_{n\to\infty} Ax_n(t) \\
&= \lim_{n\to\infty} (\eta(t, x_n(t)) + \lim_{n\to\infty} \omega(t, x_n(t)) \int_0^1 \lim_{n\to\infty} \phi\left(t, s, x_n(s), \int_0^1 \hbar(s, \theta, x_n(\theta))\, d_\theta \ltimes_2 (s, \theta)\right) d_s \ltimes_1 (t, s) \\
&= \lim_{n\to\infty} (\eta(t, x_n(t)) + \lim_{n\to\infty} \omega(t, x_n(t)) \int_0^1 \phi\left(t, s, \lim_{n\to\infty} x_n(s), \int_0^1 \hbar(s, \theta, \lim_{n\to\infty} x_n(\theta)) d_\theta \ltimes_2 (s, \theta)\right) d_s \ltimes_1 (t, s) \\
&= \eta(t, x(t)) + \omega(t, x(t)) \int_0^1 \phi\left(t, s, x(s), \int_0^1 \hbar(s, \theta, x(\theta))\, d_\theta \ltimes_2 (s, \theta)\right) d_s \ltimes_1 (t, s) = Ax(t).
\end{aligned}$$

This establishes the continuity of $Ax_n(t) \to Ax(t)$ and A.

Hence (see [30]), A has at least one fixed point $x \in Q_r$ and (2.1) (consequently, (1.1)) has at least one solution $x \in Q_r \subset C(I)$. □

2.1. *Uniqueness of the solution*

We have the following conditions for the uniqueness of the solution for (2.1).

(i*) Let $\Phi : I \times \mathbb{R}^+ \times \mathbb{R}^+ \to 2^{\mathbb{R}^+}$ satisfy the following assumptions:

(d)** The Lipschitzian set-valued map $\Phi : I \times I \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$, with a Lipschitz constant $\kappa_1 > 0$, is a nonempty compact convex subset of $2^{\mathbb{R}}$, such that

$$\|\Phi(t, s, u_1, v_1) - \Phi(t, s, u_2, v_2)\| \leqslant \kappa_1 \big(|u_1 - u_2| + |v_1 - v_2|\big).$$

Remark 2.4. Based on an assumption and Theorem [3, Sect. 9, Chap. 1, Th. 1], we can conclude the set of Lipschitz selections of Φ is not empty and there exists $\phi \in \Phi$, with

$$|\phi(t, s, u_1, v_1) - \phi(t, s, u_2, v_2)| \leqslant \kappa_1(|u_1 - u_2| + |v_1 - v_2|).$$

(ii*) The continuous function $\hbar : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition

$$|\hbar(t, s, u) - \hbar(t, s, v)| \leqslant \kappa_2 |u - v|.$$

Conditions (i)* and (ii)*, give

$$|\phi(t, s, u(s), v(s))| \leqslant \kappa_1(|u| + |v|) + \varpi_1, \quad \text{and} \quad |\hbar(t, s, u(s))| \leqslant \kappa_2 |u| + \varpi_2,$$

where

$$\varpi_1 = \sup_{t\times s \in I\times I} |\phi(t, s, 0, 0)|, \quad \text{and} \quad \varpi_2 = \sup_{t\times s\in I\times I} |\hbar(t, s, 0)|,$$

with $\varpi = \max\{\varpi_1, \varpi_2\}$, and $\kappa = \max\{\kappa_1, \kappa_2\}$.

Theorem 2.5. *Assume that conditions* (i)*-(ii)* *and* (iii)-(vi) *hold, if*

$$\big[l_1 + (l_2\, r + G)(\mu\kappa + \kappa^2\mu^2) + l_2\big(\varpi + \kappa(r + (\varpi + \kappa r)\ \mu)\big)\mu\big] \leqslant 1.$$

Then $x \in C(I)$ *is unique solution of equation* (2.1).

Proof. For x_1, x_2 be solutions of (2.1)

$$\begin{aligned}
|x_1(t) - x_2(t)| &= \Big|\eta(t, x_1) + \omega(t, x_1)\int_0^1 \phi\big(t, s, x_1(s), \int_0^1 \hbar(s, \theta, x_1(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s) \\
&\quad - \eta(t, x_2) + \omega(t, x_2)\int_0^1 \phi\big(t, s, x_2(s), \int_0^1 \hbar(s, \theta, x_2(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s)\Big| \\
&\leqslant |\eta(t, x_1) - \eta(t, x_2)| + \omega(t, x_1)\int_0^1 \phi\big(t, s, x_1(s), \int_0^1 \hbar(s, \theta, x_1(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s) \\
&\quad - \omega(t, x_1)\int_0^1 \phi\big(t, s, x_2(s), \int_0^1 \hbar(s, \theta, x_2(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s) \\
&\quad + \omega(t, x_1)\int_0^1 \phi\big(t, s, x_2(s), \int_0^1 \hbar(s, \theta, x_2(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s) \\
&\quad - \omega(t, x_2)\int_0^1 \phi\big(t, s, x_2(s), \int_0^1 \hbar(s, \theta, x_2(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s) \\
&\leqslant |\eta(t, x_1) - \eta(t, x_2)| + |\omega(t, x_1)|\int_0^1 \Big|\phi\big(t, s, x_1(s), \int_0^1 \hbar(s, \theta, x_1(\theta))\ d_\theta \ltimes_2 (s,\theta)\big) \\
&\quad - \phi\big(t, s, x_2(s), \int_0^1 \hbar(s, \theta, x_2(\theta))\ d_\theta \ltimes_2 (s,\theta)\big)\Big| d_s \ltimes_1 (t,s) \\
&\quad + |\omega(t, x_1(t)) - \omega(t, x_2(t))|\int_0^1 \Big|\phi\Big(t, s, x_2(s), \int_0^1 \hbar(s, \theta, x_2(\theta))\ d_\theta \ltimes_2 (s,\theta)\Big)\Big| d_s \ltimes_1 (t,s) \\
&\leqslant l_1|x_1(t) - x_2(t)| + [l_2|x_1(t)| + G] \\
&\quad \times \int_0^1 \kappa_1\Big(|x_1(s) - x_2(s)| + \int_0^1 |(\hbar(s, \theta, x_1(\theta)) - \hbar(s, \theta, x_2(\theta)))| d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t,s) \\
&\quad + l_2|x_1(t) - x_2(t)|\int_0^1 \Big(\varpi_1(t,s) + \kappa_1(t,s)\big(|x_2(t)| + \int_0^1 |\hbar(s, \theta, x_2(\theta))| d_\theta \ltimes_2 (s,\theta)\big)\Big) d_s \ltimes_1 (t,s) \\
&\quad \leqslant l_1|x_1(t) - x_2(t)| \\
&\quad + [l_2|x_1(t)| + G]\int_0^1 \kappa_1\big(|x_1(s) - x_2(s)| + \int_0^1 \kappa_2(|x_1(\theta) - x_2(\theta)|)\ d_\theta \ltimes_2 (s,\theta)\big) d_s \ltimes_1 (t,s) \\
&\quad + l_2|x_1(t) - x_2(t)| \\
&\quad \times \int_0^1 \Big(\varpi_1(t,s) + \kappa_1(t,s)\big(|x_2(t)| + \int_0^1 (\varpi_2(s,\theta) + \kappa_2(s,\theta)|x_2(\theta)| d_\theta \ltimes_2 (s,\theta))\big)\Big) d_s \ltimes_1 (t,s) \\
&\leqslant l_1|x_1(t) - x_2(t)| + [l_2|x_1(t)| + G]\int_0^1 \kappa_1(|x_1(s) - x_2(s)| + \kappa_2\|x_1 - x_2\|\mu) d_s \ltimes_1 (t,s) \\
&\quad + l_2|x_1(t) - x_2(t)|\int_0^1 \big(\varpi_1(t,s) + \kappa_1(t,s)(|x(t)| + (\varpi + \kappa r)\ \mu\big) d_s \ltimes_1 (t,s) \\
&\leqslant l_1\|x_1 - x_2\| + (l_2\, r + G)\big[\kappa\|x_1 - x_2\|\mu + \kappa^2\|x_1 - x_2\|\mu^2\big] \\
&\quad + l_2\|x_1 - x_2\|\big(\varpi + \kappa(r + (\varpi + \kappa r)\ \mu)\big)\mu.
\end{aligned}$$

Hence, we have

$$\|x_1 - x_2\| \leqslant \big[l_1 + (l_2\, r + G)(\mu\kappa + \kappa^2\mu^2) + l_2\big(\varpi + \kappa(r + (\varpi + \kappa r)\ \mu)\big)\mu\big]\|x_1 - x_2\|,$$

and

$$\left(1-\left[l_1+(l_2\ r+G)(\mu\kappa+\kappa^2\mu^2)+l_2\left(\varpi+\kappa(r+(\varpi+\kappa r)\ \mu)\right)\mu\right]\right)\|x_1\ -x_2\|\leqslant 0,$$

which implies

$$x_1(t)=x_2(t).$$

□

2.2. Application

As an application of the nonlinear set-valued functional integral equations (1.1). Allow for the use of the functions $\ltimes_i$ specified by

$$\ltimes_1(t,s)=\begin{cases} t\ln\frac{t+s}{t}, & \text{for } t\in(0,1], \quad s\in I,\\ 0, & \text{for } t=0, \quad s\in I,\end{cases}$$

and

$$\ltimes_2(s,\theta)=\begin{cases} s\ln\frac{s+\theta}{s}, & \text{for } s\in(0,1], \quad \theta\in I,\\ 0, & \text{for } s=0, \quad \theta\in I.\end{cases}$$

Let $\hbar(t,s,x(s))=\sigma_2(s)\ x(s)$, and $\Phi\big(t,s,x(s),z(s)\big)=\Phi\big(\sigma_1(s)\ x(s),z(s)\big)$, where

$$z(s)=\int_0^s\frac{s}{s+\theta}\sigma_2(s)x(\theta)d\theta,$$

in (1.1). The nonlinear Chandrasekhar functional integral inclusion is also obtained by utilizing the fact that functions $\ltimes_i$ meet conditions (iv)-(vi),

$$x(t)\in\eta(t,x)+\omega(t,x)\int_0^1\frac{t}{t+s}\Phi\left(\sigma_1(s)x(s),\int_0^1\frac{s}{s+\theta}\sigma_2(s)x(\theta)d\theta\right)ds\quad t,\in I. \tag{2.2}$$

We can now formulate the existence result for (2.2) as follows.

Theorem 2.6. *Assume that conditions of Theorem 2.5 hold, then there must be at least one continuous solution* $x\in C(I)$ *for inclusion* (2.2).

2.3. Example

Consider the following Chandrasekhar nonlinear functional integral inclusion

$$\frac{x(t)-\frac{e^{-t}}{20+t^2}+\frac{1}{2}|\sin(\sqrt{x})|}{\frac{1}{\sqrt{16+t}}+\frac{e^{-t}}{200}|x(t)|}\in\int_0^1\frac{t}{t+s}\frac{\sqrt{\pi}\,e^{-3t}x(s)}{(\pi+e^t)(s^2+1)}\int_0^1\frac{s}{s+\theta}\frac{\sqrt{s}}{e^s}x(\theta)d\theta ds\quad t,\in[0,1]. \tag{2.3}$$

If we choose $\Phi:[0,1]\times\mathbb{R}\to 2^{\mathbb{R}^+}$ in (2.3) as

$$\Phi\big(b_1(s)x(s),y(s)\big)=\left[0,\frac{\sqrt{\pi}\,e^{-3t}x(s)}{(\pi+e^t)(s^2+1)}\int_0^1\frac{s}{s+\theta}(s+\frac{\sqrt{s}}{e^s}\ x(\theta))\ d\theta ds\right],$$

this inclusion, as we can see, is a special example of inclusion (2.2).

Further, let us notice that a list of terms used in (2.2) as

$$\eta(t,x(t))=\frac{e^{-t}}{20+t^2}+\frac{1}{2}|\sin(\sqrt{x})|,\qquad \omega(t,x(t))=\frac{1}{\sqrt{16+t}}+\frac{e^{-t}}{200}|x(t)|,$$

$$y(s)=\int_0^1\frac{s}{s+\theta}\frac{\sqrt{s}}{e^s}x(\theta)d\theta,\qquad \hbar(t,s,x(s))=\frac{t+\sqrt{t}}{e^t}\ x(s),$$

with $\sigma_1(s) = \frac{1}{s^2+1}$, $\sigma_2(s) = \frac{\sqrt{s}}{e^{s+1}}$, $l_1 = \frac{1}{2}$, $l_2 = \frac{1}{200e}$, $\kappa_1 = \frac{\sqrt{\pi}}{e^3(\pi+1)}$, and $\kappa_2 = \frac{1}{e}$.

Define the continuous map $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$, notice that for $\phi \in S_\Phi$, then we have

$$|\phi(\sigma_1(s)x_1(s), y_1(s)) - \phi(\sigma_1(s)x_2(s), y_2(s))| \leqslant \frac{\sqrt{\pi}}{e^3\,(\pi+1)}|x_1 - x_2|,$$

and

$$\left|\hbar(t,s,x_1(t)) - \hbar(t,s,x_2(t))\right| \leqslant \frac{1}{e}|x_1 - x_2|.$$

As a result, conditions (i) and (ii)* are met with $\kappa = \frac{1}{e}$, $\varpi = 1$, $\mathcal{H} = \frac{1}{20e}$ and $G = \frac{1}{4}$. By considering the above-mentioned facts, we might conclude that condition (vii*) with form

$$[(1+\kappa)\; l_2\; \mu^2]\; r^2 \;\; + \;(l_1 + (1+\kappa\;\mu)\;\varpi\;\mu\; l_2 + (1+\kappa)G\;\kappa\;\mu^2 - 1)r + [\mathcal{H} + (1+\;\kappa\;\mu)G\;\varpi\;\mu] = 0$$

has r as a positive solution. Thus, if we choose $r \approx 6.83214$, or $r \approx 355.459$, then assumption (vii*) will be hold by assuming one of this values. As all requirements of Theorem 2.6 are hold, so inclusion (2.3) has at least one solution $x \in C(I)$.

2.3.1. *Continuous dependence on the functions* $\ltimes_i$

Definition 2.7. The solutions of (2.1) are continuously dependent on the functions $\ltimes_i(t,s)$, $i = 1,2$, if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, then

$$|\ltimes_i(t,s) - \ltimes_i^*(t,s)| \leqslant \delta \;\Rightarrow\; \|x - x^*\| \leqslant \epsilon.$$

Theorem 2.8. *Assume that conditions of Theorem 2.5 hold. Then the solution of (2.1) is continuously dependent on* $\ltimes_i(t,s)$, $i = 1,2$.

Proof. For given $\delta > 0$, with $|\ltimes_i(t,s) - \ltimes_i^*(t,s)| \leqslant \delta$, $\forall t \geqslant 0$, we obtain

$$\begin{aligned}
|x(t) - x^*(t)| = {}& |\eta(t,x) + \omega(t,x) \int_0^1 \phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\; d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\
& - \eta(t,x^*) + \omega(t,x^*) \int_0^1 \phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2^*(s,\theta)\right) d_s \ltimes_1^*(t,s))| \\
\leqslant {}& |\eta(t,x) - \eta(t,x^*)| + \omega(t,x)| \int_0^1 \phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\; d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\
& - \omega(t,x^*) \int_0^1 \phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\
& + \omega(t,x^*) \int_0^1 \phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s)) \\
& - \omega(t,x^*) \int_0^1 \phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2^*(s,\theta)\right) d_s \ltimes_1^*(t,s)| \\
\leqslant {}& |\eta(t,x) - \eta(t,x^*)| + |\omega(t,x)| \int_0^1 \left|\phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\; d_\theta \ltimes_2(s,\theta)\right)\right. \\
& \left. - \phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2(s,\theta)\right)\right| d_s \ltimes_1(t,s) \\
& + |\omega(t,x) - \omega(t,x^*)| \int_0^1 \left|\phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2(s,\theta)\right)\right| d_s \ltimes_1(t,s) \\
& + \omega(t,x^*) \int_0^1 \phi\left(t,s,x^*(s), \int_0^1 \hbar(s,\theta,x^*(\theta))\; d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s)
\end{aligned}$$

$$
\begin{aligned}
&\quad -\omega(t,x^*)\int_0^1 \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)\Big) d_s \ltimes_1^*(t,s)\Big| \\
&\leqslant l_1|x(t)-x^*(t)| + [l_2|x(t)|+G]\int_0^1 \kappa_1\Big(|x(s)-x^*(s)| \\
&\quad + \int_0^1 |\hbar(s,\theta,x(\theta))-\hbar(s,\theta,x^*(\theta))|\ d_\theta \ltimes_2(s,\theta)\Big) d_s \ltimes_1(t,s) \\
&\quad + l_2|x(t)-x^*(t)|\int_0^1 \varpi_1+\kappa_1\Big[|x^*(s)|+\int_0^1 |\hbar(s,\theta,x^*(\theta))|\ d_\theta \ltimes_2(s,\theta)\Big)\Big] d_s \ltimes_1(t,s) \\
&\quad + |\omega(t,x^*)|\Big|\int_0^1 \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2(s,\theta)\Big) d_s \ltimes_1(t,s)) \\
&\quad - \int_0^1 \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)\Big) d_s \ltimes_1(t,s)) \\
&\quad + \int_0^1 \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)\Big) d_s \ltimes_1(t,s)) \\
&\quad - \int_0^1 \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)) d_s \ltimes_1^*(t,s))\Big| \\
&\leqslant l_1|x(t)-x^*(t)| + [l_2|x(t)|+G]\int_0^1 \kappa\Big(|x(s)-x^*(s)|+\int_0^1 \kappa|x(\theta)-x^*(\theta)|\ d_\theta \ltimes_2(s,\theta)\Big) d_s \ltimes_1(t,s) \\
&\quad + l_2\big|x(t)-x^*(t)\big|\int_0^1\Big(\varpi_1+\kappa_1\big(|x^*(t)|+\int_0^1(\varpi_2+\kappa_2|x^*(\theta)|)d_\theta \ltimes_2(s,\theta)\big)\Big) d_s \ltimes_1(t,s) \\
&\quad + [l_2|x^*(t)|+G]\int_0^1 \Big|\phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2(s,\theta)\Big) \\
&\quad - \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)\Big)\Big| d_s \ltimes_1(t,s)) \\
&\quad + [l_2|x^*(t)|+G]\int_0^1 \Big|\phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta g_2^*(s,\theta)\Big)\Big|[d_s \ltimes_1(t,s)-d_s \ltimes_1^*(t,s)] \\
&\leqslant l_1|x(t)-x^*(t)| \\
&\quad + [l_2|x(t)|+G]\int_0^1 \kappa\Big(|x(s)-x^*(s)|+\int_0^1 \kappa|x(\theta)-x^*(\theta)|\ d_\theta \ltimes_2(s,\theta)\Big) d_s \ltimes_1(t,s) \\
&\quad + l_2\big|x(t)-x^*(t)\big|\int_0^1\Big(\varpi_1+\kappa_1\big(|x^*(t)|+\int_0^1(\varpi_2+\kappa_2|x^*(\theta)|)d_\theta \ltimes_2(s,\theta)\big)\Big) d_s \ltimes_1(t,s) \\
&\quad + [l_2|x^*(t)|+G] \\
&\quad \times \int_0^1 \kappa\Big(\Big|\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2(s,\theta)-\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)\Big|\Big) d_s \ltimes_1(t,s)) \\
&\quad + [l_2|x^*(t)|+G]\int_0^1 \Big|\phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2^*(s,\theta)\Big)\Big|[d_s \ltimes_1(t,s)-d_s \ltimes_1^*(t,s)] \\
&\leqslant l_1|x(t)-x^*(t)| + [l_2|x(t)|+G]\int_0^1 \kappa\Big(|x(s)-x^*(s)|+\int_0^1 \kappa|x(\theta)-x^*(\theta)|\ d_\theta \ltimes_2(s,\theta)\Big) d_s \ltimes_1(t,s) \\
&\quad + l_2\big|x(t)-x^*(t)\big|\int_0^1\Big(\varpi_1+\kappa_1\big(|x^*(t)|+\int_0^1(\varpi_2+\kappa_2|x^*(\theta)|)d_\theta \ltimes_2(s,\theta)\big)\Big) d_s \ltimes_1(t,s) \\
&\quad + [l_2|x^*(t)|+G]\int_0^1 \kappa\Big(\int_0^1 \big|\hbar(s,\theta,x^*(\theta))\big|[\ d_\theta \ltimes_2(s,\theta)-\ d_\theta \ltimes_2^*(s,\theta)]\Big) d_s \ltimes_1(t,s)) \\
&\quad + [l_2|x^*(t)|+G]
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^1 \left[\varpi + k\big(|x^*(s)| + \int_0^1 |\hbar(s,\theta,x^*(\theta))|\, d_\theta \ltimes_2^* (s,\theta)\big)\right][d_s \ltimes_1 (t,s) - d_s \ltimes_1^* (t,s)] \\
\leqslant\ & l_1|x(t)-x^*(t)| \\
&+ [l_2|x(t)|+G]\int_0^1 \kappa\left(|x(s)-x^*(s)| + \int_0^1 \kappa|x(\theta)-x^*(\theta)|\, d_\theta \ltimes_2 (s,\theta)\right) d_s \ltimes_1 (t,s) \\
&+ l_2\big|x(t)-x^*(t)\big|\int_0^1\left(\varpi_1 + \kappa_1\big(|x^*(t)| + \int_0^1 (\varpi_2 + \kappa_2|x^*(\theta)|) d_\theta \ltimes_2 (s,\theta)\big)\right) d_s \ltimes_1 (t,s) \\
&+ [l_2|x^*(t)|+G]\int_0^1 \kappa\Big(\int_0^1 [\varpi + \kappa|x^*(\theta)|][\, d_\theta \ltimes_2 (s,\theta) - \ d_\theta \ltimes_2^* (s,\theta)]\Big) d_s \ltimes_1 (t,s) + [l_2|x^*(t)|+G] \\
&\times \int_0^1 \left[\varpi + k\big(|x^*(s)| + \int_0^1 [\varpi + \kappa|x^*(\theta)|]\, d_\theta \ltimes_2^* (s,\theta)\big)\right][d_s \ltimes_1 (t,s) - d_s \ltimes_1^* (t,s)] \\
\leqslant\ & l_1\|x-x^*\| + [l_2\ r\ +G]\big[\kappa\mu\|x-x^*\| + \kappa^2\mu^2\|x-x^*\|\big] \\
&+ l_2\|x-x^*\|[(\varpi+\kappa\|x\|)\mu + ((\varpi+\kappa\|x\|)\mu)\mu] + [l_2\ r\ +G]\kappa[\varpi+\kappa r]\mu[\ltimes_2(s,1) - \ \ltimes_2^* (s,1)] \\
&+ [l_2\ r\ +G][\varpi + \kappa[r+\varpi+\kappa r]]\mu[\ltimes_1(t,1) - \ \ltimes_1^* (t,1)],
\end{aligned}
$$

taking supremum over $t \in I$, we obtain

$$
\begin{aligned}
\|x-x^*\| \leqslant\ & l_1\|x-x^*\| + [l_2\ r\ +G]\big[\kappa\mu\|x-x^*\| + \kappa^2\mu^2\|x-x^*\|\big] \\
&+ l_2\|x-x^*\|[(\varpi+\kappa\|x\|)\mu + ((\varpi+\kappa\|x\|)\mu)\mu] + [l_2\ r\ +G]\kappa[\varpi+\kappa r]\mu\delta \\
&+ [l_2\ r\ +G][\varpi+\kappa[r+\varpi+\kappa r]]\mu\delta,
\end{aligned}
$$

then

$$
\|x-x^*\| \leqslant \frac{[l_2\ r\ +G]\kappa[\varpi+\kappa r]\mu\delta + [l_2\ r\ +G][\varpi+\kappa[r+\varpi+\kappa r]]\mu\delta}{1-(l_1+[l_2\ r\ +G][\kappa\mu+\kappa^2\mu^2] + l_2[(\varpi+\kappa\|x\|)\mu + ((\varpi+\kappa\|x\|)\mu)\mu])} = \epsilon.
$$

As a result, we can deduce the solution of (2.1) is continuously dependence on $\ltimes_i,\ i=1,2$. □

Now, we have the following corollary.

Corollary 2.9. *Let assumptions of Theorem 2.8 be satisfied. Then the solutions of the inclusions (2.1) continuously dependence on functions* $\ltimes_i,\ i=1,2$.

2.3.2. Continuous dependence on the set of selection S_Φ

Definition 2.10. The solutions of the inclusion (1.1) depend continuously on the set S_Φ, if $\forall\ \epsilon>0,\ \exists\ \delta>0$, such that

$$
|\phi(t,s,x,y) - \phi^*(t,s,x,y)| < \delta,\quad \phi,\ \phi^* \in S_\Phi\ ,\quad t \in I.
$$

Then $\|x-x^*\| < \epsilon$, where x and x^* are two solutions of inclusion (1.1) corresponding to the two selections $\phi,\ \phi^* \in S_\Phi$.

Theorem 2.11. *Let assumptions of Theorem 2.5 be satisfied. Then the solutions of the inclusion (1.1) depend continuously on the set* S_Φ *of all Lipschitzian selections of* Φ.

Proof. Let $x(t)$ and $x^*(t)$ be the two solutions of inclusion (1.1) corresponding to the two selections $\phi,\ \phi^* \in S_\Phi$, we have

$$
\begin{aligned}
|x(t)-x^*(t)| = |&\eta(t,x) + \omega(t,x)\int_0^1 \phi\left(t,s,x(s),\int_0^1 \phi(s,\theta,x(\theta))\ d_\theta \ltimes_2 (s,\theta)\right) d_s \ltimes_1 (t,s) \\
&- \eta(t,x^*) + \omega(t,x^*)\int_0^1 \phi^*\left(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))\ d_\theta \ltimes_2 (s,\theta)\right) d_s \ltimes_1 (t,s)|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant |\eta(t,x)-\eta(t,x^*)| + \Big|\omega(t,x)\int_0^1 \phi\Big(t,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\Big) \\
&\quad - \omega(t,x^*)\phi^*\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))d_\theta \ltimes_2(s,\theta))\Big)d_s \ltimes_1(t,s)\Big| \\
&\leqslant |\eta(t,x)-\eta(t,x^*)| + |\omega(t,x)|\int_0^1 \Big|\phi\Big(t,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))d_\theta \ltimes_2(s,\theta)\Big) \\
&\quad - \phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))d_\theta \ltimes_2(s,\theta)\Big)\Big|d_s \ltimes_1(t,s) \\
&\quad + |\omega(t,x)|\int_0^1 \Big|\phi\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))d_\theta \ltimes_2(s,\theta)\Big) \\
&\quad - \phi^*\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))d_\theta \ltimes_2(s,\theta)\Big)\Big|d_s \ltimes_1(t,s) \\
&\quad + |\omega(t,x)-\omega(t,x^*)|\int_0^1 \Big|\phi^*\Big(t,s,x^*(s),\int_0^1 \hbar(s,\theta,x^*(\theta))d_\theta \ltimes_2(s,\theta)\Big)\Big|d_s \ltimes_1(t,s) \\
&\leqslant l_1|x(t)-x^*(t)| + [l_2|x(t)|+G] \\
&\quad \times \int_0^1 \kappa_1\Big(|x(s)-x^*(s)| + \int_0^1 \Big|\hbar(s,\theta,x(\theta))-\hbar(s,\theta,x^*(\theta))\Big|d_\theta \ltimes_2(s,\theta)\Big)d_s \ltimes_1(t,s) \\
&\quad + [l_2|x(t)|+G]\delta\int_0^1 d_s \ltimes_1(t,s) \\
&\quad + l_2|x(t)-x^*(t)|\int_0^1\Big(\varpi_1+\kappa_1\big(|x^*(t)| + \int_0^1 |\hbar(s,\theta,x^*(\theta))|d_\theta \ltimes_2(s,\theta)\big)\Big)d_s \ltimes_1(t,s) \\
&\leqslant l_1|x(t)-x^*(t)| + [l_2|x(t)|+G] \\
&\quad \times \int_0^1 \kappa_1\Big(|x(s)-x^*(s)| + \int_0^1 \kappa_2|x(\theta)-x^*(\theta)|d_\theta \ltimes_2(s,\theta)\Big)d_s \ltimes_1(t,s) \\
&\quad + [l_2|x(t)|+G]\delta\int_0^1 d_s g_1(t,s) \\
&\quad + l_2|x(t)-x^*(t)|\int_0^1\Big(\varpi_1+\kappa_1\big(|x^*(t)| + \int_0^1 (\varpi_2+\kappa_2|x^*(\theta)|)d_\theta \ltimes_2(s,\theta)\big)\Big)d_s \ltimes_1(t,s) \\
&\leqslant l_1\|x-x^*\| + [l_2\|x\|+G][\kappa\mu\|x-x^*\| + k^2\mu^2\|x-x^*\|] + [l_2\|x\|+G]\delta\mu \\
&\quad + l_2\|x-x^*\|[(\varpi+\kappa\|x\|)\mu + ((\varpi+\kappa\|x\|)\mu)\mu].
\end{aligned}
$$

Hence

$$
\|x-x^*\| \leqslant \frac{[l_2\|x\|+G]\delta\mu}{1-(l_1+[l_2\|x\|+G][\kappa\mu+\kappa^2\mu^2]+l_2[(\varpi+\kappa\|x\|)\mu+((\varpi+\kappa\|x\|)\mu)\mu])} = \epsilon.
$$

As a result of the previous inequality, we obtain

$$
\|x-x^*\| \leqslant \epsilon.
$$

This establishes the solution's continuous dependence on the set S_Φ. □

Now, we have the following corollary.

Corollary 2.12. *Let assumptions of Theorem 2.11 be satisfied. Then the solutions of the inclusion (2.2) depend continuously on the set of selections* S_Φ.

3. Existence of solutions II

Conditions (i) (d) and (vii) are now being replaced by following.

(i**)

d*) There exist two continuous functions $\varpi_1,\ \kappa_1 : I \times I \to \mathbb{R}$, such that

$$\|\Phi(t,s,u,v)\| = \sup\{|\phi| : \phi \in \Phi(t,s,u,v)\} \leqslant \varpi_1(t,s) + \kappa_1(t,s)(|u|.|v|).$$

Note: From the assumptions (a)-(d*) we can deduce that (see [1, 3, 5, 11]) there exists $\phi \in \Phi(t,s,u,v)$, such that

$$\phi(t,s,u,v)\} \leqslant \varpi_1(t,s) + \kappa_1(t,s)(|u| \cdot |v|).$$

(vii**) The following cubic algebraic equation

$$(\kappa^2\ \mu^2\ l_2)\ l^3 + (\varpi\ \kappa\ \mu^2\ l_2\ G\ \varpi\ \mu + G\ \kappa^2\ \mu^2)\ l^2 + (l_1\ + G\ \varphi\ \kappa\ \mu^2 + \varpi\ \mu\ l_2\ - 1)\ l + \mathcal{H} = 0,$$

has a positive root l.

Theorem 3.1. *Let assumptions of Theorem 2.3 be satisfied with* (i) (d) *and* (vii) *being replaced by* (i**) (d**) *and* (vii*)*, respectively, then, there exists at least one solution* $x \in C(I)$ *for equation* (2.1)*.*

Proof. The A^* operator is defined as

$$A^*x(t) = \eta(t,x) + \omega(t,x) \int_0^1 \phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s),\ t \in I,$$

and the set Q_l by

$$Q_l = \{x \in R :\ |x| \leqslant l\} \subseteq C(I),$$

with l is an algebraic cubic equation's positive root of

$$(k^2\ \mu^2\ l_2)\ l^3 + (m\ \kappa\ \mu^2\ l_2\ G\ \varpi\mu + G\ k^2\ \mu^2)\ l^2 + (l_1\ + G\ \varpi\ k\ \mu^2 + \varpi\ \mu\ l_2\ - 1)\ l + \mathcal{H} = 0.$$

The set Q_l is obvious to be nonempty, closed, bounded, and convex.

For $x \in Q_l$, we have

$$\begin{aligned}
|A^*x(t)| &= |\eta(t,x) + \omega(t,x) \int_0^1 \phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s)| \\
&\leqslant |\eta(t,x)| + |\omega(t,x)| \int_0^1 \left|\phi\left(t,s,x(s), \int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \ltimes_2(s,\theta)\right)\right| d_s \ltimes_1(t,s) \\
&\leqslant [l_1|x(t)| + \mathcal{H}] + [l_2|x(t)| + G] \\
&\quad \times \int_0^1 \left(\varpi_1(t,s) + \kappa_1(t,s)(|x(t)| \cdot \left|\int_0^1 \hbar(s,\theta,x(\theta))\ d_\theta \varpi_2(s,\theta))\right)\right| d_s \ltimes_1(t,s) \\
&\leqslant [l_1|x(t)| + \mathcal{H}] + [l_2|x(t)| + G] \\
&\quad \times \int_0^1 \left(\varpi_1(t,s) + \kappa_1(t,s)(|x(t)| \cdot \int_0^1 (\varpi_2(s,\theta) + \kappa_2(s,\theta)|x(\theta)|\ d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\
&\leqslant (\ell_1\ l + \mathcal{H}) + (l_2\ l + G) \int_0^1 (\varpi_1(t,s) + \kappa_1(t,s)(|x(t)| \cdot (\varpi + \kappa\ l)\ \mu) d_s \ltimes_1(t,s) \\
&\leqslant (l_1\ l + \mathcal{H}) + (l_2\ l + G)(\varpi + \kappa(l \cdot (\varpi + \kappa l)\ \mu))\mu \leqslant l.
\end{aligned}$$

This demonstrates that $A^*: Q_l \to Q_l$ and on Q_l class $\{A^*x\}$ is uniformly bounded. Now define sets for $x \in Q_r$ and $z(s) = \int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)$,

$$\theta^*_\phi(\delta) = \sup\{|\phi(t_2,s,x,z) - \phi(t_1,s,x,z)| : \ t_1,t_2,s \in I,\ t_1 < t_2, |t_2 - t_1| < \delta,\ |x| \leqslant l,\ |y| \leqslant l\},$$
$$\theta^*_\eta(\delta) = \sup\{|\eta(t_2,x) - \eta(t_1,x)| : \ t_1,t_2 \in I,\ |t_1 - t_2| \leqslant \delta,\ \ |x| \leqslant l\},$$
$$\theta^*_\omega(\delta) = \sup\{|\omega(t_2,x) - \omega(t_1,x)| : \ t_1,t_2 \in I,\ |t_1 - t_2| \leqslant \delta,\ \ |x| \leqslant l\}.$$

Then, based on the function $\phi : I \times I \times Q_l \times Q_l \to \mathbb{R}$, uniform continuity and assumptions (i) and (ii), we deduce that $\theta^*_\phi(\delta) \to 0$, $\theta^*_\eta(\delta) \to 0$, and $\theta^*_\omega(\delta) \to 0$, as $\delta \to 0$ independent of $x, y \in Q_l$.

Now, let $t_2,\ t_1 \in [0,1]$, with $|t_2 - t_1| < \delta$.

$$\begin{aligned}
|A^*x(t_2) - A^*x(t_1)| &= \Big|\eta(t_2,x(t_2)) + \omega(t_2,x(t_2)) \int_0^1 \phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad - \eta(t_1,x(t_1)) - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_1,s,x(s),\int_0^1 h(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\leqslant |\eta(t_2,x(t_2)) - \eta(t_1,x(t_1))| \\
&\quad + \Big|\omega(t_2,x(t_2)) \int_0^1 \phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad + \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_2,s) \\
&\quad - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\quad + \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\quad - \omega(t_1,x(t_1)) \int_0^1 \phi\Big(t_1,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) d_s \ltimes_1 (t_1,s)\Big| \\
&\leqslant \big|\eta(t_2,x(t_2)) - \eta(t_2,x(t_1))\big| + \big|\eta(t_2,x(t_1)) - \eta(t_1,x(t_1))\big| \\
&\quad + \big[\big|\omega(t_2,x(t_2)) - \omega(t_2,x(t_1))\big| + \big|\omega(t_2,x(t_1)) - \omega(t_1,x(t_1))\big|\big] \\
&\quad \times \int_0^1 \Big|\phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big)\Big| \ltimes_1 (t_2,s)\big| + \big|\omega(t_1,x(t_1))\big| \\
&\quad \times \int_0^1 \Big|\phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big)\Big| \big|d_s \ltimes_1 (t_2,s) - d_s \ltimes_1 (t_1,s)\big| \\
&\quad + \big|\omega(t_1,x(t_1))\big| \int_0^1 \Big|\phi\Big(t_2,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big) \\
&\quad - \phi\Big(t_1,s,x(s),\int_0^1 \hbar(s,\theta,x(\theta))\, d_\theta \ltimes_2 (s,\theta)\Big)\Big| \ltimes_1 (t_1,s) \\
&\leqslant l_1|x(t_2) - x(t_1)| + \theta_\eta(\delta_1) \\
&\quad + \big[l_2|x(t_2) - x(t_1)| + \theta_\omega(\delta_2)\big] \int_0^1 \big(\varpi_1(t,s) + \kappa_1(t,s)(|x| + |y|)\big) \ \ltimes_1 (t_2,s) \\
&\quad + [l_2|x(t_1)| + G] \int_0^1 (\varpi_1(t,s) + \kappa_1(t,s)(|x| \cdot |y|)) d_s[\ltimes_1(t_2,s) - \ltimes_1(t_1,s)] \\
&\quad + \int_0^1 \theta(\delta) d_s g_1(t_2,s) + \int_0^1 (\varpi_1(t,s) + \kappa_1(t,s)(|x| \cdot |y|)) d_s[\ltimes_1(t_2,s) - \ltimes_1(t_1,s)].
\end{aligned}$$

The inequality above indicates the equicontinuous for class of functions $\{A^*x\}$. A^* is compact as a result of the Arzela-Ascoli Theorem [33].

Assume $\{x_n\} \subset Q_l, \quad x_n \to x$. Then

$$Ax_n(t) = \eta(t, x_n(t)) + \omega(t, x_n(t)) \int_0^1 \phi\left(t, s, x_n(s), \int_0^1 \hbar(s, \theta, x_n(\theta))\ d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s),$$

$$\lim_{n\to\infty} Ax_n(t) = \lim_{n\to\infty} (\eta(t, x_n(t)) + \omega(t, x_n(t)) \int_0^1 \phi\left(t, s, x_n(s), \int_0^1 \hbar(s, \theta, x_n(\theta)) d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s)).$$

Thus we can get from assumption (ii*) that (see [30])

$$\begin{aligned}\lim_{n\to\infty} A^*x_n(t) &= \lim_{n\to\infty} (\eta(t, x_n(t)) \\ &\quad + \lim_{n\to\infty} \omega(t, x_n(t)) \int_0^1 \lim_{n\to\infty} \phi\left(t, s, x_n(s), \int_0^1 \hbar(s, \theta, x_n(\theta))\ d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\ &= \lim_{n\to\infty} (\eta(t, x_n(t)) \\ &\quad + \lim_{n\to\infty} \omega(t, x_n(t)) \int_0^1 \phi\left(t, s, \lim_{n\to\infty} x_n(s), \int_0^1 \hbar(s, \theta, \lim_{n\to\infty} x_n(\theta)) d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\ &= \eta(t, x(t)) + \omega(t, x(t)) \int_0^1 \phi\left(t, s, x(s), \int_0^1 \hbar(s, \theta, x(\theta))\ d_\theta \ltimes_2(s,\theta)\right) d_s \ltimes_1(t,s) \\ &= A^*x(t).\end{aligned}$$

This establishes the continuity of A^* with $A^*x_n(t) \to A^*x(t)$. As a result (see [30]), the operator A^* has at least one fixed point $x \in Q_l$ and there exists at least one solution $x \in Q_l \subset C(I)$ for (2.1). □

3.1. *Application*

1- Assume $\phi(t, s, u, v) = \phi(t, s, u.v)$, $\hbar(t, s, u(s)) = \sigma_2(t)u(s)$,

$$\ltimes_1(t,s) = \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0,1], \quad s \in I, \\ 0, & \text{for } t = 0, \quad s \in I, \end{cases}$$

and

$$\ltimes_2(s,\theta) = \begin{cases} s \ln \frac{s+\theta}{s}, & \text{for } s \in (0,1], \quad \theta \in I, \\ 0, & \text{for } s = 0, \quad \theta \in I, \end{cases}$$

in equation (2.1), then we can see that $\ltimes_1$, $\ltimes_2$ satisfies our assumptions (iv)-(vi). We get the nonlinear Chandrasekhar functional integral equation as a result,

$$x(t) = \eta(t, x(t)) + \omega(t, x(t)) \int_0^1 \frac{t}{t+s} \phi\left(t, s, x(s). \int_0^1 \frac{s}{s+\theta} \sigma_2(s) x(\theta) d\theta\right) ds. \tag{3.1}$$

2- Let $\phi(t, s, x(s).z(s)) = \sigma_1(s)x(s) \cdot z(s)$, $\omega(t, x(t)) = 1$, and

$$z(s) = \int_0^1 \frac{s}{s+\theta} \sigma_2(s) x(\theta) d\theta,$$

in equation (3.1), the Chandrasekhar quadratic functional integral equation of the following form is obtained as a result

$$x(t) = \eta(t, x(t)) + \int_0^1 \frac{t}{t+s} \sigma_1(s) x(s) \cdot \left(\int_0^1 \frac{s}{s+\theta} \sigma_2(s) x(\theta) d\theta\right) ds. \tag{3.2}$$

For $\omega(t, x(t)) = x(t)$, the Chandrasekhar cubic functional integral equation of the following form is obtained as a result

$$x(t) = \eta(t, x(t)) + x(t) \int_0^1 \frac{t}{t+s} \sigma_1(s) x(s) \cdot \left(\int_0^1 \frac{s}{s+\theta} \sigma_2(s) x(\theta) d\theta \right) ds, \tag{3.3}$$

that are special cases of equation (2.1). The Chandrasekhar quadratic and cubic functional integral equations (3.2) and (3.3) have at least one solution $x \in C(I)$ given the conditions of Theorem 3.1 hold.

3.2. *Example*

Take into account the hybrid integral inclusion of Chandrasekhar quadratic type

$$\frac{x(t) - \frac{5e^t}{7} \frac{|\cos x(t)|}{1+|\cos x(t)|}}{\frac{e^{-\pi}}{1+t} + \frac{e^{-\ln(t+1)}|x(t)|}{1+|x(t)|}} \in \left[0, \int_0^1 \left[\frac{e^{-s}}{9+e^t} + \frac{t}{t+s} \cdot \frac{2\cos(s)x(s)}{7e^{2s}\ (1+\cos^2(s))} \cdot \left(\int_0^1 \frac{s}{s+\theta} \cdot \frac{\sin(s)}{4(1+\sin^2(s))}\ x(\theta) d\theta \right) \right] ds \right]. \tag{3.4}$$

At the beginning let us notice that (3.4) is a particular case of (3.2) if we put

$$\begin{aligned}
\eta(t, x) &= \frac{1}{2(t+1)} + \frac{5e^{-t}}{7} \frac{|\cos x(t)|}{1+|\cos x(t)|}, \\
\omega(t, x) &= \frac{e^{-\pi t}}{1+t} + \frac{e^{-\ln(t+1)}|x(t)|}{1+|x(t)|}, \\
h(t, s, x(s)) &= \frac{\sin(t)}{4(1+\sin^2(t))}\ x(s), \\
\phi(t, s, x(s), y(s)) &= \frac{e^{-s}}{9+e^t} + \frac{2\cos(s)x(s)}{7e^{2s}\ (1+\cos^2(s))} \cdot y(s), \\
y(s) &= \int_0^1 \frac{s}{s+\theta} \frac{\sin(s)}{4(1+\sin^2(s))} x(\theta) d\theta,
\end{aligned}$$

$\sigma_1(s) = \frac{2\cos(s)}{7e^{2s}\ (1+\cos^2(s))}$, $\sigma_2(s) = \frac{\sin(s)}{4(1+\sin^2(s))}$, with $\kappa_1 = \frac{2}{7}$, $\kappa_2 = \frac{1}{4}$, $l_1 = \frac{5}{7e}$, and $l_2 = \frac{1}{e}$.

As a result, (i), (ii*), and (iii) are hold with $\kappa = \frac{1}{4}$ and $\varpi = \frac{1}{10}$, and $\mathcal{H} = \frac{1}{2}$, and $G = \frac{1}{e^\pi}$. We can derive from the above facts that the condition (vi*) of form

$$(\kappa^2\ \mu^2\ l_2)\ l^3 + (\varpi\ \kappa\ \mu^2\ l_2\ G\ \varpi\mu + G\ \kappa^2\ \mu^2)\ l^2 + (l_1\ + G\ \varpi\ k\ \mu^2 + \varpi\ \mu\ l_2\ - 1)\ l + \mathcal{H} = 0,$$

has an l positive solution . If we choose one of these numbers, for example, $l \approx 5.191049018$, or $l \approx 0.6637589$, then assumption (vi*) will be verified.

Equation (3.4) has at least one solution $x \in C(I)$, since all of the requirements of Theorem 3.1 are fulfilled.

4. Conclusions

Here, we proved two existence results for the nonlinear hybrid integral Urysohn-Stieltjes inclusion (1.1). As applications, we discuss the nonlinear Chandrasekhar functional integral inclusion (1.2) and a nonlinear hybrid Chandrasekhar functional integral equation and for the nonlinear cubic Chandrasekhar integral equation (1.3). The continues dependence of solution functions $\ltimes_i$ $(i = 1, 2)$ and on the set S_Φ have been obtained. Additionally, illustrative examples for the obtained results are constructed.

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