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Stabilities and instabilities of additive-quadratic 3D functional equations in paranormed spaces

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Abstract

This paper deals with the Ulam-Hyers stability of the following additive-quadratic mixed type functional equation:

$$f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right)=\frac{7}{8}\left(f(u-v)+f(v-w)+f(w-u)\right)-\frac{1}{8}\left(f(v-u)+f(w-v)+f(u-w)\right)$$

in paranormed spaces by direct method.

Keywords: Ulam-Hyers stability, additive functional equation, quadratic functional equation, paranormed spaces.

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1. Introduction

A classical question in the theory of functional equations is the following “whether it is true that a function which approximately satisfies a functional equation ϵ must be close to an exact solution ϵ? If the problem accepts a solution, can we say that the equation ϵ is stable”.

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. The stability question is: how do the solutions of the inequality differ from those of the given functional equation?

In the fall of 1940, Ulam [65] gave a wide-ranging talk before a mathematical colloquium at the university of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exists $\delta(\epsilon) > 0$ such that if $h : G_1 \to G_2$ satisfies the inequality

$$d\left(h(x * y), h(x) \diamond h(y)\right) < \delta, \quad \forall\ x, y \in G_1,$$

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then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

In the next year, Hyers [20] gave a affirmative answer to this question for additive groups under the assumption that groups are Banach spaces. He brilliantly answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated as follows.

Theorem 1.1. *Let $f : E_1 \to E_2$ be a function between Banach spaces such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon \tag{1.1}$$

for all $x, y \in E_1$ and $\epsilon > 0$ is a constant. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{1.2}$$

exists for each $x \in E_1$ and $A : E_1 \to E_2$ is unique additive mapping satisfying

$$\|f(x) - A(x)\| \leqslant \epsilon$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Taking this famous result into consideration, the additive Cauchy functional equation $f(x + y) = f(x) + f(y)$ is said to have the *Hyers-Ulam stability* on (E_1, E_2) if for every function $f : E_1 \to E_2$ satisfying the inequality (1.1) for some $\epsilon \geqslant 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.

The method in (1.2) provided by Hyers which produces the additive function A will be called a *direct method*. This method is the most important and powerful tool to study the stability of various functional equations.

It is possible to prove a stability result similar to Hyers functions that do not have bounded Cauchy difference. Aoki (1950) [5] first generalized the Hyers theorem for unbounded Cauchy difference having sum of norms $(\|x\|^p + \|y\|^p)$.

The same result was rediscovered by Rassias [53] in 1978 and proved a generalization of Hyers theorem for additive mappings. This stability result is named *Hyers-Ulam-Rassias stability* or *Hyers-Ulam-Aoki-Rassias stability* for the functional equation.

In 1982, Rassias [54], followed the innovative approach of Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^q$ with $p + q \neq 1$. Later this stability result was called *Ulam-Gavruta-Rassias stability* of functional equation.

In 1990, Rassias during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem in [56] can also be proved for value of p greater or equal to 1. In 1991, Gajda [18] provided an affirmative solution to Rassias' question for p strictly greater than one.

In 1994, Găvruţa [19] provided a further generalization of Rassias [53] theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. This stability result is called *generalized Hyers-Ulam-Rassias stability* of functional equation.

In 2008, a special case of Găvruţa's theorem for the unbounded Cauchy difference was obtained by Ravi et al. [62] by considering the summation of both the sum and the product of two p-norms in the sprit of Rassias approach and is named *Rassias Stability* of functional equation.

In 2021, Rassias and Karthikeyan [60] dealt with the Ulam-Hyers stability of the additive-quadratic mixed type functional equation in modular spaces using direct method. Recently, Karthikeyan et al. [35] proved the Ulam-Hyers stability of the additive-quartic functional equation in modular spaces by using the direct method.

During the last eight decades in the history of functional equations the mathematicians introduced and investigated the solution and stability of various types of functional equations (see [4, 6–9, 11, 13–15, 21, 23–29, 32–37, 39, 48–50, 52, 59, 61, 66]).

The additive functional equation is

$$f(x+y) = f(x) + f(y). \tag{1.3}$$

Since $f(x) = kx$ is the solution of the functional equation (1.3), every solution of the additive functional equation is called an additive mapping.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.4}$$

is said to be a quadratic functional equation. Since the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.4), every solution of the quadratic functional equation is called a quadratic mapping.

This paper presents the Ulam-Hyers stability of the additive-quadratic mixed type functional equation of the form

$$\begin{aligned} & f\left(\frac{u+v}{2} - w\right) + f\left(\frac{v+w}{2} - u\right) + f\left(\frac{w+u}{2} - v\right) \\ & \quad = \frac{7}{8}\left(f(u-v) + f(v-w) + f(w-u)\right) - \frac{1}{8}\left(f(v-u) + f(w-v) + f(u-w)\right) \end{aligned} \tag{1.5}$$

in paranormed spaces by direct method.

2. Paranormed space stability results

This section deals with the stability results of the functional equation (1.5) in paranormed spaces using direct method.

Now, we recall the basic definitions and notations in paranormed space. The concept of statistical convergence for sequences of real numbers was introduced by Fast [16] and Steinhaus [64] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [17, 31, 43, 44, 63]). This notion was defined in normed spaces by Kolk [38].

We recall some basic facts concerning Fréchet spaces.

Definition 2.1 ([67])**.** Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(P1) $P(0) = 0$;
(P2) $P(-x) = P(x)$;
(P3) $P(x+y) \leqslant P(x) + P(y)$ (triangle inequality);
(P4) if $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

Definition 2.2 ([67])**.** The paranorm is called total if, in addition, we have

(P5) $P(x) = 0$ implies $x = 0$.

Definition 2.3 ([67])**.** A Fréchet space is a total and complete paranormed space.

2.1. *Paranormed spaces: direct method*

This section deals with the Ulam-Hyers stability results of additive-quadratic functional equation (1.5) using Hyers' direct method in paranormed spaces.

Throughout this section, let $(\mathcal{U}, P)$ be a Fréchet space and $(\mathcal{V}, \|\cdot\|)$ be a Banach space.

For the convenience, we define the mappings $Df, Df_o, Df_q : \mathcal{U}^3 \to \mathcal{V}$ by

$$Df(u,v,w) = f\left(\frac{u+v}{2} - w\right) + f\left(\frac{v+w}{2} - u\right) + f\left(\frac{w+u}{2} - v\right)$$

$$-\frac{7}{8}\left(f(u-v)+f(v-w)+f(w-u)\right)+\frac{1}{8}\left(f(v-u)+f(w-v)+f(u-w)\right)$$

for all $u, v, w \in \mathcal{U}$,

$$D\ f_o(u,v,w)=f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right)-\left(f(u-v)+f(v-w)+f(w-u)\right)$$

for all $u, v, w \in \mathcal{U}$ and

$$D\ f_q(u,v,w)=f\left(\frac{u+v}{2}-w\right)+f\left(\frac{v+w}{2}-u\right)+f\left(\frac{w+u}{2}-v\right)-\frac{3}{4}\left(f(u-v)+f(v-w)+f(w-u)\right)$$

for all $u, v, w \in \mathcal{U}$.

2.2. *Additive stability results*

In this subsection, the authors discussed the Ulam-Hyers stability results of additive functional equation (1.5) using Hyers' direct method in paranormed spaces.

Theorem 2.4. *Let $\xi : \mathcal{U} \to [0, \infty)$ be a function with the condition*

$$\sum_{n=0}^{\infty}\frac{1}{2^n}\xi\left(2^n u, 2^n v, 2^n w\right) < +\infty \tag{2.1}$$

for all $u, v, w \in \mathcal{U}$. Suppose that a mapping $f_o : \mathcal{U} \to \mathcal{V}$ satisfies the following inequality

$$P\left(Df_o(u,v,w)\right) \leqslant \xi(u,v,w) \tag{2.2}$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_o(u)-\mathcal{A}(u)\right) \leqslant \sum_{n=0}^{\infty}\frac{1}{2^n}\xi\left(2^n u, 2^n u, -2^n u\right) \tag{2.3}$$

for all $u \in \mathcal{U}$. The mapping $\mathcal{A}(u)$ is defined by

$$P\left(\lim_{n\to\infty}\frac{1}{2^n}f_o\left(2^n u\right)-\mathcal{A}(u)\right) \to 0 \tag{2.4}$$

for all $u \in \mathcal{U}$.

Proof. Replacing (u, v, w) by $(u, u, -u)$ in (2.2), we get

$$P\left(2f_o\left(2u\right)-f_o(u)\right) \leqslant \xi(u,u,-u)$$

for all $u \in \mathcal{U}$. For any $m, n > 0$, we simplify

$$P\left(\frac{f_o\left(2^m u\right)}{2^{(m-1)}}-\frac{f_o\left(2^n u\right)}{2^n}\right) \leqslant \sum_{\ell=m}^{n-1}\frac{1}{2^\ell}\xi\left(2^\ell u, 2^\ell u, -2^\ell u\right) \tag{2.5}$$

for all $u \in \mathcal{U}$. It follows from (2.5) that the sequence $\left\{\frac{f_o\left(2^n u\right)}{2^n}\right\}$ is Cauchy sequence. Since $\mathcal{V}$ is complete, there exists a mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ by

$$P\left(\lim_{n\to\infty}\frac{f_o\left(2^n u\right)}{2^n}-\mathcal{A}(u)\right) \to 0$$

for all $u \in \mathcal{U}$. By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty} t_n \frac{f_o(2^n u)}{2^n} - t\mathcal{A}(u)\right) \to 0$$

for all $u \in \mathcal{U}$. Letting $m = 0$ and $n \to \infty$ in (2.5), we see that (2.3) holds for all $u \in \mathcal{U}$. To show that $\mathcal{A}$ satisfies (1.5), replacing (u, v, w) by $(2^n u, 2^n v, 2^n w)$ in (2.2), we get

$$P\left(\frac{1}{2^n} Df_o(2^n u, 2^n v, 2^n w)\right) \leqslant \frac{1}{2^n}\xi(2^n u, 2^n v, 2^n w)$$

for all $u, v, w \in \mathcal{U}$. Letting $n \to \infty$ in the above inequality and using the definition of $\mathcal{A}(u)$, we see that

$$P(D\mathcal{A}(u, v, w)) = 0 \tag{2.6}$$

for all $u, v, w \in \mathcal{U}$. Using condition (P5) in (2.6), we obtain

$$\begin{aligned}
&\mathcal{A}\left(\frac{u+v}{2} - w\right) + \mathcal{A}\left(\frac{v+w}{2} - u\right) + \mathcal{A}\left(\frac{w+u}{2} - v\right) \\
&\quad = \frac{7}{8}(\mathcal{A}(u-v) + \mathcal{A}(v-w) + \mathcal{A}(w-u)) - \frac{1}{8}(\mathcal{A}(v-u) + \mathcal{A}(w-v) + \mathcal{A}(u-w))
\end{aligned}$$

for all $u, v, w \in \mathcal{U}$. Hence $\mathcal{A}$ satisfies (1.5) for all $u, v, w \in \mathcal{U}$. In order to prove that $\mathcal{A}(u)$ is unique, let $\mathcal{A}'(u)$ be another additive mapping satisfying (1.5) and (2.3). Then

$$\begin{aligned}
P(\mathcal{A}(u) - \mathcal{A}'(u)) &= \left\{P\left(\frac{\mathcal{A}(2^m)}{2^m} - \frac{\mathcal{A}'(2^m)}{2^m}\right)\right\} \\
&\leqslant \left\{P\left(\frac{\mathcal{A}(2^m)}{2^m} - \frac{f_o(2^m)}{2^m}\right) + P\left(\frac{f_o(2^m)}{2^m} - \frac{\mathcal{A}'(2^m)}{2^m}\right)\right\} \\
&\leqslant \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell+m}}\xi\left(2^{\ell+m}u, 2^{\ell+m}v, 2^{\ell+m}w\right) \to 0 \text{ as } m \to \infty
\end{aligned}$$

for all $u \in \mathcal{U}$. Thus $P(\mathcal{A}(u) - \mathcal{A}'(u)) = 0$ for all $u \in \mathcal{U}$. Hence, we have $\mathcal{A}(u) = \mathcal{A}'(u)$. Therefore $\mathcal{A}(u)$ is unique. Thus the mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ is a unique additive mapping. This completes the proof. □

The following corollaries express the instant significance of the Theorem 2.4 concerning the Ulam-Hyers, Hyers- Ulam-Rassias, Ulam-Gavruta-Rassias and Rassias stability results of the functional equation (1.5).

Corollary 2.5. *Let $f_o : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P(Df_o(u, v, w)) \leqslant \sigma$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ such that

$$P(f_o(u) - \mathcal{A}(u)) \leqslant \frac{\sigma}{|1|}$$

for all $u \in \mathcal{U}$.

Corollary 2.6. *Let $f_o : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P(Df_o(u, v, w)) \leqslant \sigma\{P(u)^s + P(v)^s + P(w)^s\}, \qquad s \neq 1$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ such that

$$P(f_o(u) - \mathcal{A}(u)) \leqslant \frac{3\sigma P(u)^s}{|2 - 2^s|}$$

for all $u \in \mathcal{U}$.

Corollary 2.7. *Let $f_o : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P\left(Df_o(u,v,w)\right) \leqslant \sigma\, P(u)^s P(v)^s P(w)^s, \qquad s \neq \tfrac{1}{3}$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_o(u) - \mathcal{A}(u)\right) \leqslant \frac{\sigma P(x)^{3s}}{|2 - 2^{3s}|}$$

for all $u \in \mathcal{U}$.

Corollary 2.8. *Let $f_o : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P\left(Df_o(u,v,w)\right) \leqslant \sigma \left\{P(u)^s P(v)^s P(w)^s + \left\{P(u)^{3s} + P(v)^{3s} + P(w)^{3s}\right\}\right\}, \qquad s \neq \tfrac{1}{3}$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_o(u) - \mathcal{A}(u)\right) \leqslant \frac{4\sigma P(u)^{3s}}{|2 - 2^{3s}|}$$

for all $u \in \mathcal{U}$.

2.3. *Quadratic stability results*

In this subsection, the authors discussed the Ulam-Hyers stability results of quadratic functional equation (1.5) using Hyers' direct method in paranormed spaces.

Theorem 2.9. *Let $\xi : \mathcal{U} \to [0, \infty)$ be a function with the condition*

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \xi\left(2^n u, 2^n v, 2^n w\right) < +\infty \tag{2.7}$$

for all $u, v, w \in \mathcal{U}$. Suppose that a mapping $f_q : \mathcal{U} \to \mathcal{V}$ satisfies the following inequality

$$P\left(Df_q(u,v,w)\right) \leqslant \xi(u,v,w) \tag{2.8}$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_q(u) - \mathcal{Q}(u)\right) \leqslant \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \xi\left(2^n u, 2^n u, -2^n u\right) \tag{2.9}$$

for all $u \in \mathcal{U}$. The mapping $\mathcal{Q}(u)$ is defined by

$$P\left(\lim_{n\to\infty} \frac{1}{2^{2n}} f_q\left(2^n u\right) - \mathcal{Q}(u)\right) \to 0 \tag{2.10}$$

for all $u \in \mathcal{U}$.

Proof. Replacing (u, v, w) by $(u, u, -u)$ in (2.8), we get

$$P\left(2^2 f_q\left(2u\right) - f_q(u)\right) \leqslant \xi(u, u, -u)$$

for all $u \in \mathcal{U}$. For any $m, n > 0$, we simplify

$$P\left(\frac{f_q\left(2^m u\right)}{2^{2(m-1)}} - \frac{f_q\left(2^n u\right)}{2^{2n}}\right) \leqslant \sum_{\ell=m}^{n-1} \frac{1}{2^{2\ell}} \xi\left(2^\ell u, 2^\ell u, -2^\ell u\right) \tag{2.11}$$

for all $u \in \mathcal{U}$. It follows from (2.11) that the sequence $\left\{\frac{f_q(2^n u)}{2^{2n}}\right\}$ is Cauchy sequence. Since $\mathcal{V}$ is complete, there exists a mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ by

$$P\left(\lim_{n\to\infty} \frac{f_q(2^n u)}{2^{2n}} - \mathcal{Q}(u)\right) \to 0$$

for all $u \in \mathcal{U}$. By continuity of multiplication, we have

$$P\left(\lim_{n\to\infty} t_n \frac{f_q(2^n u)}{2^{2n}} - t\mathcal{Q}(u)\right) \to 0$$

for all $u \in \mathcal{U}$. Letting $m = 0$ and $n \to \infty$ in (2.11), we see that (2.9) holds for all $u \in \mathcal{U}$. To show that $\mathcal{Q}$ satisfies (1.5), replacing (u, v, w) by $(2^n u, 2^n v, 2^n w)$ in (2.8), we get

$$P\left(\frac{1}{2^{2n}} Df_q(2^n u, 2^n v, 2^n w)\right) \leqslant \frac{1}{2^{2n}} \xi(2^n u, 2^n v, 2^n w)$$

for all $u, v, w \in \mathcal{U}$. Letting $n \to \infty$ in the above inequality and using the definition of $\mathcal{Q}(u)$, we see that

$$P(D\mathcal{Q}(u, v, w)) = 0 \tag{2.12}$$

for all $u, v, w \in \mathcal{U}$. Using condition (P5) in (2.12), we obtain

$$\begin{aligned}
&\mathcal{Q}\left(\frac{u+v}{2} - w\right) + \mathcal{Q}\left(\frac{v+w}{2} - u\right) + \mathcal{Q}\left(\frac{w+u}{2} - v\right) \\
&\quad = \frac{7}{8}(\mathcal{Q}(u-v) + \mathcal{Q}(v-w) + \mathcal{Q}(w-u)) - \frac{1}{8}(\mathcal{Q}(v-u) + \mathcal{Q}(w-v) + \mathcal{Q}(u-w))
\end{aligned}$$

for all $u, v, w \in \mathcal{U}$. Hence $\mathcal{Q}$ satisfies (1.5) for all $u, v, w \in \mathcal{U}$. In order to prove that $\mathcal{Q}(u)$ is unique, let $\mathcal{Q}'(u)$ be another quadratic mapping satisfying (1.5) and (2.9). Then

$$\begin{aligned}
P(\mathcal{Q}(u) - \mathcal{Q}'(u)) &= \left\{P\left(\frac{\mathcal{Q}(2^m)}{2^{2m}} - \frac{\mathcal{Q}'(2^m)}{2^{2m}}\right)\right\} \\
&\leqslant \left\{P\left(\frac{\mathcal{Q}(2^m)}{2^{2m}} - \frac{f_q(2^m)}{2^{2m}}\right) + P\left(\frac{f_q(2^m)}{2^{2m}} - \frac{\mathcal{Q}'(2^m)}{2^{2m}}\right)\right\} \\
&\leqslant \sum_{\ell=0}^{\infty} \frac{1}{2^{2(\ell+m)}} \xi(2^{\ell+m} u, 2^{\ell+m} v, 2^{\ell+m} w) \to 0 \text{ as } m \to \infty
\end{aligned}$$

for all $u \in \mathcal{U}$. Thus $P(\mathcal{Q}(u) - \mathcal{Q}'(u)) = 0$ for all $u \in \mathcal{U}$. Hence, we have $\mathcal{Q}(u) = \mathcal{Q}'(u)$. Therefore $\mathcal{Q}(u)$ is unique. Thus the mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ is a unique quadratic mapping. This completes the proof. □

The following corollaries express the instant significance of the Theorem 2.9 concerning the Ulam-Hyers, Hyers- Ulam-Rassias, Ulam-Gavruta-Rassias and Rassias stability results of the functional equation (1.5).

Corollary 2.10. *Let $f_q : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P(Df_q(u, v, w)) \leqslant \sigma$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P(f_q(u) - \mathcal{Q}(u)) \leqslant \frac{\sigma}{|3|}$$

for all $u \in \mathcal{U}$.

Corollary 2.11. *Let $f_q : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P\left(Df_q(u,v,w)\right) \leqslant \sigma\{P(u)^s + P(v)^s + P(w)^s\}, \qquad s \neq 2$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_q(u) - \mathcal{Q}(u)\right) \leqslant \frac{3\sigma P(u)^s}{|2^2 - 2^s|}$$

for all $u \in \mathcal{U}$.

Corollary 2.12. *Let $f_q : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P\left(Df_q(u,v,w)\right) \leqslant \sigma\, P(u)^s P(v)^s P(w)^s, \qquad s \neq \tfrac{2}{3}$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_q(u) - \mathcal{Q}(u)\right) \leqslant \frac{\sigma P(x)^{3s}}{|2^2 - 2^{3s}|}$$

for all $u \in \mathcal{U}$.

Corollary 2.13. *Let $f_q : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P\left(Df_q(u,v,w)\right) \leqslant \sigma\left\{P(u)^s P(v)^s P(w)^s + \left\{P(u)^{3s} + P(v)^{3s} + P(w)^{3s}\right\}\right\}, \qquad s \neq \tfrac{2}{3}$$

for all $u, v, w \in \mathcal{U}$, then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f_q(u) - \mathcal{Q}(u)\right) \leqslant \frac{4\sigma P(u)^{3s}}{|2^2 - 2^{3s}|}$$

for all $u \in \mathcal{U}$.

2.4. *Additive-quadratic mixed stability results*

This subsection deals with the Ulam-Hyers stability results of additive-quadratic mixed type functional equation (1.5) using Hyers' direct method in paranormed spaces.

Theorem 2.14. *Let $\xi : \mathcal{U} \to [0, \infty)$ be a function with the conditions (2.1) and (2.7) for all $u, v, w \in \mathcal{U}$. Suppose that a mapping $f : \mathcal{U} \to \mathcal{V}$ satisfies the following inequality*

$$P\left(Df(u,v,w)\right) \leqslant \xi(u,v,w)$$

for all $u, v, w \in \mathcal{U}$, then there exist a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ and unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P\left(f(u) - A(u) - \mathcal{Q}(u)\right) \leqslant \frac{1}{2}\left\{\sum_{n=0}^{\infty} \frac{1}{2^n}\left(\xi\left(2^n u, 2^n u, -2^n u\right) + \xi\left(-2^n u, -2^n u, 2^n u\right)\right)\right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{1}{2^{2n}}\left(\xi\left(2^n u, 2^n u, -2^n u\right) + \xi\left(-2^n u, -2^n u, 2^n u\right)\right)\right\}$$

for all $u \in \mathcal{U}$. The mapping $\mathcal{A}$ and $\mathcal{A}$ are respectively defined in (2.4) and (2.10) for all $u \in \mathcal{U}$.

Proof. Let $f_q(u) = \frac{1}{2}\{f(u) + f(-u)\}$ for all $u \in V$. Then $f_q(0) = 0, f_q(u) = f_q(-u)$. Hence

$$\begin{aligned} P(Df_q(u,v,w)) &= \frac{1}{2}\{P(Df(u,v,w) + Df(-u,-v,-w))\} \\ &\leqslant \frac{1}{2}\{P(Df(u,v,w)) + P(Df(-u,-v,-w))\} \\ &\leqslant \frac{1}{2}\{\xi(u,v,w) + \xi(-u,-v,-w)\} \end{aligned}$$

for all $u, v, w \in \mathcal{U}$. Hence from Theorem 2.9, there exists a unique quadratic mapping $Q : \mathcal{U} \to \mathcal{V}$ such that

$$P(f_q(u) - Q(u)) \leqslant \frac{1}{2}\left\{\sum_{n=0}^{\infty} \frac{1}{2^n}\left(\xi(2^n u, 2^n u, -2^n u) + \xi(-2^n u, -2^n u, 2^n u)\right)\right\} \tag{2.13}$$

for all $u \in \mathcal{U}$.

Again $f_o(u) = \frac{1}{2}\{f(u) - f(-u)\}$ for all $u \in V$. Then $f_o(0) = 0, f_o(u) = -f_o(-u)$. Hence

$$\begin{aligned} P(Df_o(u,v,w)) &= \frac{1}{2}\{P(Df(u,v,w) + Df(u,v,w))\} \\ &\leqslant \frac{1}{2}\{P(Df(u,v,w)) + P(Df(-u,-v,-w))\} \\ &\leqslant \frac{1}{2}\{\xi(u,v,w) + \xi(-u,-v,-w)\} \end{aligned}$$

for all $u, v, w \in \mathcal{U}$. Hence from Theorem 2.4, there exists a unique additive mapping $A : \mathcal{U} \to \mathcal{V}$ such that

$$P(f_o(u) - A(u)) \leqslant \frac{1}{2}\left\{\sum_{n=0}^{\infty} \frac{1}{2^{2n}}\left(\xi(2^n u, 2^n u, -2^n u) + \xi(-2^n u, -2^n u, 2^n u)\right)\right\} \tag{2.14}$$

for all $u \in \mathcal{U}$.

Since $f(u) = f_q(u) + f_o(u)$, it follows from (2.13) and (2.14), we arrive

$$\begin{aligned} P(f(u) - A(u) - Q(u)) &= P(f_o(u) + f_q(u) - A(u) - Q(u)) \\ &\leqslant P(f_q(u) - Q(u)) + P(f_o(u) - A(u)) \\ &\leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\xi(2^i u, 2^i u, -2^i u)}{2^i} + \frac{\xi(-2^i u, -2^i u, -2^i u)}{2^i}\right)\right. \\ &\quad \left. + \sum_{i=0}^{\infty}\left(\frac{\xi(2^i u, 2^i u, -2^i u)}{2^{2i}} + \frac{\xi(-2^i u, -2^i u, 2^i u)}{2^{2i}}\right)\right\} \end{aligned}$$

for all $u \in \mathcal{U}$. Hence this completes the proof. □

The following corollaries express the instant significance of the Theorem 2.9 concerning the Ulam-Hyers, Hyers- Ulam-Rassias, Ulam-Gavruta-Rassias and Rassias stability results of the functional equation (1.5).

Corollary 2.15. *Let $f : \mathcal{U} \to \mathcal{V}$ be a mapping and assume that there exist real numbers σ and s such that*

$$P(Df(u,v,w)) \leqslant \sigma$$

for all $u, v, w \in \mathcal{U}$, then there exist a unique additive mapping $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ and unique quadratic mapping $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ such that

$$P(f(u) - \mathcal{A}(u) - \mathcal{Q}(u)) \leqslant \sigma\left(\frac{1}{|1|} + \frac{1}{|3|}\right)$$

for all $u \in \mathcal{U}$.

Corollary 2.16. *Let* $f : \mathcal{U} \to \mathcal{V}$ *be a mapping and assume that there exist real numbers* σ *and* s *such that*

$$P\left(Df(u,v,w)\right) \leqslant \sigma\{P(u)^s + P(v)^s + P(w)^s\}, \qquad s \neq 1,2$$

for all $u, v, w \in \mathcal{U}$*, then there exist a unique additive mapping* $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ *and unique quadratic mapping* $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ *such that*

$$P\left(f(u) - \mathcal{A}(u) - \mathcal{Q}(u)\right) \leqslant 3\sigma\left(\frac{1}{|2-2^s|} + \frac{1}{|2^2-2^s|}\right) P(u)^s$$

for all $u \in \mathcal{U}$.

Corollary 2.17. *Let* $f : \mathcal{U} \to \mathcal{V}$ *be a mapping and assume that there exist real numbers* σ *and* s *such that*

$$P\left(Df(u,v,w)\right) \leqslant \sigma\, P(u)^s P(v)^s P(w)^s, \qquad s \neq \tfrac{1}{3}, \tfrac{2}{3}$$

for all $u, v, w \in \mathcal{U}$*, then there exist a unique additive mapping* $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ *and unique quadratic mapping* $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ *such that*

$$P\left(f(u) - \mathcal{A}(u) - \mathcal{Q}(u)\right) \leqslant \sigma\left(\frac{1}{|2-2^{3s}|} + \frac{1}{|2^2-2^{3s}|}\right) P(x)^{3s}$$

for all $u \in \mathcal{U}$.

Corollary 2.18. *Let* $f : \mathcal{U} \to \mathcal{V}$ *be a mapping and assume that there exist real numbers* σ *and* s *such that*

$$P\left(Df(u,v,w)\right) \leqslant \sigma\left\{P(u)^s P(v)^s P(w)^s + \left\{P(u)^{3s} + P(v)^{3s} + P(w)^{3s}\right\}\right\}, \qquad s \neq \tfrac{1}{3}, \tfrac{2}{3}$$

for all $u, v, w \in \mathcal{U}$*, then there exist a unique additive mapping* $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ *and unique quadratic mapping* $\mathcal{Q} : \mathcal{U} \to \mathcal{V}$ *such that*

$$P\left(f(u) - \mathcal{A}(u) - \mathcal{Q}(u)\right) \leqslant 4\sigma\left(\frac{1}{|2-2^{3s}|} + \frac{1}{|2^2-2^{3s}|}\right) P(u)^{3s}$$

for all $u \in \mathcal{U}$.

3. Counter example for non stability cases

Now, we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = 1$ in Corollary 2.6.

Example 3.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\rho(u) = \begin{cases} \mu u, & \text{if } |u| < 1, \\ \mu, & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f_o(u) = \sum_{n=0}^{\infty} \frac{\rho(2^k u)}{2^{2k}}, \quad \text{for all } u \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df_o(u,v,w)| \leqslant \frac{15}{8}\mu\left(|u| + |v| + |w|\right) \tag{3.1}$$

for all $u, v, w \in \mathbb{R}$. Then, there does not exists a mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f_o(u) - \mathcal{A}(u)| \leqslant \beta|u|, \quad \text{for all } u \in \mathbb{R}. \tag{3.2}$$

Proof. Now,

$$|f_o(u)| \leqslant \sum_{n=0}^{\infty} \frac{|\rho(2^k u)|}{|2^k|} = \sum_{k=0}^{\infty} \frac{\mu}{2^k} = 2\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.1).

If $u = v = w = 0$, then (3.1) is trivial. If $|u|+|v|+|w| \geqslant \frac{1}{2}$ then the left hand side of (3.1) is less than $\frac{15}{8}\mu$. Now suppose that $0 < |u|+|v|+|w| < \frac{1}{2}$. Then there exists a positive integer ℓ such that

$$\frac{1}{2^{(\ell+1)}} \leqslant |u|+|v|+|w| < \frac{1}{2^{\ell}}, \tag{3.3}$$

so that $2^{(\ell-1)}|u| \leqslant \frac{1}{2}, 2^{(\ell-1)}|v| \leqslant \frac{1}{2}, 2^{(\ell-1)}|w| \leqslant \frac{1}{2}$ and consequently

$$\begin{aligned}&2^{(\ell+1)}\left(\frac{u+v}{2}-w\right), 2^{(\ell+1)}\left(\frac{v+w}{2}-u\right), 2^{(\ell+1)}\left(\frac{w+u}{2}-v\right), \frac{2^{(\ell+1)}.7}{8}(u-v),\\ &\frac{2^{(\ell+1)}.7}{8}(v-w), \frac{2^{(\ell+1)}.7}{8}(w-u), -\frac{2^{(\ell+1)}}{8}(v-u), -\frac{2^{(\ell+1)}}{8}(w-v), -\frac{2^{(\ell+1)}}{8}(u-w) \in (-1,1).\end{aligned}$$

Therefore for each $k = 0, 1, \ldots, \ell-1$, we have

$$\begin{aligned}&2^{k}\left(\frac{u+v}{2}-w\right), 2^{k}\left(\frac{v+w}{2}-u\right), 2^{k}\left(\frac{w+u}{2}-v\right), \frac{2^{k}.7}{8}(u-v),\\ &\frac{2^{k}.7}{8}(v-w), \frac{2^{k}.7}{8}(w-u), -\frac{2^{k}}{8}(v-u), -\frac{2^{k}}{8}(w-v), -\frac{2^{k}}{8}(u-w) \in (-1,1).\end{aligned}$$

and

$$\begin{aligned}&\rho\left(\frac{2^k(u+v)}{2}-2^k w\right)+\rho\left(\frac{2^k(v+w)}{2}-2^k u\right)+\rho\left(\frac{2^k(w+u)}{2}-2^k v\right)\\ &-\frac{7}{8}\left(\rho\left(2^k(u-v)\right)+\rho\left(2^k(v-w)\right)+\rho\left(2^k(w-u)\right)\right)\\ &+\frac{1}{8}\left(\rho\left(2^k(v-u)\right)+\rho\left(2^k(w-v)\right)+\rho\left(2^k(u-w)\right)\right)=0\end{aligned}$$

for $k = 0, 1, \ldots, \ell-1$. From the definition of f and (3.3), we obtain that

$$\begin{aligned}\left|Df_o(u,v,w)\right| &\leqslant \sum_{k=0}^{\infty}\frac{1}{2^k}\Big|\rho\left(\frac{2^k(u+v)}{2}-2^k w\right)+\rho\left(\frac{2^k(v+w)}{2}-2^k u\right)+\rho\left(\frac{2^k(w+u)}{2}-2^k v\right)\\ &\quad-\frac{7}{8}\left(\rho\left(2^k(u-v)\right)+\rho\left(2^k(v-w)\right)+\rho\left(2^k(w-u)\right)\right)\\ &\quad+\frac{1}{8}\left(\rho\left(2^k(v-u)\right)+\rho\left(2^k(w-v)\right)+\rho\left(2^k(u-w)\right)\right)\Big|\\ &\leqslant \sum_{k=\ell}^{\infty}\frac{1}{2^k}\Big|\rho\left(\frac{2^k(u+v)}{2}-2^k w\right)+\rho\left(\frac{2^k(v+w)}{2}-2^k u\right)+\rho\left(\frac{2^k(w+u)}{2}-2^k v\right)\\ &\quad-\frac{7}{8}\left(\rho\left(2^k(u-v)\right)+\rho\left(2^k(v-w)\right)+\rho\left(2^k(w-u)\right)\right)\\ &\quad+\frac{1}{8}\left(\rho\left(2^k(v-u)\right)+\rho\left(2^k(w-v)\right)+\rho\left(2^k(u-w)\right)\right)\Big|\\ &\leqslant \frac{15}{8}\mu\left(|u|+|v|+|w|\right).\end{aligned}$$

Thus f satisfies (3.1) for all $u, v, w \in \mathbb{R}$ with $0 \leqslant |u| + |v| + |w| \leqslant \frac{1}{2}$.

We claim that the additive functional equation (1.5) is not stable for $s = 1$ in Corollary 2.6. Suppose on the contrary that there exist a mapping $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (3.2). Since f is bounded and continuous for all $u \in \mathbb{R}$, $\mathcal{A}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.4, $\mathcal{A}$ must have the form $\mathcal{A}(u) = cu$ for any u in $\mathbb{R}$. Thus we obtain that

$$|f_o(u)| \leqslant (\beta + |c|)\,|u|. \tag{3.4}$$

But we can choose a positive integer m with $m\mu > \beta + |c|$.

If $u \in \left(0, \frac{1}{2^{(m-1)}}\right)$, then $2^k x \in (0, 1)$ for all $k = 0, 1, \ldots, m-1$. For this u, we get

$$f_o(u) = \sum_{n=0}^{\infty} \frac{\rho(2^k u)}{2^k} \geqslant \sum_{n=0}^{m-1} \frac{\mu(2^k u)}{2^k} = 2m\mu u > (\beta + |c|)\,u,$$

which contradicts (3.4). Therefore, the additive functional equation (1.5) is not stable in sense of Ulam-Hyers-Rassias if $s = 1$. □

Next, we will provide an example to illustrate that the functional equation (1.5) is not stable for $s = 2$ in Corollary 2.11.

Example 3.2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\rho(u) = \begin{cases} \mu u^2, & \text{if } |u| < 1, \\ \mu, & \text{otherwise}, \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f_q(u) = \sum_{n=0}^{\infty} \frac{\rho(2^k u)}{2^{2k}}, \quad \text{for all } \; u \in \mathbb{R}.$$

Then f satisfies the functional inequality

$$|Df_q(u, v, w)| \leqslant 5\mu\left(|u|^2 + |v|^2 + |w|^2\right) \tag{3.5}$$

for all $u, v, w \in \mathbb{R}$. Then, there does not exists a mapping $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f_q(u) - \mathcal{A}(u)| \leqslant \beta |u|^2, \quad \text{for all } \; u \in \mathbb{R}. \tag{3.6}$$

Proof. Now,

$$|f_q(u)| \leqslant \sum_{n=0}^{\infty} \frac{|\rho(2^k u)|}{|2^{2k}|} = \sum_{k=0}^{\infty} \frac{\mu}{2^{2k}} = \frac{4}{3}\mu.$$

Therefore we see that f is bounded. We are going to prove that f satisfies (3.5).

If $u = v = w = 0$, then (3.5) is trivial. If $|u|^2 + |v|^2 + |w|^2 \geqslant \frac{1}{2}$, then the left hand side of (3.5) is less than 5μ. Now suppose that $0 < |u|^2 + |v|^2 + |w|^2 < \frac{1}{2}$. Then there exists a positive integer ℓ such that

$$\frac{1}{2^{2(\ell+1)}} \leqslant |u| + |v| + |w| < \frac{1}{2^{2\ell}},$$

so that $2^{2(\ell-1)}|u|^2 \leqslant \frac{1}{2^2}, 2^{2(\ell-1)}|v|^2 \leqslant \frac{1}{2^2}, 2^{2(\ell-1)}|w|^2 \leqslant \frac{1}{2^2}$ and consequently

$$2^{2(\ell+1)}\left(\frac{u+v}{2} - w\right), 2^{2(\ell+1)}\left(\frac{v+w}{2} - u\right), 2^{2(\ell+1)}\left(\frac{w+u}{2} - v\right), \frac{2^{2(\ell+1)}.7}{8}(u-v),$$

$$\frac{2^{2(\ell+1)}.7}{8}(v-w), \frac{2^{2(\ell+1)}.7}{8}(w-u), -\frac{2^{2(\ell+1)}}{8}(v-u), -\frac{2^{2(\ell+1)}}{8}(w-v), -\frac{2^{2(\ell+1)}}{8}(u-w) \in (-1,1).$$

Therefore for each $k = 0, 1, \ldots, \ell-1$, we have

$$2^k\left(\frac{u+v}{2}-w\right), 2^k\left(\frac{v+w}{2}-u\right), 2^k\left(\frac{w+u}{2}-v\right), \frac{2^k.7}{8}(u-v),$$
$$\frac{2^k.7}{8}(v-w), \frac{2^k.7}{8}(w-u), -\frac{2^k}{8}(v-u), -\frac{2^k}{8}(w-v), -\frac{2^k}{8}(u-w) \in (-1,1)$$

and

$$\rho\left(\frac{2^k(u+v)}{2}-2^k w\right)+\rho\left(\frac{2^k(v+w)}{2}-2^k u\right)+\rho\left(\frac{2^k(w+u)}{2}-2^k v\right)$$
$$-\frac{7}{8}\left(\rho\left(2^k(u-v)\right)+\rho\left(2^k(v-w)\right)+\rho\left(2^k(w-u)\right)\right)$$
$$+\frac{1}{8}\left(\rho\left(2^k(v-u)\right)+\rho\left(2^k(w-v)\right)+\rho\left(2^k(u-w)\right)\right)=0$$

for $k = 0, 1, \ldots, \ell-1$. From the definition of f and (3.3), we obtain that

$$\begin{aligned}\left|Df_q(u,v,w)\right| &\leqslant \sum_{k=0}^{\infty}\frac{1}{2^{2k}}\Big|\rho\left(\frac{2^k(u+v)}{2}-2^k w\right)+\rho\left(\frac{2^k(v+w)}{2}-2^k u\right)+\rho\left(\frac{2^k(w+u)}{2}-2^k v\right)\\ &\quad-\frac{7}{8}\left(\rho\left(2^k(u-v)\right)+\rho\left(2^k(v-w)\right)+\rho\left(2^k(w-u)\right)\right)\\ &\quad+\frac{1}{8}\left(\rho\left(2^k(v-u)\right)+\rho\left(2^k(w-v)\right)+\rho\left(2^k(u-w)\right)\right)\Big|\\ &\leqslant \sum_{k=\ell}^{\infty}\frac{1}{2^{2k}}\Big|\rho\left(\frac{2^k(u+v)}{2}-2^k w\right)+\rho\left(\frac{2^k(v+w)}{2}-2^k u\right)+\rho\left(\frac{2^k(w+u)}{2}-2^k v\right)\\ &\quad-\frac{7}{8}\left(\rho\left(2^k(u-v)\right)+\rho\left(2^k(v-w)\right)+\rho\left(2^k(w-u)\right)\right)\\ &\quad+\frac{1}{8}\left(\rho\left(2^k(v-u)\right)+\rho\left(2^k(w-v)\right)+\rho\left(2^k(u-w)\right)\right)\Big|\\ &\leqslant 5\mu\left(|u|^2+|v|^2+|w|^2\right).\end{aligned}$$

Thus f satisfies (3.5) for all $u, v, w \in \mathbb{R}$ with $0 \leqslant |u|^2+|v|^2+|w|^2 \leqslant \frac{1}{2^2}$.

We claim that the additive functional equation (1.5) is not stable for $s = 2$ in Corollary 2.11. Suppose on the contrary that there exist a mapping $\mathcal{Q}: \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (3.6). Since f is bounded and continuous for all $u \in \mathbb{R}$, $\mathcal{Q}$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.9, $\mathcal{Q}$ must have the form $\mathcal{Q}(u) = cu^2$ for any u in $\mathbb{R}$. Thus we obtain that

$$|f_q(u)| \leqslant (\beta+|c|)\,|u|^2. \tag{3.7}$$

But we can choose a positive integer m with $m\mu > \beta + |c|$.

If $u \in \left(0, \frac{1}{2^{2(m-1)}}\right)$, then $2^k x \in (0,1)$ for all $k = 0, 1, \ldots, m-1$. For this u, we get

$$f_q(u) = \sum_{n=0}^{\infty}\frac{\rho(2^k u)}{2^{2k}} \geqslant \sum_{n=0}^{m-1}\frac{\mu(2^k u)}{2^{2k}} = \frac{4m}{3}\mu u^2 > (\beta+|c|)\,u^2$$

which contradicts (3.7). Therefore, the quadratic functional equation (1.5) is not stable in sense of Ulam-Hyers-Rassias if $s = 2$. □

4. Conclusion

This article has proved the stability results of the additive functional equation, quadratic functional equation, and additive-quadratic mixed type functional equations in paranormed spaces with suitable counterexamples.

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